On the Sectional Curvatures of the Time-like Generalized Ruled Surface in IR_1^n

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In IR_1^n, $(k+1)-$dimensional time-like ruled surface is defined parametrically as follows

$$\varphi(t, u_1, \ldots, u_k) = \alpha(t) + \sum_{\nu=1}^{k} u_\nu e_\nu(t)$$

and denoted as M, where the base curve α of M ruled surface is time-like curve, generating space $E_k(t)$ is space-like subspace. If otherwise mentioned, $(k+1)-$dimensional time-like ruled surface M is supposed to have a $(k-m+1)-$dimensional central ruled surface.

Two-dimensional subspace Π of $(k+1)-$dimensional time-like ruled surface at the point $\xi \in T_M(\xi)$ is called tangent section of M at point ξ.

If $\vec{v}$ and $\vec{w}$ form a basis of the tangent section Π, then $Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2$ is a non-zero quantity if and only if Π is non-degenerate. This quantity represents the square of the Lorentzian area of the parallelogram determined by $\vec{v}$ and $\vec{w}$. Using the square of the Lorentzian area of the parallelogram determined by the basis vectors $\{\vec{v}, \vec{w}\}$, one has the following classification for the tangent sections of the time-like ruled surfaces:

$$\begin{array}{lcl} Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 < 0 & , & (\text{time}-\text{like} \quad \text{plane}) \\ Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 = 0 & , & (\text{degenerate} \quad \text{plane}) \\ Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2 > 0 & , & (\text{space}-\text{like} \quad \text{plane}) \end{array}$$

For the non-degenerate tangent section Π given by the basis $\{\vec{v}, \vec{w}\}$ of M at the point ξ, the definition

$$K(\vec{v}, \vec{w}) = \frac{\langle R_{\vec{v}\vec{w}} \vec{v}, \vec{w} \rangle}{\langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2} = \frac{\sum R_{ijkm} \beta_i \gamma_j \beta_k \gamma_m}{\langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2}$$

is called sectional curvature of M at the point ξ, where $\vec{v} = \sum \beta_i \frac{\partial}{\partial x_i}$ and $\vec{w} = \sum \gamma_j \frac{\partial}{\partial x_j}$. Here the coordinates of the basis vectors $\vec{v}$ and $\vec{w}$ are $(\beta_0, \beta_1, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_k)$, respectively.

A normal tangent vector which is orthogonal to the generating space $E_k(t)$ of M.

$$n=\sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma(t)\, a_{k+\sigma}(t)+\eta_{m+1} a_{k+m+1}(t) \quad , \quad \left(\eta_{m+1}\neq 0\right)$$

is time-like or space-like vector. This means that the tangent sectional (e_ν, n), $1\le\nu\le k$, at the point $\forall\xi\in M$ time-like or space-like. This tangent section is called ν^{th} principal tangent section of M. Thus, whether ν^{th} principal tangent section is time-like or space-like, we can give following theorems and corollaries.

Theorem 1 *Let M be a generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in IR_1^n.Curvature of (e_ν, n), $1\le\nu\le k$, non-degenerate ν^{th}principal sectional curvature of M, at the point $\forall\xi\in M$ is*

$$K_\nu(\xi)=-\frac{1}{2g}\frac{\partial^2 g}{\partial u_v^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_v}\right)^2 \quad , \quad 1\le\nu\le k.$$

Corallary 1 *Let M be a generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in IR_1^n. σ^{th}, $1\le\sigma\le m$, principal sectional curvature and $(m+\rho)^{\text{th}}$, $1\le\rho\le k-m$, principal sectional curvature of M, at the point $\forall\xi\in M$ are*

$$K_\sigma(\xi)=-\frac{(\kappa_\sigma)^2\left[\sum\limits_{\iota=1}^{m}(u_\iota\kappa_\iota)^2-\eta_{m+1}^2-(u_\sigma\kappa_\sigma)^2\right]}{\left(\sum\limits_{\iota=1}^{m}(u_\iota\kappa_\iota)^2-\eta_{m+1}^2\right)^2} \quad , \quad 1\le\sigma\le m,$$
$$K_{m+\rho}(\xi)=0 \quad , \quad 1\le\rho\le k-m,$$

respectively.

Corallary 2 *Generalized time-like ruled surface with central ruled surface has no $(m+\rho)^{\text{th}}$, principal sectional curvature at the point $\forall\xi\in M$ for, $1\le\rho\le k-m$, in IR_1^n.*

Theorem 2 *Let M be a generalized time-like ruled surface with central ruled surface and n be non-degenerate normal tangent vector in IR_1^n. σ^{th}, $1\le\sigma\le m$, principal sectional curvature and $(m+\rho)^{\text{th}}$, $1\le\rho\le k-m$, principal sectional curvature of M, at the central point $\forall\zeta\in\Omega$ are*

$$K_\sigma(\zeta)=\tfrac{1}{P_\sigma^2} \quad , \quad 1\le\sigma\le m,$$
$$K_{m+\rho}(\zeta)=0 \quad , \quad 1\le\rho\le k-m,$$

respectively, where $P_\sigma=\frac{\eta_{m+1}}{\kappa_\sigma}$, $1\le\sigma\le m$, is the σ^{th} principal distribution parameter of M.

In IR_1^n, if one-dimensional generator $h_\sigma = Sp\{e_\sigma\}$, $1 \leq \sigma \leq m$, (σ^{th} principal ray) moves along the orthogonal trajectory of M, then $2-$dimensional ray surface is obtained. This surface is is called σ^{th} principal ray surface and denoted by M_σ, $1 \leq \sigma \leq m$. A parameterization of M_σ is

$$\varphi_\sigma(t,u) = \alpha(t) + u e_\sigma(t) \quad , \quad 1 \leq \sigma \leq m.$$

Now the theorem about non-degenerate sectional curvature of principal ray surfaces can be given in the following.

Theorem 3 *Let M be a generalized time-like ruled surface with central ruled surface and M_σ, $1 \leq \sigma \leq m$, be $2-$dimensional time-like σ^{th} principal ray surface in IR_1^n. For $\zeta \in \Omega \subset M$, $u \in IR$ the sectional curvature of M_σ at the point $\zeta + u e_\sigma$ on generator $h_\sigma = Sp\{e_\sigma\}$ is*

$$K_{\zeta + u e_\sigma}(e_\sigma, n) = \frac{P_\sigma^2}{(u^2 - P_\sigma^2)^2} \quad , \quad 1 \leq \sigma \leq m \tag{1}$$

where P_σ, $1 \leq \sigma \leq m$, is the σ^{th} principal distribution parameter of M.

Let M_σ, $1 \leq \sigma \leq m$, be σ^{th} principal ray surface produced by $h_\sigma = Sp\{e_\sigma\} \subset E_k(t)$ along the orthogonal trajectory of M and P_σ, be the principal distribution parameter of M in IR_1^n. The sectional curvature given in equation (1) is the generalized form of the Lamarle formula in IR_1^3, which is the relationship between the Gaussian curvature and principal parameter. Thus, equation (1) is named as *Generalized Lorentzian Lamarle Formula* by us.

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