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# HOLONOMY AND MONODROMY GROUPOIDS

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**Abstract.** We outline the construction of the holonomy groupoid of a locally Lie groupoid and the monodromy groupoid of a Lie groupoid. These specialize to the well known holonomy and monodromy groupoids of a foliation, when the groupoid is just an equivalence relation.

**Introduction.** The holonomy and monodromy groupoids of foliations are well known, and with their smooth structure are usually attributed to Winkelnkemper [41] and Phillips [37]. The purpose of this paper is to advertise the fact, due to Pradines in 1966 [38], that these constructions are special cases of constructions which apply to wide classes of structured groupoids, where the foliation case is essentially that where the groupoid is the equivalence relation determined by the leaves of the foliation. In the final section, we suggest a number of wider questions and possible directions for investigation, in particular the possible relation with generalised Galois theory, and the potentiality of higher dimensional analogues.

An important feature of Pradines' work is that these constructions of holonomy and monodromy groupoids come with universal properties of a local-to-global form. The association of monodromy with a universal principle is classical, see for example Chevalley

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[18]. The monodromy principle asserts roughly that, in a simply connected situation, for example a simply connected group, or an equivalence relation on a simply connected space, a local morphism extends to a global morphism. More generally, a local morphism can be lifted to a global morphism on the universal cover.

The association of holonomy with a universal principle is less well known. It is stated in terms of an adjoint pair of functors, but not explained in detail, in [38]. It involves the notion of what Pradines called 'un morceau d'un groupoïde différentiables' and which we prefer to call a 'locally Lie groupoid'. This is a groupoid $G$ and a subset $W$ of $G$ containing the identities and such that $W$ has the structure of a manifold. Conditions are imposed so that the groupoid structure is as 'smooth as possible' on $W$. There is a kind of 'holonomy principle' that, in the 'locally sectionable' case (see below), the manifold structure on $W$ extends to a Lie groupoid structure not on $G$ but on an overgroupoid $Hol(G, W)$ of $G$, and in which $W$ is an open subspace.

The case when $Hol(G, W) = G$ is also of interest, since this gives a condition for the pair $(G, W)$ to be *extendible*. This is used crucially to obtain a Lie structure on the monodromy groupoid of a Lie groupoid. Thus whereas usually the holonomy groupoid is constructed as a quotient of the monodromy groupoid, here we regard the holonomy construction as fundamental. This difference of approach seems of interest.

Another question arising from this work is the applicability of the notion of locally Lie groupoid for encapsulating ideas of local structures. It is proven by Brown and Mucuk in [17] that the charts of a foliation on a paracompact manifold gives rise to a locally Lie groupoid. This process is generalised by Brown and İçen in [9] to the case of a local subgroupoid. We also note recent work of Claire Debord [20] which studies the case of singular foliations, and has constructions whose relation to those given here would be interesting to determine.

One aim for Pradines of this notion of what we call a locally Lie groupoid was as a half way house between a Lie algebroid and a Lie groupoid. We have not found a clear statement of which Lie algebroids give rise to a locally Lie groupoid, but the two steps of holonomy and monodromy groupoid were designed to model two of the three steps in getting an essentially unique Lie group from a Lie algebra, namely: produce from the Lie algebra a locally Lie group; from this produce a Lie group; finally, take the universal cover of this Lie group. It is remarkable that Pradines' intuitions on these steps was so strong.

The main ideas of the results and proofs for the holonomy and monodromy groupoids were described by Pradines to Brown in the early 1980s, and an incomplete account was written in [5]. A full account of the holonomy construction and related material was given in Aof's Bangor thesis [2] and published in [3]. A full account of the monodromy construction was given in Mucuk's Bangor thesis [36] and published in [17]. It should be emphasised that this gives useful conditions for the groupoid $M(G)$, obtained from a Lie groupoid by taking the universal covers of the stars of $G$ at the identities, to be given the structure of Lie groupoid so that the projection $M(G) \to G$ gives the universal covering map on each star.

A key aspect of the construction is that $M(G)$ is initially defined by a universal construction which ensures that it comes with a monodromy principle on the extendibility to $M(G)$ of certain local morphisms on $G$. The problem is to get a topology on $M(G)$

and this, remarkably, is solved by the holonomy construction, but in the case where the holonomy is trivial. This seems a roundabout method. The point, however, seems to be that the construction of the topology involves local smooth admissible sections, and the proof that this method works seems to be no simpler in the case of trivial holonomy than in the general case. Thus it is important to be clear about the general method.

