Geometric calculus associated to second-order dynamics and applications

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Abstract

The main part of the lecture is a review of the geometric calculus I have been using for about 15 years in the study of (primarily) second-order ordinary differential equations, plus a sketch of its most successful applications.

First, we recall the notion of forms and vector fields along the tangent bundle projection $\tau : TM \to M$, the general concept of derivation of such forms and the need for a connection in the classification problem. A second-order system provides such a connection in a canonical way (Sode-connection). We further motivate the calculus along τ by showing how most geometrical objects of interest in applications arise from tensor fields along τ via suitable lifting operations, and conversely, the decomposition of tensor fields on the full tangent bundle into horizontal and vertical components, identifies the essential ingredients of the theory as being sections of an appropriate pullback bundle. An important class of derivations are the self-dual derivations of degree zero. Such derivations play a key role in the linearization of the Sode-connection, which gives rise to a so-called connection of Berwald type on the pullback bundle $\tau^* TM \to TM$. Other important ingredients of the geometric calculus are the *dynamical covariant derivative* ∇ (another self-dual derivation of degree zero) and the *Jacobi endomorphism* Φ (a type (1,1) tensor field along τ). They are the main tools, for example, in the "most economical" description of symmetries and adjoint symmetries of the given dynamical system, which is a generalization of the equation of geodesic deviation in Riemannian geometry, plus its dualization. Another field of application is the study of the multiplier problem in the inverse problem of the calculus of variations. But perhaps the most successful application so far concerns the geometric characterization of decoupling of second-order equations, both into real or complex single equations, where the theory at the same time provides algorithmic procedures for the construction of coordinates in which the decoupling takes place.

We will briefly indicate also, how the original theory for autonomous second-order equations has been extended to time-dependent systems, to higher order differen-

tial equations, to mixed first and second-order equations with applications to non-holonomic mechanics, and even to what are called second-order equations in the context of dynamics on Lie algebroids.

Finally, we will mention some applications in progress where the tools of our geometrical calculus make their appearance also in studies which are as diverse as: an exterior differential systems approach to the inverse problem of Lagrangian mechanics, the direct construction of a hierarchy of first integrals for the geodesic flow of a Finsler manifold, and the geometric characterization of so called "driven cofactor pair systems".