THE EQUIVALENCE PRINCIPLE AND PROJECTIVE STRUCTURE IN 4-DIMENSIONAL LORENTZ MANIFOLDS

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The purpose of this lecture is to try and show how far one can proceed in the finding of the metric on a 4-dimensional Lorentz manifold if the projective structure of its Levi-Civita connection is given. This work has applications to the general theory of relativity. It represents joint work with Dr David Lonie in Aberdeen.

Let M be a smooth 4-dimensional manifold and let g and g' be Lorentz metrics on M. Let ∇ and ∇' be the Levi-Civita connections associated with g and g', respectively. Suppose that, for each $p \in M$, there exists an open subset G_p of the tangent space T_pM to M at p such that (i) each member of G_p is timelike with respect to g and g' and (ii) for each $u \in G_p$ the (unparametrised) ∇-geodesic starting at p and with initial tangent u is also an (unparametrised) geodesic for ∇'. The restriction to the subsets G_p simulates the restricted experimental evidence of the *principle of equivalence*, that the paths of "free" particles will follow such geodesics. It can be shown from this that ∇ and ∇' in fact share their geodesics and are thus *projectively related*. In showing this, the assumption (i), introduced for obvious physical reasons, is not used. Thus, ∇ and ∇' determine a global 1-form ψ on M (the projective 1-form) and, because they are metric connections, ψ is a closed 1-form [1]. There is also a convenient relationship between the curvature tensors R and R' from ∇ and ∇', respectively, in terms of the 1-form ψ. There are two convenient equivalent statements of this projective equivalence between ∇ and ∇' [1, 2], one relating these connections themselves and a second one which relates the ∇ covariant derivative of g' to g' and ψ.

To obtain information relating g and g' under the above conditions, an approach has been devised in [3, 4] which uses the algebraic nature of the curvature tensor, together with the equivalent projective conditions given above. Another method may also be applied when one of the metrics admits certain symmetries [5] because one can then take advantage of the nice relation between the symmetries of projectively related metrics. An example of the first method arises in the important situation for general relativity when one of the metrics, say g, is a vacuum (Ricci-flat) metric. (The physical assumption that (M, g) is non-flat is also imposed, that is, the curvature tensor R does not vanish over any non-empty open subset of M). In this case, one may appeal to Petrov's algebraic classification of the vacuum curvature tensor R [6]. In fact, one may topologically decompose M disjointly into five open subsets, one for each Petrov type, (such that each subset contains only points of that particular Petrov type) together with a closed nowhere dense set F. It can then be shown, using the convenient algebraic properties of the curvature tensor for each of the Petrov types, that ∇ and ∇' agree on the open dense subset, $M \setminus F$, of M and hence on M. It follows from this that *g' is also a vacuum metric on M*. Use of holonomy theory and the fact that M is a connected manifold then shows, with one highly specialised case excepted, that *g and g' are conformally related on M by a constant conformal factor*. (The special case concerns a subclass of the so-called *pp-waves* and can be easily handled separately.) Thus, from the physical viewpoint, knowledge of the geodesics (as described above) essentially uniquely determines the metric up to "units of measurement". As an example of the second method, let

g be a member of the important class of Friedmann-Robertson-Walker-Lemaitre (FRWL) cosmological metrics. In this case the high degree of symmetry possessed by g is useful. Here, the known result that, with the somewhat unphysical Einstein static and de-Sitter type metrics excluded, the dimension of the Lie algebra of projective vector fields for (M, g) is at most seven, is useful [7, 8]. From this, and the fact that projectively related connections have the same Lie algebras of projective vector fields, one can show that if g' is projectively related to g, in the sense defined above, then g' is also an FRWL metric sharing the same space slices of constant cosmic time and having the same Killing algebra as g and is conveniently related to g (but not so tightly as in the vacuum case).

References

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