The use of local admissible sections for these constructions seems essential. To see this we contrast with the group case. If $G$ is a topological group, then left multiplication $L_g$ by an element $g$ of $G$ maps open sets of $G$ to open sets, and in fact $L_g$ is a homeomorphism of $G$. This is no longer the case if $G$ is topological groupoid, for obvious domain reasons.

To remedy this situation, Ehresmann introduced the notion of 'smooth local continuous admissible section' $\sigma$ of a Lie groupoid $G$. This is a smooth section of the source map $\alpha$ defined on some open set $U$ of the object space $O_G$ and such that $\beta\sigma$ maps $U$ diffeomorphically to an open set of $O_G$. Then left multiplication $L_\sigma$ can be defined on $G$ and does map open sets of $G$ to open sets of $G$. We say that left multiplication by an element has to be 'localised', that is 'spread' to a local area. Intuitively, we regard $\sigma$ and its associated $L_\sigma$ as a 'local procedure' on the Lie groupoid $G$.

In the case of a locally Lie groupoid $(G, W)$ there is a new twist. We can say that $\sigma$ is smooth only if the image of $\sigma$ lies in $W$, since only $W$ has a manifold structure. We call such a $\sigma$ a 'local procedure'. The composition in the groupoid $G$ extends to a composition of local admissible sections, and so such a composition can be regarded as a 'composite of local procedures', but such a composition may not have values in $W$ and so is not a 'local procedure'. In fact in the literature, more so in physics than in mathematics, the notion of holonomy is regarded as an iteration of local procedures which returns to the starting point but not to the starting value. We will see this interpreted as a germ $[\sigma]_x$ of such a composite for which $\sigma(x) = 1_x$ but there is no neighbourhood $U$ of $x$ for which $\sigma(U)$ is contained in $W$ and $\sigma|U$ is smooth. That is, the iteration does not even locally give a local procedure.

The convenient formal description of the above is in terms of inverse semigroups and groupoids of germs. The nice point is that the formal description does exactly encapsulate the intuition, and it is the intention of this paper to convey this point.

Now we give some precise definitions.

**1. Definitions.** We fix our notation. A groupoid consists of a set $G$ and two functions, the source and target maps, $\alpha, \beta : G \to G$ such that $\alpha\beta = \beta, \beta\alpha = \alpha$ (whence $\alpha^2 = \alpha, \beta^2 = \beta$, and $\alpha$ and $\beta$ have the same image). We often write $g : \alpha g \to \beta g$. Further, there is a multiplication written, say, $gh$, for $g, h \in G$, with the property that $gh$ is defined if and only if $\beta g = \alpha h$, and then $\alpha(gh) = \alpha g, \beta(gh) = \beta h$. The set $\alpha G$ is called the set of identities, or objects, of the groupoid $G$, and is written $O_G$. If $x \in \alpha G$ one often writes $1_x$ for $x$ to emphasise that such an $x$ acts as an identity. We also require associativity of the multiplication, and the existence of an inverse to every element of $G$. It is often convenient to think of $O_G$ as disjoint from $G$. Thus a groupoid is also a small category in which every morphism is an isomorphism.

A groupoid in which $\alpha = \beta$ is called a *bundle of groups*, while a groupoid in which the *anchor map* $(\alpha, \beta) : G \to O_G \times O_G$ is injective is just an equivalence relation.

In order to cover both the topological and differentiable cases, we use the term $\mathcal{C}^r$ manifold for $r \geqslant -1$, where the case $r = -1$ deals with the case of topological spaces and continuous maps, with no local assumptions, while the case $r \geqslant 0$ deals as usual with $\mathcal{C}^r$ manifolds and $\mathcal{C}^r$ maps. Of course, a $\mathcal{C}^0$ map is just a continuous map. We then abbreviate $\mathcal{C}^r$ to *smooth*. The terms *Lie group* or *Lie groupoid* will then involve smoothness in this extended sense.

The following definition is due to Ehresmann [24].

