Higher vector bundles and multi-graded manifolds

Janusz Grabowski

Institute of Mathematics, Polish Academy of Sciences

(joint work with M. Rotkiewicz)

A natural condition is given assuring that an action of the multiplicative monoid of non-negative reals on a manifold F comes from homoteties of a vector bundle structure on F, or, equivalently, from an Euler vector field. This is used in showing that double (or higher) vector bundles known in the literature can be equivalently defined as manifolds with a family of commuting Euler vector fields. Higher vector bundles can be therefore defined as manifolds admitting certain $\mathbb{N}^n$-gradation in the structure sheaf.

The n-vector bundles F admit canonical lifts to the tangent and to the cotangent bundles TF and T^*F. In particular, the iterated tangent and cotangent bundles are canonical examples of higher vector bundles. The cotangent bundle T^*F is of particular interest, since it is canonically fibred not only over F but also over all duals $F^*_{(k)}$ of F with respect to all its vector bundle structures $F \to F_{[k]}$. The side bundles $F_{[k]}$ are canonically $(n-1)$-vector bundles themselves. We prove the existence of a canonical identifications $T^*F \simeq T^*F^*_{(k)} \simeq T^*F^*_{(l)}$ which are additionally symplectomorphisms. They can be viewed as a generalization of the celebrated "universal Legendre transformation" $T^*TM \simeq T^*T^*M$. Moreover, the set of higher vector bundles $\{F, F^*_{(1)}, \ldots, F^*_{(n)}\}$ is closed with respect to duality (under natural identifications) for all vector bundle structures on them. This is a phenomenon observed first for double and triple vector bundles by K. Konieczna, P. Urbański and K. C. H. Mackenzie.

Next, we prove that symplectic n-vector bundles, i.e. n-vector bundles with a symplectic form which is linear (1-homogeneous) with respect to all vector bundle structures, are always of the form T^*F for certain $(n-1)$-vector bundle F. This, in turn, generalizes the known fact that any vector bundle equipped with a linear symplectic form is, in fact, T^*M.

Consequently, multi-graded (super)manifolds are canonically associated with higher vector bundles which is an equivalence of categories. Of particular interest are symplectic multi-graded manifolds which are proven to be associated with cotangent bundles. The symplectic multi-graded manifolds, equipped with certain homological Hamiltonian vector fields, lead to an alternative to D. Roytenberg's picture generalization of Lie bialgebroids, Courant brackets, Drinfeld doubles and can be viewed as higher BRST and Batalin-Vilkovisky formalisms.