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Hilbert diffeomorphism groups and their geometry for open manifolds

Abstract

Let M^n be closed, $\mathcal{D}(M) \equiv \text{Diff}(M)$ the group of smooth diffeomorphisms. For many applications in PDE theory one needs a completed version $\mathcal{D}^{p,r}(M)$, $1 \leq p, r$ Sobolev index. For $r > \frac{n}{p}$ and M^n closed, this easily can be done. One defines $\mathcal{D}^{p,r}$ by means of a finite cover $\mathfrak{U} = \{U_\alpha, \varphi_\alpha\}_{1 \leq \alpha \leq m}$ and imposes Euclidean Sobolev conditions,

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \longrightarrow \mathbb{R}^n \text{ is in } W^{p,r}(\varphi(U), \mathbb{R}^n).$$

It easily follows that $\mathcal{D}^{p,r}$ is independent of the choice of the finite cover. For open manifolds, this is totally wrong. We defined for open manifolds (M^n, g) of bounded geometry completed diffeomorphism groups $\mathcal{D}^{p,r}(M^n, g)$ satisfying the following conditions

1) $\mathcal{D}^{p,r}$ is a Banach manifold, for $p = 2$ it is a Hilbert manifold,

2) it depends only on the component $\text{comp}(g) \subset \mathcal{M}^{p,r}(I, B_k)$, the completed space of metrics of bounded geometry,

3) if (M^n, g) is compact then our definition coincides with all other definitions and is completely independent of g.

In a second step, we construct Hilbert submanifolds of volume (element) preserving, symplectic, contact and gauge diffeomorphisms, define for them a (weak) Riemannian structure and calculate their curvature. All this has many applications in mathematical physics.