Ivan Bernoulli Series Universalissima

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Abstract

One states, recalls here that the textbooks formula named as celebrated Taylor formula pertains historically to Johann Bernoulli (1667-1748). When Ivan Bernoulli‘s *Series Universalisima* was published in Acta Eruditorum in Leipzig in 1694 Brook Taylor was nine years old. As on the 01.01.2006 one affirms 258th anniversary of death of Johann Bernoulli this calendarium-article is *Pro*posed *Pro hac vice, Pro memoria, Pro nunc and Pro opportunitate* - as circumstances allow.

Key Words: Bernoulli-Taylor formula, Graves-Heisenberg-Weyl (GHW)algebra, umbral calculus.

1. The first historical remarks

Principal Data

1.I. Johann **Bernoulli** [1667-1748]- ”*Archimedes of his age*”. Johann Bernoulli attained great fame already in his lifetime. He was elected a fellow of the academies of St Petersburg, Paris, Berlin, London and Bologna. He was called the ”*Archimedes of his age*” and this is what is inscribed on his tombstone. He is also famous as a Master of the brilliant pupil Leonhard.

1.II. Leonhard **Euler**[1707-1783]. Leonhard Euler - also born in Switzerland -studied under Giovanni Bernoulli at Basel. Euler and Bernoullies became also Personalities of the Great Imperial St. Petersburg of those times. Euler supported a lifelong friendship with Ivan Bernoulli’s sons Daniel. Jean or Giovanni or Johann is Ivan there in Russia‘s Capital St. Peterburg as in Sonin‘s article title (See Sonin,1897).

Note: Johann Bernoulli‘s ”Series universalissima” was already at work in his Acta Eruditorum paper published in 1694 in Leipzig. Complete list is available via address below.

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Bibliography-Data: Johann (Giovanni, Jean, Ivan) Bernoulli (27.01.1667-01.01.1748)

List of publications of Johann Bernoulli including post mortem papers with quotations from and on Johann Bernoulli are attainable via Brandenburgische Akademie der Wissenschaften Akademiebibliothek in

`http://bibliothek.bbaw.de/kataloge/literaturnachweise/bernou-1/literatur.pdf`

1.III. Brook **Taylor** [1685-1731].

Note: Taylor's Methodus incrementorum directa et inversa (1715) had given birth to what is now called the "calculus of finite differences". It also contained the celebrated formula known as Taylor's expansion, the importance of which was recognized only in 1772 when Lagrange proclaimed it the basic principle of the differential calculus.

1.IV. N.Y. **Sonin**. Nikolay Yakovlevich Sonin [1849-1915] was the pupil of Nicolai Vasilievich Bugaev and went on to make a major contribution to mathematics. He taught at the **University of Warsaw** where he obtained a doctorate in 1874. Then in 1876 N.Y. Sonin was appointed to a chair in the University of Warsaw. In 1894 Sonin moved to St Petersburg. Together with A A Markov, Sonin prepared a two volume edition of Chebyshev's works in French and Russian. Here for us of primary importance is Sonin's article on Ivan Bernoulli. The source of detailed information and historian like investigation paper. This is Academician at that time Sonin's article(Sonin, 1897) dated from **1897**. There (p. 341) he quotes and supports the conviction of an anonymous author of *Epistola pro eminente mathematico, Dn. Johanno Bernulio, contra quedam ex Anglia antagonistam scripta* in Acta Eruditorum 1716 p.307 -the conviction that one deals here with plagiarism in Taylor's Methodus (1715).

According to Salvatore Anastasio (see more in the third section) "Historians have since shown that Taylor was not guilty of plagiarism, but only of having failed to keep up on the literature from the Continent." In the same spirit in one of their internet articles

http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Taylor.html

by J. J. O'Connor and E. F. Robertson the authors indicate that what we now call Taylor series is the one which *Taylor was the first to discover*. James Gregory [1638-1675], Newton [1643-1724], Leibniz [1646-1716], Johann Bernoulli [1667-1748] and de Moivre [1667-1754] had all known *variants of Taylor's Theorem*. James Gregory, for example, knew that *arctanx* = *x* - *x*3/3+*x*5/5 - x7/7+.... J. J. O'Connor and E. F. Robertson state that "*all of these mathematicians had made their discoveries independently, and Taylor's work was also independent of that of the others. The importance of Taylor's Theorem remained unrecognised until 1772 when Lagrange proclaimed it the basic principle of the differential calculus. The term "Taylor's series" seems to have used for the first time by Lhuilier in 1786* ".

