ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SOME SETS IN $\ell_1$

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Abstract. For a sequence $x \in \ell_1 \setminus c_{00}$, one can consider the set $E(x)$ of all subsums of series $\sum_{n=1}^{\infty} x(n)$. Guthrie and Nymann proved that $E(x)$ is one of the following types of sets:

- $(I)$ a finite union of closed intervals;
- $(C)$ homeomorphic to the Cantor set;
- $(MC)$ homeomorphic to the set $T$ of subsums of $\sum_{n=1}^{\infty} b(n)$ where $b(2n-1) = 3/4^n$ and $b(2n) = 2/4^n$.

By $I$, $C$ and $MC$ denote the sets of all sequences $x \in \ell_1 \setminus c_{00}$, such that $E(x)$ has the property $(I)$, $(C)$ and $(MC)$, respectively. In this note we show that $I$ and $C$ are strongly $c$-algebrable and $MC$ is $c$-lineable. We show that $C$ is a dense $G_\delta$-set in $\ell_1$ and $I$ is a true $F_\sigma$-set. Finally we show that $I$ is spaceable while $C$ is not spaceable.

1. Introduction

1.1. Lineability, algebrability and spaceability. Having a linear algebra $A$ and its subset $E \subset A$ one can ask if $E \cup \{0\}$ contains a linear subalgebra $A'$ of $A$. Roughly speaking if the answer is positive, then $E$ is algebrable. It is a recent trend in Mathematical Analysis to establish the algebrability of sets $E$ which are far from being linear, that is $x, y \in E$ does not generally imply $x + y \in E$. Such algebrability results were obtained in sequence spaces (see [7], [6] and [8]) and in function spaces (see [2], [5], [4], [12] and [13]).

1991 Mathematics Subject Classification. Primary: 40A05; Secondary: 15A03.

Key words and phrases. subsums of series, achievement set of sequence, algebrability, strong algebrability, lineability, spaceability.
Assume that $V$ is a linear space (linear algebra). A subset $E \subset V$ is called lineable (algebrable) whenever $E \cup \{0\}$ contains an infinite-dimensional linear space (infinitely generated linear algebra, respectively), see [3], [9] and [15].

For a cardinal $\kappa > \omega$, let us observe that the set $E$ is $\kappa$-algebrable (i.e. it contains $\kappa$-generated linear algebra), if and only if it contains an algebra which is a $\kappa$-dimensional linear space (see [7]). Moreover, we say that a subset $E$ of a commutative linear algebra $V$ is strongly $\kappa$-algebrable ([7]), if there exists a $\kappa$-generated free algebra $A$ contained in $E \cup \{0\}$.

Note, that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of free generators of a free algebra $A \subset E$ if and only if the set $X'$ of elements of the form $x_{\alpha_1}^{k_1}x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $X'$ are in $E \cup \{0\}$. It is easy to see that free algebras have no divisors of zero.

In practice, to prove $\kappa$-algebrability of set $E \subset V$ we have to find $X \subseteq E$ of cardinality $\kappa$ such that for any polynomial $P$ in $n$ variables and any distinct $x_1, \ldots, x_n \in X$ we have either $P(x_1, \ldots, x_n) \in E$ or $P(x_1, \ldots, x_n) = 0$. To prove the strong $\kappa$-algebrability of $E$ we have to find $X \subset E$, $|X| = \kappa$, such that for any non-zero polynomial $P$ and distinct $x_1, \ldots, x_n \in X$ we have $P(x_1, \ldots, x_n) \in E$.

In general, there are subsets of linear algebras which are algebrable but not strongly algebrable. Let $c_{00}$ be a subset of $c_0$ consisting of all sequences with real terms equal to zero from some place. Then the set $c_{00}$ is algebrable in $c_0$ but is not strongly 1-algebrable [7].

