OLGA KRAVCHUK
KHMELNYTSKYI STATE
UNIVERSITY (UKRAINE)

TO INVARIANT NORMALES OF VECTOR FIELD IN FOUR-DIMENSIONAL AFFINE SPACE

Abstract.

In four- dimensional affine space invariant points, straights, hypersurfices and two-dimensional spaces associated with vector field have been built.

The received data are transformed into spaces of indefinite size.

P.1. Differential equations of consistent geometric objects of vector field in A_4.

Let`s consider four-dimensional affine space A_4, related to moving reper

$$d\vec{A} = \omega^{\alpha}\vec{e}_{\alpha i} \qquad (1.1)$$

$$d\vec{e}_{\alpha} = \omega^{\alpha}_{\beta i}\vec{e}_{\beta}$$

Equations of space A_4 structure are taken into consideration are following

$$D\omega^{\alpha} = \left[\omega^{\beta}\omega^{\alpha}_{\beta}\right], D\omega^{\alpha}_{\beta} = \left[\omega^{\gamma}_{\beta}\omega^{\alpha}_{\gamma}\right], \ \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \ \ldots = \overline{1,4} \qquad (1.2)$$

Definition: Vector element of A_4 space is called a set which consists of point A and vector a, for this one point A is beginning. Vector element will be define as (A, $\vec{a}$). In this case point A will be called as the beginning of vector element.

It is obvious that a $\vec{a}$ є V_4 and A є A_4

Definition: Vector field in A_4 is the set in which every point of A_4 space is coordinated in some way definite vector $\vec{a}$ with beginning in this point.

In all vector elements' set vector fields define a kind of semi-polytype.

Remark, that vector field can be done as in whole space A_4 but also in a separate its region.

In future we'll late into account that beginning of vector coincides in A point then $\boldsymbol{\delta A} = 0$ forms ω^{α} are the main (central).

We'll express vector $\vec{a}$ through vectors of basis $\vec{e}^{\alpha}$ in the form

$$\vec{a} = a^{\alpha}\vec{e}_{\alpha} \tag{1.3}$$

Coordinates of vector $\vec{a}$ will satisfy differential equations

$$da^{\alpha} + a^{\beta}\omega_{\beta}^{\alpha} = a_{\beta}^{\alpha}\omega^{\beta} \tag{1.4}$$

Continuing equations (1.4) receive

$$da_{\beta}^{\alpha} = a_{j}^{\alpha}\omega_{\beta}^{\gamma} - a_{\beta}^{\chi}\omega_{j}^{\alpha} + a_{\beta j}^{\alpha}\omega^{\gamma} \tag{1.5}$$

while $a_{\beta j}^{\alpha} = a_{j\beta}^{\alpha}$

Equation (1.5)can be presented in form:

$$\begin{aligned}
&da_{j}^{i} - a_{k}^{i}\omega_{j}^{k} - a_{4}^{i}\omega_{j}^{4} + a_{j}^{k}\omega_{k}^{i} + a_{j}^{4}\omega_{4}^{i} = a_{j\beta}^{i}\omega^{2}\\
&da_{4}^{i} - a_{k}^{i}\omega_{4}^{k} - a_{4}^{i}\omega_{4}^{4} + a_{4}^{k}\omega_{k}^{i} + a_{4}^{4}\omega_{4}^{i} = a_{4\alpha}^{i}\omega^{2}\\
&da_{j}^{4} - a_{j}^{4}\omega_{i}^{j} - a_{4}^{4}\omega_{i}^{4} + a_{i}^{j}\omega_{j}^{4} + a_{i}^{4}\omega_{4}^{4} = a_{i\alpha}^{4}\omega^{2}\\
&da_{4}^{4} - a_{j}^{4}\omega_{4}^{j} - a_{4}^{i}\omega_{j}^{4} = a_{4\alpha}^{4}\omega^{2},\ \left(i, j, k = \overline{1,3}\right)
\end{aligned} \tag{1.6}$$