Well. Those were the times without e-mails though famous margins and covers were prosperously used...and here again the same authors about James Gregory: *"In February 1671 he discovered Taylor's theorem (not published by Taylor until 1715), and the theorem is contained* *in a letter sent to Collins on 15 February 1671. The notes Gregory made in discovering this result still exist written on the back of a letter sent to Gregory on 30 January 1671 by an Edinburgh bookseller. Collins wrote back to say that Newton had found a similar result and Gregory decided to wait until Newton had published before he went into print. He still felt badly about his dispute with Huygens and he certainly did not wish to become embroiled in a similar dispute with Newton*". The above quotation comes from

`http : //www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Gregory.html`

The similar story is with Graves and GHW algebra. Though this is another story of the History. *History of Mathematics.*

1.V. C.Graves (1853-1857). The Graves-Heisenberg-Weyl (GHW) algebra is called by physicists the Heisenberg-Weyl algebra. See: Kwaśniewski (2004,2005) and references therein). We shall deal with GHW algebra in the sequel. Before that let us continue with the main story.

The first historical remark with nowadays form of formulas. Here are the famous examples of expansion

$$\partial_0 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \frac{d^n}{dx^n}$$

or

$$\epsilon_0 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \frac{d^n}{dx^n}$$

where ∂_0 is the divided difference operator while ϵ_0 is at the zero point evaluation functional. If one compares these with "*series universalissima*" of J.Bernoulli from *Acta Erudicorum* (1694) and with

$$exp\{yD\} = \sum_{k=0}^{\infty} \frac{y^k D^k}{k!}, D = \frac{d}{dx},$$

then confrontation with B.Taylor's "*Methodus incrementorum directa et inversa*" (1715), London; entitles one to call the expansion formulas considered above and in this note "*Bernoulli-Taylor formulas*" or (for $n \to \infty$) "*Bernoulli-Taylor series*" See: Kwaśniewski (2004,2005,2003). In support to this conviction postulate- facts come to become the support. **Facts**. In 1694 Johann Bernoulli published articles with his series univeralissima applied to several functions:

[J.Bernoulli *a*] Modus generalis construendi omnes Aequationes differentiales primigradus. Lipsiae 1694 in: Acta Eruditorum, 1694. S.435-437.

[J.Bernoulli *b*] Additamentum effectionis omnium Quadraturarum et Rectificationum Curvarum per seriem quandam generalissimam. Lipsiae 1694 in: Acta Eruditorum, 1694. S.437-441

From Sonin (1897, p.341) we learn more. In August of 1695 in the letter to Leibniz Johann Bernoulli among others- also considered the function $f(x) = x^x$ and calculated its integral over [0, 1] interval using his series universalissima. This was highly appraised by Leibniz who quoted this result without proof in his article ”Principia Calculi Exponentialium, seu Percurrentium” on mart of 1697 in Acta eruditorum, Leipzig.

Well, after then series universalissima was not unknown and more . It was really ”universalissima” in the operational spirit of Leibniz approach to Calculithe up today modern and efficient attitude to Calculi. It is so for example in combinatorics with Laplace‘s generating functions‘ rigorous apparatus that had been harnessed to work efficiently since its firmly established birth in Bernoulli‘s Acta Eruditorum articles.

2. The second historical remark on Johann Bernoulli and Brook Taylor as both are Great

Information 2.1. Johann **Bernoulli** was elected a fellow of the academy of St Petersburg. Johann Bernoulli- the Discoverer of *Series Universalissima* was ”**Archimedes of his age**” and this is indeed inscribed on his tombstone.