Definition 1.1. Let $G$ be a groupoid and let $X = O_G$ be a smooth manifold. An *admissible local section* of $G$ is a function $\sigma : U \to G$ from an open set in $X$ such that

(i) $\alpha\sigma(x) = x$ for all $x \in U$;
(ii) $\beta\sigma(U)$ is open in $X$, and
(iii) $\beta\sigma$ maps $U$ diffeomorphically to $\beta\sigma(U)$.

Let $W$ be a subset of $G$ and let $W$ have the structure of a smooth manifold such that $X$ is a submanifold. We say that $(\alpha, \beta, W)$ is *locally sectionable* if for each $w \in W$ there is an admissible local section $\sigma : U \to G$ of $G$ such that (i) $\sigma\alpha(w) = w$, (ii) $\sigma(U) \subseteq W$ and (iii) $\sigma$ is smooth as a function from $U$ to $W$. Such a $\sigma$ is called a *smooth admissible local section.*

The following definition is due to Pradines [38] under the name "*morceau de groupoïde différentiable*". Recall that if $G$ is a groupoid then the difference map $\delta$ is $\delta : G \times_\beta G \to G, (g, h) \mapsto gh^{-1}$.

Definition 1.2. A *locally Lie groupoid* is a pair $(G, W)$ consisting of a groupoid $G$ and a smooth manifold $W$ such that:
$G_1$) $O_G \subseteq W \subseteq G$;
$G_2$) $W = W^{-1}$;
$G_3$) the set $W(\delta) = (W \times_\beta W) \cap \delta^{-1}(W)$ is open in $W \times_\beta W$ and the restriction of $\delta$ to $W(\delta)$ is smooth;
$G_4$) the restrictions to $W$ of the source and target maps $\alpha$ and $\beta$ are smooth and the triple $(\alpha, \beta, W)$ is locally sectionable;
$G_5$) $W$ generates $G$ as a groupoid.

Note that, in this definition, $G$ is a groupoid but does not need to have a topology. The locally Lie groupoid $(G, W)$ is said to be *extendible* if there can be found a topology on $G$ making it a Lie groupoid and for which $W$ is an open submanifold.

The main result of [17] (which was known to Pradines) is that a foliation $\mathcal{F}$ on a paracompact manifold $M$ gives rise to a locally Lie groupoid $(G, W)$ where $G$ is the equivalence relation of the leaves of the foliation, and $W$ is constructed from a refinement of the local charts of the foliation. In general such $(G, W)$ are not extendible. A standard example is the foliation of the Möbius Band $M$ by circles. In this case the equivalence relation determined by the leaves is not a submanifold of $M \times M$ [17]. Foliations have also been shown to lead to *local equivalence relations* [40].

Here is an example of non extendibility due to Pradines [2].

Example 1.3. Consider the bundle of groups $F$ given by the first projection $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where the real line $\mathbb{R}$ is considered as a topological abelian group under addition. We regard $F$ as a groupoid, and in fact as a topological groupoid in the obvious sense. Now let $N$ be the subgroupoid of $F$ generated by $(x, 0)$ if $x < 0$ and $(x, 1)$ if $x \geqslant 0$.

Fig. 1

Let $G$ be the quotient groupoid $F/N$, and let $q : F \to G$ be the quotient morphism of groupoids. Then the stars $\alpha^{-1}(x)$ of $G$ are bijective with $\mathbb{R}$ if $x < 0$ and with $\mathbb{R}/\mathbb{Z}$ if $x \geqslant 0$. Let $W'$ be the subspace $\mathbb{R} \times (-1/4, 1/4)$ of $F$, and let $W = qW'$. The topology on $W'$ may easily be transferred to a topology on $W$ so that $(G, W)$ becomes a locally Lie groupoid. However, it is not possible to extend this topology so as to get even a topological groupoid structure on $G$, for which $W$ is an open subspace. This can be seen by noting that the section $s$ of the map $\alpha$ of $G$ given by $x \mapsto q(x, 1/8)$ is continuous but $9s$ is not. Instead, there is another groupoid $H = Hol(G, W)$, called the holonomy groupoid of the locally topological groupoid $(G, W)$, which is a topological groupoid, and which contains $W$ as an open subspace. The groupoid $H$ is equipped with a surjective morphism $\phi : H \to G$ which is the identity on objects. In this case the kernel of $\phi$ is non trivial only at 0 and is there of the form $\{0\} \times \mathbb{Z}$.

