ANR-PROPERTY OF HYPERSPACES WITH THE ATTOUCH-WETS TOPOLOGY

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In the given report we give necessary and sufficient conditions on a metric space X under which the hyperspaces $\mathrm{Cld}_{AW}(X)$, $\mathrm{Bdd}_{AW}(X)$, $\mathrm{Comp}_{AW}(X)$ and $\mathrm{Fin}_{AW}(X)$ are absolute neighborhood retracts (briefly ANR's). This completes a row of results concerning the ANR-property of hyperspaces with the Attouch-Wets topology. For the Attouch-Wets topology see [Be], [BKS].

First, we give some necessary definitions, see [Vo], [BV]. We call a metric space (X, d):

- *chain connected* if for any points $x, y \in X$ and any $\eta > 0$ there is a sequence $x = x_0, x_1, \dots, x_l = y$ of points of X such that $d(x_i, x_{i-1}) < \eta$ for all $i \le l$; such a sequence $x = x_0, x_1, \dots, x_l = y$ is called an *η-chain* linking x and y and l is the *length* of this chain;
- *chain equi-connected* if for any $\eta > 0$ there is a number $l \in \mathbb{N}$ such that any points $x, y \in X$ can be connected by an η-chain of length $\le l$;
- *uniformly locally chain equi-connected* [briefly (ulcec)] (at a subset $X_0 \subset X$) if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \eta > 0\ \exists l \in \mathbb{N}$ such that any points x, y in X (in X_0) with $d(x, y) < \delta$ can be connected by an η-chain of diameter $< \varepsilon$ and length $\le l$;
- *continuum connected at infinity* if any bounded subset of (X, d) is contained in a bounded subset complement of which has only unbounded continuum-connected components;
- *path connected at infinity* if any bounded subset of (X, d) is contained in a bounded subset complement of which has only unbounded path-connected components.

Let (X,d) be a metric space with a fixed point $x_0 \in X$. We introduce here the following notion. We say that a metric space satisfies the property ‡ if $\forall r > 0\ \exists B \supset B(x_0, r)\ \forall R > 0$: $B \subset B(x_0, R)\ \forall \eta > 0\ \exists l \in \mathbb{N}$ such that each point in $X \setminus B$ can be connected by an η-chain of length $\le l$ with some point in $X \setminus B(x_0, R)$.

Theorem 1. *The hyperspace* $\mathrm{Cld}_{AW}(X)$ *is an ANR (an AR) if and only if X is uniformly locally chain equi-connected at each bounded subset and satisfies the property ‡ (and contains no bounded chain-connected components).*

Theorem 2. *The hyperspace* $\mathrm{Bdd}_{AW}(X)$ *is an ANR if and only if* $\mathrm{Cld}_{AW}(X)$ *is an ANR if and only if X is uniformly locally chain equi-connected at each bounded subset and satisfies the property ‡*

For the hyperspace of all non-empty compact subsets of a metric space we have the following theorem.

Theorem 3. *The hyperspace* $\mathrm{Comp}_{AW}(X)$ *is an ANR (an AR) if and only if X is locally continuum connected and continuum connected at infinity (and contains no bounded connected components).*

Theorem 4. *The hyperspace* $\mathrm{Fin}_{AW}(X)$ *is an ANR (an AR) if and only if X is locally path connected and path connected at infinity (and contains no bounded connected components).*

References

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