In fixation of main parameters differential equations (1.4) and (1.5) take correspondently form

$$\delta a^{\alpha} + a^{\beta}\pi_{\beta}^{\alpha} = 0 \tag{1.4'}$$

$$\delta a_{\beta}^{\alpha} = a_{j}^{\alpha}\pi_{\beta}^{\gamma} - a_{\beta}^{\gamma}\pi_{j}^{\alpha} = 0 \tag{1.5'}$$

From the previous data it is clear, that fundamental object of the first order $\{a^{\alpha}\}$ is tensor and fundamental object of the second order $\{a^{\alpha}, a_{\beta}^{\alpha}\}$consists of two tensors a^{α} and a_{β}^{α}.

Continuing differential equations (1.5) we'll receive a set of fundamental objects $\{a^{\alpha}, a_{\beta}^{\alpha}, a_{\beta j}^{\alpha}, a_{\beta j\delta}^{\alpha}, \ldots\}$, which lies in basis of differential geometry of vector field in four-dimensional equiaffine space A_4.

P.2. Some fields of invariant geometric objects are united with vector field.

Let's find differential equations of some invariant geometric objects joined to vector field.

2.1. Field of points. Let's consider point $P(x^{\alpha})$in affine space A$_4$. When $\vec{P}$ – radius – vector of this point in this case related to affine reper $(\vec{A},\vec{e}_{\alpha})$ it can be done with the following phrase

$$\vec{P}=\vec{A}+x^{\alpha}\vec{e}_{\alpha} \tag{2.1}$$

Differentiating (2.1) taking into account equation of structure, we'll receive

$$dx^{\alpha}+x^{\beta}\omega_{\beta}^{\alpha}=x_{\beta}^{\alpha}\omega^{\beta} \tag{2.2}$$

or in fixation of main parameters

$$\delta x^{\alpha}+x^{\beta}\pi_{\beta}^{\alpha}=0 \tag{2.3}$$

Let's consider values $N^{\alpha}=a_{\beta}^{\alpha}a^{\beta}$ (2.4)

If differential equations consider values (2.4) have the form $dN^{\alpha}+N^{\alpha}\omega_{j}^{\beta}=N_{\beta}^{\alpha}\omega^{\beta}$ according to (2.2) they define invariant point.

2.2. Field of straights. Straight, which cross the A point with directed vector $R=\nu^{\alpha}e_{\alpha}$ define as $l=[A,R]$

Conditions of invariantness of straight will be

$$\delta R=QR\,,\ dQ=0 \tag{2.5}$$

From the previous data $\delta\nu^{\alpha}+\nu^{\beta}\pi_{\beta}^{\alpha}=Q\nu^{2}$, (2.6)

$$\text{or}\quad \begin{aligned}&\delta\nu^{i}+\nu^{i}\pi_{j}^{i}+\nu^{n}\pi_{n}^{i}=Q\nu^{i}\\&\delta\nu^{n}+\nu^{i}\pi_{i}^{n}+\nu^{n}\pi_{n}^{n}=Q\nu^{n}\end{aligned} \tag{2.7}$$

Writing down correspondence (2.6) we'll have

$$\begin{aligned}&d\nu^{1}+\nu^{1}\omega_{1}^{1}+\nu^{2}\omega_{2}^{1}+\nu^{3}\omega_{3}^{1}+\nu^{4}\omega_{4}^{1}=Q\nu^{1}\\&d\nu^{2}+\nu^{1}\omega_{1}^{2}+\nu^{2}\omega_{2}^{2}+\nu^{3}\omega_{3}^{2}+\nu^{4}\omega_{4}^{2}=Q\nu^{2}\\&d\nu^{3}+\nu^{1}\omega_{1}^{3}+\nu^{2}\omega_{2}^{3}+\nu^{3}\omega_{3}^{3}+\nu^{4}\omega_{4}^{3}=Q\nu^{3}\\&d\nu^{4}+\nu^{1}\omega_{1}^{4}+\nu^{2}\omega_{2}^{4}+\nu^{3}\omega_{3}^{4}+\nu^{4}\omega_{4}^{4}=Q\nu^{4}\end{aligned} \tag{2.7'}$$