Example 1.4. There is a variant of this last example in which $F$ is as above, but this time $N$ is the union of the groups $\{x\} \times (1 + |x|)\mathbb{Z}$ for all $x \in \mathbb{R}$. One defines $W$ as before, but this time considers $W$ as a differential manifold. The topological structure on $W$ can be extended to give a topological groupoid structure on the quotient $G = F/N$. The differential structure, however, cannot be so extended, because the section $s$ given as in the previous example is such that $9s$ is not smooth. In this case one gets a differential holonomy groupoid, with a projection morphism $\phi : H \to G$ whose kernel is as in the previous example. One can get similar examples with varying degrees of differentiability considered. In this and the previous example, the holonomy groupoids constructed are non-Hausdorff topological (or Lie) groupoids.

**2. The holonomy construction.** The main result of Aof and Brown [3] is a version of Théorème 1 of Pradines [38] and was stated in the topological case. In the smooth case it states:

THEOREM 2.1 (Pradines [38], Aof and Brown [3]) (Globalisability theorem). *Let $(G, W)$ be a locally Lie groupoid. Then there is a Lie groupoid $H$, a morphism $\phi : H \to G$ of groupoids and an embedding $i : W \to H$ of $W$ to an open neighborhood of $O_H$ such that the following conditions are satisfied.*
i) *$\phi$ is the identity on objects, $\phi i = id_W, \phi^{-1}(W)$ is open in $H$, and the restriction $\phi_W : \phi^{-1}(W) \to W$ of $\phi$ is smooth;*
ii) *if $A$ is a Lie groupoid and $\xi : A \to G$ is a morphism of groupoids such that:*
*a) $\xi$ is the identity on objects;*
*b) the restriction $\xi_W : \xi^{-1}(W) \to W$ of $\xi$ is smooth and $\xi^{-1}(W)$ is open in $A$ and generates $A$;*
*c) the triple $(\alpha_A, \beta_A, A)$ is locally sectionable,*
*then there is a unique morphism $\xi' : A \to H$ of Lie groupoids such that $\phi\xi' = \xi$ and $\xi' a = i\xi a$ for $a \in \xi^{-1}(W)$.*

The groupoid $H$ is called the *holonomy groupoid* $Hol(G, W)$ of the locally Lie groupoid $(G, W)$. It is thus the minimal overgroupoid of $G$ which can be made into a Lie groupoid with $W$ as open subspace.

We should also say that Pradines actually states more since his is a theorem on germs of such $(G, W)$. So there is still more work to be done on giving a full account of this result and illustrating it with examples.

*Sketch of the proof of Theorem 2.1.* An important construction due to Ehresmann is a multiplication on the set $\Gamma(G)$ of local admissible sections of $G$ in which if $x \in X$

$$(\sigma\tau)(x) = (\sigma x)(\tau\beta\sigma x).$$

With this multiplication, $\Gamma(G)$ is a monoid, and in fact an inverse monoid, in the sense that every $\sigma$ has a unique (generalised) inverse $\sigma'$ such that

$$\sigma\sigma'\sigma = \sigma, \qquad \sigma'\sigma\sigma' = \sigma'.$$

Since $\sigma' x = (\sigma(\beta\sigma)^{-1}x)^{-1}$, we write $\sigma^{-1}$ for $\sigma'$. An important reason for introducing these sections is that if $G$ is a topological groupoid, then translation by a continuous local admissible section does map open sets of $G$ to open sets of $G$.

Let $\Gamma^c(W)$ be the subset of $\Gamma(G)$ consisting of local admissible sections which (i) have values in $W$ and (ii) are smooth. Of course the first condition is necessary for the second condition to make sense. Let $\Gamma^c(G, W)$ be the sub-inverse monoid of $\Gamma(G)$ generated by $\Gamma^c(W)$. At this stage it is convenient to assume that $W = W^{-1}$. It is proved in [2] that this is no loss of generality.

Now let $J^c(G, W)$ be the sheaf of germs of the elements of $\Gamma^c(G, W)$, and let $J^c(W)$ be the sheaf of germs of the elements of $\Gamma^c(W)$. The germ of a local section $\sigma$ at the point $x$ of its domain is written $[\sigma]_x$. Then the inverse monoid structure on $\Gamma(G)$ induces on $J^c(G, W)$ the structure of groupoid, in which $[\sigma]_x[\tau]_y = [\sigma\tau]_x$ is defined if and only if $y = \beta\sigma x$.

