Topological groups through the looking-glass

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A *ball structure* is a triplet $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii.* Given any $x \in X$ and $\alpha \in P$, we put $B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}$.

A ball structure X is called a *ballean* (or a *coarse structure*) if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exist $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B}_1 = (X_1, P_1, B_1), \mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

The category of balleans and $\prec$-mappings can be considered (see [1], [3]) as an asymptotic reflection of the category of uniform spaces and uniformly continuous mappings.

A family $\mathcal{I}$ of subsets of a group G is called a *Boolean group ideal* if

- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;
- $A \in \mathcal{I}, A' \subseteq A \Rightarrow A' \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow AB \in \mathcal{I}, A^{-1} \in \mathcal{I}$;
- $F \in \mathcal{I}$ for every finite subset F of G.

Every Boolean group ideal $\mathcal{I}$ determines the ballean $\mathcal{B}(G, \mathcal{I}) = (G, \mathcal{I}, B)$, where $B(g, A) = gA$. The balleans on G determined by the Boolean group ideals are the natural (see [1, Chapter 6]) counterparts of the group topologies on G.

We show that, for every countable group G, there are $2^{\mathfrak{c}}$ distinct Boolean group ideals on G, describe some fragments of the lattice of the Boolean group ideals on G, its interaction with T-sequences from [2], and apply these results to the Stone-Čech compactification βG of a discrete group G.

REFERENCES

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