Information 2.2. Brook **Taylor**. ”*There are other important ideas which are contained in the Methodus incrementorum directa et inversa of 1715 which were not recognised as important at the time. These include singular solutions to differential equations, a change of variables formula, and a way of relating the derivative of a function to the derivative of the inverse function. Also contained is a discussion on vibrating strings, an interest which almost certainly come from Taylor's early love of music.*”

Brook ”***Taylor** was a mathematician of far greater depth than many have given him credit for:- A study of Brook Taylor's life and work reveals that his contribution to the development of mathematics was substantially greater than the attachment of his name to one theorem would suggest. His work was concise and hard to follow. The surprising number of major concepts that he touched upon, initially developed, but failed to elaborate further leads one to regret that health, family concerns and sadness, or other unassessable factors, including wealth and parental dominance, restricted the mathematically productive portion of his relatively short life.*”

These extracts come from the article by J. J. O'Connor and E. F. Robertson

`http : //www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Taylor.html`

3. The third historical remark with champagne is a quotation

This third historical remark with champagne in Champagne is a quotation of an unsurpassed feature short note by Salvatore Anastasio on Bernoulli-Taylor litigation.

`http : //www.newpaltz.edu/math/fall98.pdfTheEndofaFeudby`

Quotation "*The End of a Feud*" by Sal Anastasio **(1998).**

Many students of mathematics are aware of the controversy between Isaac Newton and Gottfried Wilhem Leibniz over which of them was the first to "discover" the Calculus. For many years in the early eighteenth century, followers of Newton and Leibniz disputed bitterly over the priority issue. However, historians today are convinced that each man deserves credit as an independent "discoverer". What is less well known is the 275 year long feud between the families of Brook Taylor(1685-1731) and Johann Bernoulli(1667-1748), only recently ended. Brook Taylor (yes, he of "Taylor Series" fame) was an English mathematician deeply committed to the cause of Newton. **Bernoulli**, equally committed to Leibniz, was a Swiss mathematician who **greatly developed Leibniz' techniques (which proved far superior to Newton's and those are common today)** and was responsible for the first textbook in Calculus: L'Hospital's Analyse des Inniment Petits, 1696. ("L'Hospital's" ? Yes, but that's another story!) It appears that Taylor and Bernoulli began their own dispute in 1715, when Taylor published results in England which had already been discovered on the Continent by Leibniz and Bernoulli. (Historians have since shown that Taylor was not guilty of plagiarism, but only of having failed to keep up on the literature from the Continent.) The French probabilist, Pierre Remond de Montmort, apparently the Jimmy Carter of his day, tried very hard, but in vain, to patch things up between Taylor and Bernoulli. But, not to worry. About 8 years ago, on July 7, 1990, Francois de Montmort, a descendant of Pierre, played host, at the de Montmort ancestral chateau in Champagne, to descendants of Bernoulliand Taylor: Rene Bernoulli of Basel, Switzerland, and Chalmers Trench of Slane, Ireland.

After toasting each other with champagne (in Champagne), the two journeyed to the front lawn of the chateau with a shovel in tow. There they solemnly dug a hole (would I lie?) and successfully buried a hatchet (I kid you not) which had been provided by an American historian of science. The feud is now officially over."

4. Closing and rather technical remarks for today

Remarks to be rather skip over by non specialists. Introductory information. An extension of Ivan Bernoulli formula of a new sort with the rest term of the Cauchy type was recently derived also by the author in the case of the so called ψ-difference calculus which constitutes the representative case of extended umbral calculus. Naturally the central importance of such a type formulas is beyond any doubt - and recent publications do confirm this historically established experience.

4.I. All formulas based on Bernoulli-Taylor formula for extended umbral calculus from Kwaśniewski (2004) (and corresponding references by the author therein) may be quite easily extended to the case of any $Q \in End(P)$ linear operator that reduces by one the degree of each polynomial ; see Markowsky (1978). Namely one introduces:

Definition 4.1.

$$\hat{x}_Q \in End(P), \hat{x}_Q : F[x] \to F[x]$$

such that $(x^n) = \frac{(n+1)}{(n+1)_\psi} q_{n+1}; n \geq 0$; where $Qq_n = nq_{n-1}$.

Then $*_Q$ product of formal series and Q-integration are defined almost mnemonic analogously; see Kwaśniewski (2004,2005,2003,2002).

4.II. In 1937 Jean Delsarte (1937) had derived the general Bernoulli-Taylor formula for a class of linear operators δ including linear operators that reduce by one the degree of each polynomial. The rest term of the Cauchy-like type in his Taylor formula (I) is given in terms of the unique solution of a first order partial differential equation in two real variables. This first order partial differential equation is determined by the choice of the linear operator δ and the function f under expansion. In our Bernoulli-Taylor ψ–formula or in its straightforward $*_Q$ product of formal series and Q-integration generalizationthere is no need to solve any partial differential equation.

