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On infinite Lie pseudo-groups and filtered Lie algebras

All differentiable manifolds, vector fields and maps are assumed to be of class $\mathcal{C}^\infty$.

Let M be a manifold and Θ a sheaf of germs of local vector fields defined on M. We shall denote by $J^k\Theta$ and $J^kT(M)$, $k \geq 0$, respectively the set of all k-jets of local sections of Θ and the vector bundle of k-jets of local sections of the tangent bundle $T(M)$ of M. Given a point $a \in M$, Θ_a, $J^k_a\Theta$, $J^k_aT(M)$ denote respectively the stalk of Θ at a, the set of all k-jets $j^k_a\theta$ at the point a of a local section θ of Θ, defined in a neigborhood of a and the fiber of $J^kT(M)$ over a.

Definition. Θ is an infinitesimal Lie pseudo-group defined on M (ILPG) if,

1. For every $a \in M$, Θ_a is a Lie algebra over $\mathbb{R}$ under the Lie bracket of germs of vector fields.
2. There exists an integer k_0 satisfying following conditions:
   (a) For all $k \geq k_0$, $J^k\Theta$ is a differentiable vector sub-bundle of $J^kT(M)$
   (b) A local vector field θ is a section of Θ defined on the open set $U \subset M$ if and only if $j^{k_0}_a \in J^{k_0}_a$, for all $a \in U$.

Θ is a transitive ILPG on M if $J^0\Theta = T(M)$.

Let $\mathcal{L}_a$ be the transitive filtered Lie algebra of infinite jets of local vectors defined in a neighborhood of $a \in M$ [5]. Endowed with the topology defined by the filtration, $\mathcal{L}_a$ is also a topological Lie algebra. We denote by $\mathcal{L}(\Theta, a)$ the closure in $\mathcal{L}_a$ of $J^\infty_a\Theta \subset \mathcal{L}_a$. If Θ is transitive on M, $\mathcal{L}(\Theta, a)$ is a transitive filtered subalgebra of $\mathcal{L}_a$ and also a topological subalgebra of $\mathcal{L}_a$.

Let Θ_1 be an ILPG defined on M_1. An ILPG Θ_2 defined on M_2 is a homeomorphic prolongation of Θ_1 if there exists a submersion $\rho : M_2 \to M_1$ and for every $a_1 \in M_1$ and $a_2 \in M_2$ with $\rho(a_2) = a_1$, $(\Theta_2)_{a_2}$ is projectable by ρ onto $(\Theta_1)_{a_1}$. By definition, Θ_2 is an isomorphic prolongation of Θ_1 if, moreover, for every germ $\theta_1 \in (\Theta_1)_{a_1}$, there exists only one $\theta_2 \in (\Theta_2)_{a_2}$ whose projection is θ_1.

The ILPGs Θ_1 and Θ_2 are equivalent in the sense of E. Cartan, [1], [3], [4], if there exists an ILPG Θ_3 defined on a manifold M_3 which is an isomorphic prolongation of M_1 and M_2. We say that Θ_1 and Θ_2 are locally equivalent at points $a_1 \in M_1$ and $a_2 \in M_2$ if there are open neighborhoods U_1 and U_2 of a_1 and a_2 for which the restrictions $\Theta_1|U_1$ and $\Theta_2|U_2$ are equivalent.

In the following theorem, we assume that the manifolds M_1 and M_2 and the ILPGs Θ_1 and Θ_2 are real analytic.

Theorem [2]. *Let Θ_1 and Θ_2 be two transitive ILPGs defined on M_1 and M_2. Θ_1 and Θ_2 are locally equivalent in the sense of E. Cartan, at points $a_1 \in M_1$ and $a_2 \in M_2$ if and only if $\mathcal{L}(\Theta_1, a_1)$ and $\mathcal{L}(\Theta_2, a_2)$ are isomorphic topological Lie algebras.*

References

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