The sets $\Gamma^c(W)$ and $J^c(W)$ have a rôle as codifying a *local procedure*. The inverse monoid $\Gamma^c(G,W)$ and the groupoid $J^c(G,W)$ then codify the *iteration of local procedures*. It is in this sense that we are dealing with local-to-global techniques.

There is a morphism of groupoids, the 'final map', $\psi : J^c(G,W) \to G, [\sigma]_x \mapsto \sigma x$, which is the identity on objects. We set

$$J_0 = J^c(W) \cap (Ker\psi),$$

so that $J_0$ consists of germs $[\sigma]_x$ of continuous local admissible sections $\sigma$ with values in $W$ and such that $\sigma x = 1_x$. The aim is to define the holonomy groupoid of the locally Lie groupoid $(G,W)$ to be the quotient groupoid

$$Hol(G,W) = J^c(G,W)/J_0.$$

For this we need to prove:

LEMMA 2.2. *The set $J_0$ is a normal subgroupoid of $J^c(G,W)$.*

The main point of the proof is that because of the definitions of $J_0$ and of $J^c(G,W)$ one has only to check that if $[\rho]_x, [\sigma]_x \in J_0$ and $[\tau]_x \in J^c(W)$, then $[\rho\sigma^{-1}]_x \in J_0$, and $[\tau]_x[\sigma]_x[\tau]_x^{-1} \in J_0$. This follows from continuity considerations and the facts that

$$(\rho\sigma)x = 1_x = (\tau\sigma\tau^{-1})x.$$

Let $p : J^c(G,W) \to Hol(G,W)$ be the quotient morphism. We write $H$ for $Hol(G,W)$ and write $\langle\sigma\rangle_x$ for $p([\sigma]_x)$. Thus $H$ is a groupoid. Note that the morphism $\psi : J^c(G,W) \to G$ induces a morphism which we write $\phi : H \to G, \langle\sigma\rangle_x \mapsto \sigma x$. For this morphism to be surjective, it is sufficient to assume that $W$ generates $G$ as a groupoid, and that for every element $w$ of $W$ there is a continuous admissible local section of $G$ through $w$.

Let $f \in \Gamma^c(G,W)$. We define a partial function $\chi_f : W \to H$, by

$$w \mapsto \langle f\rangle_x\langle\sigma_w\rangle_x,$$

where $\sigma_w$ is an admissible local section of $s$ through $w$. Again, one has to assume that such a section exists for all $w \in W$, and one has to prove that this value is independent of the choice of local section $\sigma_w$, and that $\chi_f$ is injective with domain an open subset of $W$.

A key lemma is that if $f, g \in \Gamma^c(G,W)$ then $(\chi_f)^{-1}(\chi_g) = L_h$, left multiplication by the section $f^{-1}g$. This shows that $(\chi_f)^{-1}(\chi_g)$ maps an open set of $W$ diffeomorphically to an open set of $W$. This algebraic format for the change of charts is also convenient for proving $Hol(G,W)$ becomes a Lie groupoid, see [3].

We also need that every element of the holonomy groupoid arises in this way, and for this we also need that $W$ generates $G$. Such an assumption is in practice not so great a restriction. A result of Pradines (compare [2, Proposition 1.5.16]) is that if each star $\alpha^{-1}(x)$ of $G$ meets $W$ in a connected set, then any open neighbourhood of $x$ in $W$ generates $G$.

These results allow the $\chi_f$ for all $f \in \Gamma^c(G,W)$ to be used as charts for a topology on the holonomy groupoid $H$. Notice that every element of $H$ is of the form $\chi_f(x)$ for some $f \in \Gamma^c(G,W)$ and $x \in \mathcal{D}_f$. Consequently, given $f \in \Gamma^c(G,W)$, the function $x \mapsto \chi_f x$ for $x \in \mathcal{D}_f$ is a continuous admissible local section of $H$. Also, $H$ is generated as a groupoid by $\chi_{\mathbf{1}}(W)$ where **1** here denotes the identity section with domain $X$. This completes the sketch proof. ∎

Readers of the Bourbaki account for Lie groups ([4] p.210) may be puzzled by the lack of a condition involving conjugacy, of the type that for all $g \in G$ there is an open neighbourhood $U$ of $\alpha g$ such that $gUg^{-1}$ is contained in $W$. Pradines argues (private communication) that in the first place this condition is unrealistic, since it involves 'global' elements $g$ of $G$. In the second place, this condition is not needed, by virtue of the assumptions on generation.