4.III. In the magnificent paper Steffensen(1941)- Professor J. F. Steffensen, the Master of polynomials application to actuarial problems

(*see : http : //www.math.ku.dk/arkivet/jfsteff/stfarkiv.htm*)

supplied a remarkable derivation of another Bernoulli-Taylor formula with the rest of ”Q-Cauchy type” in the example presenting the ”Abel poweroids”

4.IV. The recent paper (see Ismail et. al., 2003) by Mourad E. H. Ismail and Denis Stanton, may serve as a kind of indication for pursuing further investigation. There the authors have established two new q-analogues of a Taylor series expansion for polynomials using special Askey-Wilson polynomial bases. As ”byproducts” their important paper includes also new summation theorems,quadratic transformations for q-series and new results on a q-exponential function.

4.V. Let us also draw an attention to two more different publication on the subject which are the ones referred to as Ernst(1999) and (Jing et al.,1995). The q-Bernoulli theorems are named here and there above as q-Taylor theorems. The corresponding (q, h)-Bernoulli theorem for the $\partial_{q,h}$-difference calculus of Hahn (see Kwaśniewski,2005) might be obtained the same way as the q-Bernoulli-Taylor theorem constituting here the special case of the Viskov method application. This is so because the $\partial_{q,h}$-difference calculus of Hahn may be reduced to q-calculus of Thomae-Jackson due to the following observation. Let

$$h \in \_R,\ (E_{q,h}\varphi)(x) = \varphi(qx + h)$$

and let

$$(\partial_{q,h}\varphi)(x) = \frac{\varphi(x) - \varphi(qx+h)}{(1-q)x - h} \tag{1}$$

Then (see Hann in Kwaśniewski(2005))

$$\partial_{q,h} = E_{1,\frac{-h}{1-q}} \partial_q E_{1,\frac{h}{1-q}}.$$

It is easy now to derive corresponding formulas including Bernoulli-Taylor $\partial_{q,h}$-formula obtained in Kirschenhofer (1979) now by the Viskov method (see Kwaśniewski,2005,2004,2003) which for

$$q \to 1,\ h \to 0$$

recovers the content of one of Viskov examples while for

$$q \to 1,\ h \to 1$$

one recovers the content of the another Viskov example. The case h $\to$ 0 is included in the formulas of q-calculus of Thomae-Jackson easy to be specified: see Kwa´sniewski (2005) (see also **thousands** of up-dated references there). For Bernoulli-Taylor Formula (presented during *PTM*-Convention Lodz - 2002)one may contact Kwaśniewski (2003) for its recent version.

4.VI. As in Kwaśniewski (2003) the rule $\tilde{g}(x) = g(\hat{x}_\varphi)\mathbf{1}$ defines the map $\sim : g \mapsto \tilde{g}$ which is an umbral operator $\sim : P \mapsto P$. It is mnemonic extension of the corresponding q - definition by Kirschenhofer (1979) and Cigler (1979). This umbral operator (without reference to Kirschenhofer (1979) and Cigler (1979)) had been already used in theoretical physics aiming at Quantum Mechanics on the lattice ; see Dimakis et all (1996). The similar aim is represented in Zapatrin (2000) and Brelav et all (2000) (see further references there) where incidence algebras are being prepared for that purpose. As it is well known - the classical umbral [See Doubilet et al.(1972) and Rota (1975)] and extended finite operator calculi (see Kwaśniewski, 2003) may be formulated in the reduced incidence algebra language. Hence both applications of related tools to the same goal are expected to meet at the arena of GHW algebra description of both [see Kwaśniewski(2003), and Kirschenhofer (1979)].

At the end let us again come back to 17-th and 18-th centuries springs of modern Calculi.

Recall. **Johann Bernoulli** (27.01.1667-01.01.1748). We thus affirm at the very 2007 year the **259th** anniversary of death of Johann Bernoulli **just falling on coming 01.01.2007.**

His Legacy Bibliography-Data:

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`http : //bibliothek.bbaw.de/kataloge/literaturnachweise/bernou-1/literatur.pdf`

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