A NOTE ON THE LAGRANGE DIFFERENTIAL IN BIGRADED MODULE OF VERTICAL TANGENT BUNDLE VALUED FORMS

Roman Ya. MATSYUK

Within the finite order approach to the split calculus of variations on fibred manifolds we cast the set of Euler–Lagrange expressions in the guise of a volume semi-basic differential form with values in the dual to the bundle V of vertical tangent vectors on some fibration $Y \to B$. Although this treatment in not new, it helps to establish a simple relation between the operators d_π of fiber derivative and D of semi-basic derivative in the appropriate bigraded tensor algebra of cross-sections of the vector bundle $\wedge^\bullet V_r^* \otimes_{Y_r} \wedge^\bullet T^*B$ on one hand, and the operators d_V and d_H previously introduced by Tulczyjew in the total bigraded algebra of exterior differential forms on the corresponding jet bundle prolongation $\pi_r : Y_r \to B$, on the other hand. The split bigraded algebra $Sec\left(\wedge^\bullet V_r^* \otimes_{Y_r} \wedge^\bullet T^*B\right)$ may be converted into the total algebra $\wedge^\bullet T^*Y_{r+1}$ applying the dual of the Cartan contact form morphism $T^*Y_{r+1} \to V_r$ together with subsequent alternation. By means of this mapping operators d_V and d_H correspond to the operators d_π and D respectively.

Let y^α_{N} denote standard coordinates in Y_r with multiindex N of order $\|\mathrm{N}\| \leq r$. Let $\iota^i dy^\alpha_{\mathrm{N}} = dy^\alpha_{\mathrm{N}-1_i}$ be the order-reducing operators similar to those introduced by Tulczyjew, and let $D_i =< \partial_i, D >$. The split Lagrange differential δ is first defined on the cross-sections of $\wedge^\bullet V_r^*$ as

$$\delta\varphi = \deg(d_\pi\varphi) + \sum_{\|\mathrm{N}\|\geq 0} \frac{(-1)^{\|\mathrm{N}\|}}{\mathrm{N}!} D_{\mathrm{N}}\iota^{\mathrm{N}} d_\pi\varphi\,,$$

and then extended to the total algebra $Sec\left(\wedge^\bullet V_r^* \otimes_{Y_r} \wedge^\bullet T^*B\right)$ by trivial action on the subalgebra of scalar differential forms on B.

As conventionally assumed, the variational derivative of the Lagrange density $\lambda \in Sec\left(\wedge^{\dim B} T^*B\right)$ at the extremal section υ of Y is an r^{th} order differential operator $\mathcal{E}^\upsilon$ acting from the space of cross-sections of the induced vertical tangent bundle $\upsilon^{-1}V$ to the space $Sec\left(\wedge^{\dim B} T^*B\right)$. Then the transpose operator ${}^t\mathcal{E}^\upsilon$ is given by the formula

$${}^t\left(\mathcal{E}^\upsilon\right)(1) = \left(j_{2r}\upsilon\right)^*\delta\lambda\,,$$

where $\delta\lambda \in V^* \otimes_{Y_{2r}} \wedge^{\dim B} T^*B$, and the pull-back of V^*–valued differential form $\left(j_{2r}\upsilon\right)^*\delta\lambda$ is a cross-section of the vector bundle $\upsilon^{-1}V^* \otimes \wedge^{\dim B} T^*B$.