Let $X$ be a Banach space. The subset $M$ of $X$ is spaceable if $M \cup \{0\}$ contains infinitely dimensional closed subspace $Y$ of $X$. Since every infinitely dimensional Banach space contains linearly independent set of the cardinality continuum, the spaceability implies $\mathfrak{c}$-lineability. However, the spaceability is a much stronger property than $\mathfrak{c}$-lineability. The notions of spaceability and $\mathfrak{c}$-algebrability are incomparable. We will show that even $\mathfrak{c}$-algebrable dense $\mathcal{G}_\delta$-sets in $\ell_1$ may not be spaceable. On the other hand, there are sets in $c_0$ which are spaceable but not 1-algebrable (see [7]).
1.2. The subsums of series. Let \( x \in \ell_1 \). The set of all subsums of \( \sum_{n=1}^{\infty} x(n) \), meaning the set of sums of all subseries of \( \sum_{n=1}^{\infty} x(n) \), is defined by

\[
E(x) = \{ a \in \mathbb{R} : \exists A \subset \mathbb{N} \sum_{n \in A} x(n) = a \}.
\]

Some authors call it the achievement set of \( x \). The following theorem is due to Kakeya.

**Theorem 1.** [18]. Let \( x \in \ell_1 \)

1. If \( x \not\in c_{00} \), then \( E(x) \) is a perfect compact set.
2. If \( |x(n)| > \sum_{i>n} |x(i)| \) for almost all \( n \), then \( E(x) \) is homeomorphic to the Cantor set.
3. If \( |x(n)| \leq \sum_{i>n} |x(i)| \) for \( n \) sufficiently large, then \( E(x) \) is a finite union of closed intervals. In the case of non-increasing sequence \( x \), the last inequality is also necessary to obtain \( E(x) \) being a finite union of intervals.

Moreover, Kakeya conjectured that \( E(x) \) is either nowhere dense or it is a finite union of intervals. Probably, the first counterexample to this conjecture was given (without a proof) by Weinstein and Shapiro [21] and, with a correct proof, by Ferens [11]. Guthrie and Nymann [16] showed that, for the sequence \( b \) given by the formulas \( b(2n-1)=\frac{3}{4^n} \) and \( b(2n)=\frac{2}{4^n} \), the set \( T = E(b) \) is not a finite union of intervals but it has nonempty interior.

In the same paper they formulated the following theorem

**Theorem 2.** [16] Let \( x \in \ell_1 \setminus c_{00} \), then \( E(x) \) is one of the following sets:

(i) a finite union of closed intervals;
(ii) homeomorphic to the Cantor set;
(iii) homeomorphic to the set \( T \).

A correct proof of the Guthrie and Nymann trichotomy was given by Nymann and Sáenz [20]. The sets homeomorphic to \( T \) are called Cantorvals (more precisely: M-Cantorvals). Note that Theorem 2 can be formulated as follows: The space \( \ell_1 \) is a disjoint union of the sets \( c_{00}, \mathcal{I}, \mathcal{C} \) and \( \mathcal{MC} \) where
\[ \mathcal{I} \text{ consists of sequences } x \text{ with } E(x) \text{ equal to a finite union of intervals, } \mathcal{C} \text{ consists of sequences } x \text{ with } E(x) \text{ homeomorphic to the Cantor set, and } \mathcal{MC} \text{ of } x \text{ with } E(x) \text{ being an M-Cantorval.} \]

For \( x \in \ell_1 \), let \( x' \) be an arbitrary finite modification of \( x \), and let \( |x| \) denote the sequence \( y \in \ell_1 \) such that \( y(n) = |x(n)| \). Then \( x \in \mathcal{I} \iff |x| \in \mathcal{I} \iff x' \in \mathcal{I} \). The same equivalences hold for sets \( \mathcal{C} \) and \( \mathcal{MC} \).