The above construction can be followed through to give the results of Examples 5 and 6.

There is a surprising application of the holonomy groupoid construction, namely to give a condition that a locally Lie groupoid $(G, W)$ is extendible, i.e. determines a topology on $G$ making it a Lie groupoid for which $W$ is an open subspace. In terms of previous notation, this condition is simply that $Ker\ \psi$ is contained in $J^c(W)$, which is equivalent to the condition that if $\sigma$ is any product of admissible continuous local sections about $x$ each with values in $W$, and $\sigma(x) = 1_x$, then some restriction of $\sigma$ to a neighbourhood of $x$ has values in $W$ and is smooth. It is not clear that there is any easier proof of this extendibility result than that obtained from the construction of the holonomy groupoid.

This extendibility result is used, as suggested by Pradines (see [5]), in constructing a topology on the monodromy groupoid of a topological groupoid. The basic method is as follows.

Let now $G$ be a topological groupoid and let $W$ be an open subset of $G$ containing the identities. The groupoid structure on $G$ makes $W$ into a pregroupoid, by which is meant that the product $uv$ of two elements $u, v$ of $W$ is not always defined (in $W$). There is a standard way of making any pregroupoid $W$ into a groupoid $M$ with a morphism of pregroupoids $i : W \to M$ such that any pregroupoid morphism from $W$ to a groupoid $K$ extends uniquely to a morphism $M \to K$. Since $W$ embeds in a groupoid (namely $G$), the morphism $i : W \to M$ is an embedding. Methods of [23] may be extended to show that under suitable local conditions on $G$, the topology on $W$ may be extended to a topology on each $s_M^{-1}x, x \in X$, such that each projection $s_M^{-1}x \to s_G^{-1}x$ is a universal cover. The previously mentioned universal property now gives a version of the classical Monodromy Principle [18], but stated in terms of groupoids, rather than equivalence relations or groups as in [18].

The problem is now to make $M$ into a topological groupoid so that the universal property yields a continuous morphism on $M$ if $K$ is a topological groupoid and the pregroupoid morphism $W \to K$ is continuous. The surprising, but simple to prove, result is that the pair $(M, W)$ satisfies the condition for extendibility stated above, basically because $G$ is already a Lie groupoid. So the holonomy method outlined above is used to extend the topology on $W$ to a topology on $M$, assuming that $G$ has enough continuous admissible local sections. (This is Pradines' method for Théorème 2 of [38], 1966, explained to Brown in 1981.)

The monodromy groupoid construction yields the homotopy groupoid of a foliation, discussed in [37]. It also yields this groupoid with the universal property of globalising a morphism defined locally. Once again we see a local-to-global feature which fits naturally into the context of groupoids.

These methods also give an answer to the following question: given a Lie groupoid $G$, let $Cov(G)$ be the union of the universal covers at $1_x$ of the star of $G$ at $x$, for all $x \in G$. Let $q : Cov(G) \to G$ be the projection. Assume we can choose a neighbourhood $W$ of $O_G$ so that the inclusion $W \to G$ lifts to $W \to Cov(G)$. The monodromy principle then yields a morphism of groupoids $\phi : M(G) \to Cov(G)$ which is continuous on stars. But $p : M(G) \to G$ is a covering map on each star, and so $\phi$ is a bijection on each star, and hence is an isomorphism. This isomorphism induces a topology on $Cov(G)$ making it a topological groupoid, or, in appropriate circumstances, a Lie groupoid. Such a construction is given by Mackenzie in [32] by a different method, in the locally trivial case.

The full details of the above arguments are given in [16].

The use of the monodromy groupoid and $Cov(G)$ also enables us to explain the holonomy groupoid of Example 1.3. The monodromy groupoid of $Hol(G, W)$ is the original groupoid $F$ and so $Hol(G, W)$ is isomorphic to the quotient of $F$ by the subgroupoid generated by $(x, 0)$ for $x \leqslant 0$ and $(x, 1)$ for $x > 0$.

