Underdetermined systems of ODEs – the geometric approach of E. Cartan

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Abstract

The equation

$$z' = \left(y''\right)^2 \tag{1}$$

(derivatives are with respect to an independent variable x) canNOT be solved by giving parametric expressions $x(t) = \varphi$, $y(t) = \psi$, $z(t) = \chi$, where φ, ψ, χ are certain fixed functions of t, $f(t)$, $f'(t), \ldots, f^{(n)}(t)$ and f is a free function of one variable t. (The order n of the highest derivative involved is also fixed, not depending on f.) This was shown in 1912 by D. Hilbert, [H]; the equation itself was very popular in the 1910s. Other researchers active around that time in the field of underdetermined systems of ODEs were E. Cartan, E. Goursat, P. Zervos. The first of them published in 1914, [C], a theorem giving a local characterization of the underdetermined systems, of degree of underdeterminacy 1, having the mentioned property of parametrization of solutions. And Hilbert's example (1) simply did not satisfy that characterization. (Very close topics had also been addressed by Zervos in [Z].)

That is to say, Cartan proposed a theorem subsuming Hilbert's result. Th equation (1) turns out to have a too rich geometry (so-called special Cartan '$2, 5$' geometry) precluding such parametrizations of solutions. The accent in the talk will be precisely on the geometric side of the problem, partially following a modern approach of A. Kumpera in [K]. Also an open question will be formulated, falling close to Cartan's theorem.

References

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[Z] P. Zervos; *Sur l'integration de certains systemes indetermines d'equations differentielles*, Journal de Crelle (J. reine angew. Math.) **143** (1913), 300 – 312.