2. Algebraic substructures in \( \mathcal{C}, \mathcal{I} \) and \( \mathcal{MC} \).

Jones in a very nice paper [17] gives the following example. Let \( x(n) = 1/2^n \) and \( y(n) = 1/3^n \). Then clearly \( x \in \mathcal{I} \) and \( y \in \mathcal{C} \). Moreover, \( x + y \in \mathcal{C} \) and \( x - y \in \mathcal{I} \). Since \( x = (x + y) - y \) and \( y = -(x - y) + x \), then neither \( \mathcal{I} \) nor \( \mathcal{C} \) is closed under pointwise addition. However, in the present paper we show that the sets \( \mathcal{C}, \mathcal{I} \) and \( \mathcal{MC} \) contain large (\( c \)-generated) algebraic structures. To prove the strong \( c \)-algebrability of \( \mathcal{C} \) and \( \mathcal{I} \), we will combine Theorem 1 and the method of linearly independent exponents, which was successful in [6] and [7]. In the next theorem we construct generators as the powers of one geometric series \( x_q (x(n) = q^n) \) for \( 0 < q < \frac{1}{2} \). Clearly, by Theorem 1, \( x_q \in \mathcal{C} \).

**Theorem 3.** \( \mathcal{C} \) is strongly \( c \)-algebraable.

**Proof.** Fix \( q \in (0, 1/2) \). Let \( \{r_\alpha : \alpha < c\} \) be a linearly independent (over the field of all rationals \( \mathbb{Q} \)) set of reals greater than 1. Let \( x_\alpha(n) = q^{r_\alpha n} \). We will show that the set \( \{x_\alpha : \alpha < c\} \) generates a free algebra \( \mathcal{A} \) which, except for the null sequence, is contained in \( \mathcal{C} \).

To do this, we will show that for any \( \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R} \setminus \{0\} \), any matrix \( [k_{ij}]_{i \leq m, j \leq \beta} \) of natural numbers with nonzero distinct rows, and any \( \alpha_1 < \alpha_2 < \cdots < \alpha_j < c \), the sequence \( x \) given by

\[
x(n) = P(x_{\alpha_1}, \ldots, x_{\alpha_j})(n)
\]
where

\[ P(z_1, \ldots, z_j) = \beta_1 z_1^{k_{11}} z_2^{k_{12}} \cdots z_j^{k_{1j}} + \cdots + \beta_m z_1^{k_{m1}} z_2^{k_{m2}} \cdots z_j^{k_{mj}} \]

is in \( C \). In other words,

\[ x(n) = \beta_1 q^{n(r_{\alpha_1} k_{11} + \cdots + r_{\alpha_j} k_{1j})} + \cdots + \beta_m q^{n(r_{\alpha_1} k_{m1} + \cdots + r_{\alpha_j} k_{mj})} \]

Since \( r_{\alpha_1}, \ldots, r_{\alpha_j} \) are linearly independent and the rows of \( [k_{il}]_{i \leq m, l \leq j} \) are distinct, the numbers \( r_1 := r_{\alpha_1} k_{11} + \cdots + r_{\alpha_j} k_{1j}, \ldots, r_m := r_{\alpha_1} k_{m1} + \cdots + r_{\alpha_j} k_{mj} \) are distinct. We may assume that \( r_1 < \cdots < r_m \). Then

\[
\frac{|x(n)|}{\sum_{i>n} |x(i)|} = \frac{|\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}|}{\sum_{i>n} |\beta_1 q^{ir_1} + \cdots + \beta_m q^{ir_m}|} \\
\geq \frac{|\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}|}{\sum_{i>n} (|\beta_1 q^{ir_1} + \cdots + |\beta_m q^{ir_m}|)} = \frac{|\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}|}{\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}} \\
\to \frac{1 - q^{r_1}}{q^{r_1}} > 1.
\]

Therefore there is \( n_0 \), such that \( |x(n)| > \sum_{i>n} |x(i)| \) for all \( n \geq n_0 \). Hence, by Theorem 1, we obtain that \( x \in C \).

\[ \square \]

It is obvious that the geometric sequence \( x_q \), even for \( q > \frac{1}{2} \), is not useful to construct the generators of linear algebra contained in \( \mathcal{I} \). Indeed, for sufficiently large exponent \( k \), the sequence \( x^k_q \) belongs to \( C \). So, in the next theorem we use the harmonic series.

