

TOTALLY SINGULAR LAGRANGIANS AND AFFINE HAMILTONIANS

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The duality between Lagrangians and Hamiltonians is usually associated with a duality between tangent and cotangent spaces. In the case of hyperregular Lagrangians and Hamiltonians, the duality is related by a Legendre transformation.

Unfortunately, the class of hyperregular Lagrangians and Hamiltonians is too restrictive. The constraints used by Dirac allow to perform a Legendre transformation in a large class of non-hyperregular Lagrangians and Hamiltonians. We do not use here Dirac's constraints, but Lagrangians and Hamiltonians that have null vertical Hessians, i.e. the „most singular” Lagrangians and Hamiltonians; they are called here as *totally singular* ones. The Lagrangian case is remarked in [1], being related to a classification of singular Lagrangians. Marsden and Ratiu are concerned in [2] with a special case, called here the regular case, when the vector field given by the Euler equation is uniquely determined. We consider in the paper allowed totally singular Lagrangians, i.e. totally singular Lagrangians together with a vector field that is a solution of its Euler equation (in [1] one says that the Lagrangian allows a global dynamics). This Lagrangians are in duality with some corresponding totally singular Hamiltonians, so that the Legendre map sends the integral curves of the Euler equation to integral curves of the Hamilton equation. We prove that *every allowed totally singular Lagrangian has a dual allowed totally singular Lagrangian, but the converse is true only locally*. Certain examples of totally singular Lagrangians and Hamiltonians are given.

The second part is devoted to the case of allowed totally singular Lagrangians and Hamiltonians defined on the higher order tangent space $T^k M$, when the vertical Hessians of k -velocities for Lagrangians and of momenta for affine Hamiltonians respectively are vanishing (see [3, 4] for a setting of Lagrangians and Hamiltonians of higher order). A duality between totally singular allowed Lagrangians and affine Hamiltonians is considered also in this case. Allowed totally singular Lagrangian has a dual allowed totally singular Hamiltonian, but for the converse situation, assuming some conditions, we prove that *an allowed totally singular Hamiltonian of order k has a totally singular allowed Lagrangian of order k and both can be related to ordinary dual (allowed totally singular) Lagrangians and Hamiltonians on $T^{k-1} M$* .

In order to have consistent examples of totally singular Lagrangians and Hamiltonians of higher order, lifting procedures are given. In this way, certain

examples considered in the first section, or examples considered in [2] or [1] can be lifted to totally singular Lagrangians and Hamiltonians of higher order.

References

- [1] Cantrijn F., Cariñena J.F., Crampin M., Iborat A., *Reduction of degenerate Lagrangian systems*, J. Geom. Phys, 3, (3) (1986), 353-400.
- [2] Marsden J., Ratiu T., *Introduction to Mechanics and Symmetry*, Second Edition, Springer-Verlag New York, Inc., 1999.
- [3] Popescu P., Popescu Marcela, *Affine Hamiltonians in Higher Order Geometry*, Int. J. Theor. Phys. (to appear).
- [4] Popescu P., Popescu Marcela, *Lagrangians and Hamiltonians on Affine Bundles and Higher Order Geometry*, Proc. vol. VII International Conference *Geometry and Topology of Manifolds*, The Mathematical Legacy of Charles Ehresmann on the occasion of the hundredth anniversary of his birthday (Będlewo, Poland, May 8–15, 2005), Banach Center Publications, Editors Jan Kubarski, Jean Pradines, Tomasz Rybicki, Robert Wolak, (to appear).

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