Vertex Operator construction of coupled soliton hierarchies

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The aim of this talk is present the paper [1] written together with Giovanni Ortenzi where we give a representation–theoretic interpretation of recent discovered coupled soliton equations [2],[6] [3] [5] using vertex operators construction of affinization of not simple but quadratic Lie algebras. In this setup we are able to obtain new integrable hierarchies coupled to each Drinfeld–Sokolov of A, B, C, D hierarchies and to construct their soliton solutions.

The not simple Lie algebras considered in this talk are given by the tensor product $\mathfrak{g}^{(n)} = \mathfrak{g} \otimes \mathbb{C}^{(n)}(\lambda)$ where $\mathfrak{g}$ is any Lie algebra and $\mathbb{C}^{(n)}(\lambda) = \mathbb{C}[\lambda]/(\lambda)^{n+1}$. This algebra may be identified with the Lie algebra of polynomial maps from $\mathbb{C}[\lambda]/(\lambda^{n+1})$ in $\mathfrak{g}$, hence an element $X(\lambda)$ in $\mathfrak{g}(\lambda)$ can be viewed as the mapping $X : \mathbb{C} \to \mathfrak{g}$, $X(\lambda) = \sum_{k=0}^{n} X_k \lambda^k$ where $X_k \in \mathfrak{g}$. In this setting the Lie bracket of two elements in $\mathfrak{g}(\lambda)$, $X(\lambda) = \sum_{k=0}^{n} X_k \lambda^k$ and $Y(\lambda) = \sum_{k=0}^{n} Y_k \lambda^k$ can be written explicitly as

$$[X(\lambda), Y(\lambda)] = \sum_{k=0}^{n} (\sum_{j=0}^{k} [X_j, Y_{k-j}]_{\mathfrak{g}}) \lambda^k. \tag{0.1}$$

The most important peculiarity of this obviously not simple Lie algebras is that if $\mathfrak{g}$ admits a symmetric ad–invariant non–degenerated bilinear form then roughly speaking this bilinear form is inherited by the whole Lie algebra $\mathfrak{g}^{(n)}$. Using this fact we will able to affinize such algebras, obtaining infinite dimensional Lie algebras with a multidimensional central extensions. The key point to link these infinite dimensional Lie algebras with hierarchies of soliton equations is that, despite to the fact that they are not affine Kac–Moody Lie algebras, they still admit representations trough Vertex Operators Algebras. These representations may be lifted to representations of infinite dimensional Lie groups explicitly defined during the talk. This fact finally allows us to implement the theory developed by Kac, Peterson and Wakimoto to construct corresponding generalized Hirota bilinear equation and their multisoliton solution in term of τ–functions. Explicit examples of such construction will be presented during the talk, namely those of the case of the of the coupled AKP BKP and their reduction to Lie algebras generalizing the algebras $A_1^{(1)}$, $A_2^{(1)}$, $A_1^{(2)}$, and $B_2^{(1)}$ making the direct connection whit the hierarchies of such type already presented in the literature. Finally it will be briefly sketched how we intend together with professor Johan van de Leur proceed further in this direction [4] .

AMS Subject Classification: Primary 17B69, 37K10, Secondary 37K30

References

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