Orbits structure of co-adjoint action and bystages hypothesis

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We shall describe the structure of orbits of the coadjoint action of a connected Lie group on the dual space to its Lie algebra. Classical results of Kostant give a fairly complete invariant-theoretic picture of the (co)adjoint action in the case of a reductive Lie group. Our result is a generalization of the results of Rawnsley [1], Guillemin and Sternberg [2] and Raïs [3] obtained for semi-direct product of a Lie group and a linear space. This linear space is a normal commutative subgroup. We shall consider the general case of a non-commutative normal subgroup.

Let G be a connected real (or complex) Lie group with the Lie algebra $\mathfrak{g}$. Consider the coadjoint representation Ad^* of the Lie group G on the dual space $\mathfrak{g}^*$. Let $\mathcal{O}_\alpha = \mathrm{Ad}^*(G)\alpha$ be an orbit in $\mathfrak{g}^*$ through an arbitrary covector $\alpha \in \mathfrak{g}^*$.

Suppose that G is not a simple Lie group. Let $A \subset G$ be a normal closed subgroup of G with the Lie algebra $\mathfrak{a} \subset \mathfrak{g}$. Since the subalgebra $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, the adjoint representations of G in $\mathfrak{g}$ induces the representation φ of G in $\mathfrak{a}$, and consequently the action of G in $\mathfrak{a}^*$ (associated with the dual representation φ^*). Denote by $\mathcal{O}'_\nu$ the corresponding orbit in $\mathfrak{a}^*$ through the element (restriction) $\nu = \alpha|\mathfrak{a}$. Then $\mathcal{O}'_\nu = G/G_\nu$, where G_ν is the isotropy group of ν.

We shall prove that the G-orbits $\mathcal{O}_\alpha$ in $\mathfrak{g}^*$ are fibered over the G-orbits $\mathcal{O}'_\nu$ in $\mathfrak{a}^*$; its fibre type is a direct product of some vector space V_ν and a coadjoint orbit of some (not necessary connected) Lie group Q_ν, which is a factor group of the isotropy group G_ν. There is a one to one correspondence between the set of all G-orbits $\mathcal{O}_\alpha$ with $\alpha|\mathfrak{a} = \nu$ and the set of all coadjoint Q_ν-orbits in the dual space of the Lie algebra of Q_ν. The space V_ν is the same for all these orbits $\mathcal{O}_\alpha$ (with $\alpha|\mathfrak{a} = \nu$) and has a dimension equal to the codimension of the Lie group $G_\nu \cdot A$ in G.

Let $\mathfrak{g}_\nu$ be the Lie algebra with the Lie group G_ν. As a corollary, from the proof of the above described result, we obtain the following statement ("bystages hypotesis" [4]):

(BH) Let σ_1, σ_2 be covectors from $\mathfrak{g}^*$ such that for its restrictions to subspaces $\mathfrak{a}$ and $\mathfrak{g}_\nu$ we have $\sigma_1|\mathfrak{a} = \sigma_2|\mathfrak{a} = \nu$ and $\sigma_1|\mathfrak{g}_\nu = \sigma_2|\mathfrak{g}_\nu = \tau$. Then $\sigma_2 = \mathrm{Ad}_g^* \sigma_1$ for some $g \in (G_\nu)_\tau$, where $(G_\nu)_\tau$ is the isotropy group of $\tau \in \mathfrak{g}_\nu^*$ for the coadjoint action of G_ν on $\mathfrak{g}_\nu^*$.

1. J. H. Rawnsley, *Representation of a semi-direct product by quantization*, Math. Proc. Cambr. Phil. Soc., **78** (1975), 345–350.

2. V. Guillemin and S. Sternberg, *The moment mat and collective motion*, Annals of Physics, **127** (1980), 220–253.

3. M. Raïs, *L'indice des produits semi-directs $E \times_\rho \mathfrak{g}$*, C. R. Acad. Sci. Paris Ser. A, **287** (1978), 195–197.

4. J. Marsden, G. Misiołek, M. Perlmutter, and T. Ratiu, *Symplectic reduction for semidirect products and central extensions*, Diff. Geom. and its Appl. **9** (1998), 173212.