Sometimes it is convenient to promote norm of vector $\vec{R}$ in which$\nu^{n}=1$. Then

$$Q=\omega_{n}^{n}+\nu^{i}\omega_{i}^{n} \tag{2.8}$$

Putting (2.8) in the first equation (2.6) we'll have

$$\delta\nu^{i}+\nu^{j}\pi_{j}^{i}-\nu^{i}\pi_{n}^{n}-\nu^{j}\nu^{k}\pi_{k}^{n}+\pi_{n}^{i}=0 \tag{2.9}$$

Thus, differential equations of invariantness of straight will have a form

$$d\nu^i + \nu^j\omega_j^i - \nu^i\omega_n^n - \nu^i\nu^k\omega_k^n + \omega_n^i = \nu_\alpha^i\omega^\alpha \qquad (2.10)$$

Let's build values b_α^β (in condition of $def \ // \ a_\beta^\alpha \ // \ \neq 0$) $a_\beta^\gamma b_\gamma^\beta = \delta_\alpha^\beta$ (2.11)

And with their help values

$$\begin{aligned} M^\alpha &= b_\beta^\alpha a^\beta \\ dM^\alpha &= -M^\beta\omega_\beta^\alpha + M_\gamma^\alpha\omega^\gamma \end{aligned} \qquad (2.12)$$

If differential equations of (2.12) have the equations' structure (2.6), the pair $[A,M]$, where $\vec{M} = M^\alpha\vec{e}_\alpha$ defines invariant straight.

2.3 Field of hypersurfices. Equation invariantness of hypersurfice $\nu_\alpha x^\alpha + \nu = 0$ related to moving reper ($\vec{A},\vec{e}$) has the following form

$$\begin{aligned} \delta\nu_\alpha + \nu_\beta\pi_\alpha^\beta &= Q\nu_\alpha \\ \delta\nu &= Q\nu \end{aligned} \qquad (2.13)$$

Q - a linear form, which means dQ=0. Two cases are possible.

2.3.1. Hypersurfice doesn't cross point A.

It's possible to put in this case $\nu = 1$, then $Q = 0$

Conditions of invariantness of hypersurfices will take the form

$$\delta\nu_\alpha - \nu_\beta \ \pi_\alpha^\beta = 0 \qquad (2.14)$$

2.3.2. Hypersurfice crosses point A.

In this case $\nu = 0$. Conditions of its invariantness have the form

$$\delta\nu_\alpha - \nu_\beta \ \pi_\alpha^\beta = Q\nu_\alpha \qquad (2.15)$$

Putting $\nu_n = 1$ conditions (2.15) will have the form

$$\delta\nu_i - \nu_j\pi_i^j - \nu_i\pi_n^n - \nu_i\nu_j\pi_n^j - \pi_i^n = 0 \qquad (2.16)$$

We'll build values

$$g^{\alpha\beta} = N^\alpha M^\beta$$

In condition of $def \, // \, g^{\alpha\beta} \, // \neq 0$ introduce values $g^{\alpha\gamma}g_{\gamma\beta} = \delta_\beta^\alpha$ and with their help

$$\begin{aligned} g_\alpha &= g_{\alpha\beta}g^\beta \\ \delta g_\alpha - g_\beta\pi^\beta &= 0 \end{aligned}$$

If differential equations of value g_α have structure of differential (2.14) then these values define hyperspace which doesn’t cross point A in the form $g_\alpha x^\alpha + 1 = 0$

Also have been built series of different values, which define invariant points, straights, Hypersurfice and two-dimensional surfaces associated with vector field in space A_4.

The received data are transformed into spaces of indefinite size.