**Theorem 4.** \( \mathcal{I} \) is strongly \( t \)-algebrable.

**Proof.** Let \( K \) be a linearly independent subset of \((1, \infty)\) of cardinality \( c \). For \( \alpha \in K \), let \( x_\alpha \) be a sequence given by the formula \( x_\alpha(n) = \frac{1}{n^\alpha} \). We will show that the set \( \{x_\alpha : \alpha \in K\} \) generates a free algebra \( \mathcal{A} \) which is contained in \( \mathcal{I} \cup \{0\} \). To do this, we will show that for any \( \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R} \setminus \{0\} \), any matrix \( [k_{il}]_{i \leq m, l \leq j} \) of natural numbers with nonzero distinct rows, and any \( \alpha_1 < \alpha_2 < \cdots < \alpha_j \), the sequence \( x \) defined by

\[
x = P(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_j})
\]
\[ x(n) = P(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_j})(n) \]

Note that \( p_1, \ldots, p_m \) are distinct. Assume that \( p_1 < p_2 < \cdots < p_m \). We have

\[
\sum_{k>n}|x(k)| \leq \frac{\beta_1}{\sum_{k>n} \frac{1}{n^{\frac{k}{p_1}}}} + \beta_2 \frac{1}{n^{\frac{2}{p_2}}} + \cdots + \beta_m \frac{1}{n^{\frac{m}{p_m}}}
\]

Observe that the first inequality holds for \( n \) large enough. Therefore there is \( n_0 \) such that \( |x(n)| \leq \sum_{i>n} |x(i)| \) for any \( n \geq n_0 \). Hence, by Theorem 1 we obtain that \( x \in I \).

The method described in the next lemma belongs to the mathematical folklore and was used to construct sequences \( x \)'s with \( E(x) \) being Cantorvals. We present its proof since we did not find it explicitly formulated in the mathematical literature.

**Lemma 5.** Let \( x \in \ell_1 \) be such that

(i) \( E(x) \) contains an interval;

(ii) \( |x(n)| > \sum_{i>n} |x(i)| \) for infinitely many \( n \);

(iii) \( |x_n| \geq |x_{n+1}| \) for almost all \( n \).

Then \( x \in MC \).
Proof. By (ii)-(iii), the point \( x \) does not belong to \( \mathcal{I} \). By (i), the point \( x \) does not belong to \( \mathcal{C} \). Hence, by Theorem 2 we get \( x \in \mathcal{MC} \). \( \square \)

Up to last years, there were only known a few examples of sequences belonging to \( \mathcal{MC} \). These examples were not very useful to construct a large number of linearly independent sequences. Recently, Jones in [17] has constructed a one-parameter family of sequences in \( \mathcal{MC} \). We shall use some modification of the example given by Jones in the proof of our next theorem.

**Theorem 6.** \( \mathcal{MC} \) is \( c \)-lineable.

**Proof.** Let

\[
x_q = (4, 3, 2, 4q, 3q, 2q, 4q^2, 3q^2, 2q^2, 4q^3, \ldots)
\]

and

\[
y_q = (1, 1, 1, 1, q, q, q, q, q^2, q^2, q^2, q^2, q^3, \ldots)
\]

for \( q \in \left[\frac{1}{6}, \frac{2}{11}\right) \).

Observe that the sequences \( x_q, q \in \left[\frac{1}{6}, \frac{2}{11}\right) \) are linearly independent. We need to show that each non-zero linear combination of sequences \( x_q \) fulfils the assumptions (i)–(iii) of Lemma 5 and therefore it is actually in \( \mathcal{MC} \). To prove this, let us fix \( q_1 > q_2 > \cdots > q_m \in \left[\frac{1}{6}, \frac{2}{11}\right), \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R} \) and define sequences \( x \) and \( y \) by

\[
x(n) = \beta_1 x_{q_1}(n) + \beta_2 x_{q_2}(n) + \cdots + \beta_m x_{q_m}(n)
\]

and

\[
y(n) = \beta_1 y_{q_1}(n) + \beta_2 y_{q_2}(n) + \cdots + \beta_m y_{q_m}(n).
\]

At first, we will check that for almost all \( n \)

(1) \( 2|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n| > 9 \sum_{k>n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k| \).