**3. Local subgroupoids.** There is considerable work on local equivalence relations, part of the motivation being that a foliation on a manifold $M$ determines a local equivalence relation on $M$ [40]. Now an equivalence relation on $M$ is just a subgroupoid of the indiscrete groupoid $M \times M$ which has the object set $M$ (this is also known as a *wide* subgroupoid of $M \times M$). It thus seems natural to consider an arbitrary groupoid $Q$ with object set $M$ and to consider the sheaf $p : \mathcal{L}_Q \to M$ associated to the presheaf $U \mapsto L_Q(U)$ where $L_Q(U)$ is the set of wide subgroupoids of $Q|U$. This notion is studied in [9, 11]. In [9] there are given conditions on a local subgroupoid of a Lie groupoid so that it leads to a locally Lie groupoid and hence to holonomy and monodromy groupoids. In particular, this leads to a monodromy principle for local subgroupoids.

**4. Questions**

Question 4.1. We have already mentioned the question of extending the work on holonomy and monodromy to germs, thus giving a complete account of the theorems in the first of Pradines' notes [38], which are stated as the existence of adjoint functors. Some remarks on this are given in [39].

Question 4.2. It would be interesting to know (i) how useful is the notion of locally Lie groupoid in formulating local properties, and (ii) what is its relation to the notion of Lie algebroid.

Question 4.3. The following question could be of interest. Grothendieck has developed extensive work on the fundamental group in the context of algebraic geometry. The notion of monodromy is also often vital in these arithmetic questions. Can the above approach to monodromy and covering spaces be of use in these arithmetic problems? This would be an interesting further vindication of Pradines' approach, and is related to the next question.

QUESTION 4.4. It is known that covering spaces and Galois theory are closely related, see for example [22, 12]. The last paper relates the generalised Galois theory of Janelidze [30] to covering space theory. It would be very interesting to tie in notions of monodromy for groupoids with these broader aspects of Galois theory and of descent [31].

QUESTION 4.5. For the present writers, the most intriguing, and possibly the most difficult, question is that of higher dimensional analogues of these results. Background to the idea that multiple groupoids form candidates for 'higher dimensional groups' is given in [8]. A starting point was that since higher homotopy groups are abelian because 'group objects in groups' are just abelian groups, it is therefore natural to look at objects of the type of 'group objects in groupoids' or 'groupoid objects in groupoids'. These are more complicated objects than groups, and the complication of $n$-fold groupoids increases directly with $n$. Indeed, it is known that $n$-fold groupoids model homotopy $n$-types. Such $n$-dimensional structures lend themselves to the consideration of 'algebraic inverses to subdivision'; since subdivision is a fundamental process in local-to-global questions, the possibility of detailed algebraic control over the inverse process in certain circumstances would be expected to lead to surprising new results, and involving essentially non abelian considerations. These objects do arise naturally in homotopy theory, where they lead to new algebraic constructions such as a non abelian tensor product of groups and to calculations in homotopy theory not possible by other means [14, 8]. These algebraic objects, or analogous ones, also arise in many other algebraic and geometric situations [15, 13, 33, 35].

Thus it is natural to consider the possibility of higher dimensional forms of holonomy and monodromy. A tentative step in this direction is given in [10], which covers part of [29]. The basic intuition is that for a groupoid an admissible section can also be considered as a homotopy. A reasonable generalisation of an admissible section should therefore be a notion of a homotopy, i.e. a deformation. This notion exists for various forms of double groupoid. Thus the existence of multiple geometric structures (double foliations, foliated bundles, etc.) should in principle be properly reflected by multiple algebraic structures.

It is surely intuitively significant in this respect that multiple categories arise in the context of concurrency in computer science, where the multiple processors are thought of as each giving another time dimension. The algebraic analysis seems naturally to involve a generalisation of the notion of free category on a graph to a certain notion of a free cubical $\omega$-category on a cubical set. The analysis of this situation is still incomplete, but is studied in [28, 27].

It is possible that a description of the relation between holonomy in the sense of this paper and holonomy for principal bundles with connection, and hence the relation with curvature, requires some higher dimensional algebraic treatment.

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