We have

\[
\frac{2|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|} \geq \frac{2|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n} |\beta_1 q_1^k| + |\beta_2 q_2^k| + \cdots + |\beta_m q_m^k|}
\]
= \frac{2|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{9(\|\beta_1 q_1^{n+1} + \beta_2 q_2^{n+1} + \cdots + \beta_m q_m^{n+1}\|)} \rightarrow \frac{2}{9} \frac{1 - q_1}{q_1} > \frac{2}{9} \frac{1 - \frac{2}{11}}{\frac{2}{11}} = 1.

Note that if \( n \) is not divisible by 3, then \(|x(n)| \geq |x(n + 1)|\). On the other hand, if \( n = 3l \), then

\[ |x(n)| = 2|\beta_1 q_1^l + \cdots + \beta_m q_m^l| \]

and

\[ |x(n + 1)| = 3|\beta_1 q_1^{l+1} + \cdots + \beta_m q_m^{l+1}| \leq 9 \sum_{k > l} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|. \]

Hence by (1) we obtain \(|x(n)| \geq |x(n + 1)|\) for almost all \( n \). By (1) we also have \(|x(n)| > \sum_{i > n} |x(i)|\) for infinitely many \( n \).

Now we will show that

\begin{equation}
(2) \quad |\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n| \leq \frac{5}{5} \sum_{k > n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|.
\end{equation}

We have

\[
\begin{aligned}
\frac{|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{5 \sum_{k > n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|} &\leq \frac{|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{5 |\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|} \\
&= \frac{|\beta_1 + \beta_2 q_1^n + \cdots + \beta_m q_m^n|}{5 |\beta_1 \sum_{i > 0} q_1^i + \beta_2 q_2^i + \cdots + \beta_m q_m^n|} \\
&\rightarrow_{n \rightarrow \infty} \frac{1}{5} \cdot \frac{1 - q_1}{q_1} \leq \frac{1}{5} \cdot \frac{1 - \frac{1}{6}}{\frac{1}{6}} = 1.
\end{aligned}
\]

By (2) we obtain that \(|y(n)| \leq \sum_{k > n} |y(k)|\) for almost all \( n \). Therefore by Theorem 1, the set \( E(y) \) is a finite union of closed intervals. Thus \( E(y) \) has non-empty interior.

To end the proof we need to show that \( E(x) \) has non-empty interior. We will prove that

\[ 2 \sum_{n=0} |\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n| + E(y) \subseteq E(x). \]

Let \( t \in \sum_{n=0} |\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n| + E(y). \)
Note that any element $t$ of $E(y)$ is of the form
\[ t = k_0(\beta_1 + \beta_2 + \cdots + \beta_m) + k_1(\beta_1 q_1 + \beta_2 q_2 + \cdots + \beta_m q_m) + k_2(\beta_1 q_1^2 + \beta_2 q_2^2 + \cdots + \beta_m q_m^2) + \cdots \]
where $k_n \in \{0, 1, 2, 3, 4, 5\}$. Thus $t$ is of the form
\[ t = \sum_{n=0}^{\infty} (\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n) + \]
\[ + k_0(\beta_1 + \beta_2 + \cdots + \beta_m) + k_1(\beta_1 q_1 + \beta_2 q_2 + \cdots + \beta_m q_m) + \]
\[ + k_2(\beta_1 q_1^2 + \beta_2 q_2^2 + \cdots + \beta_m q_m^2) + \cdots \]
Note that each number from $\{2, 3, 4, 5, 6, 7\}$, that is every number of the form $2 + k_n$, can be written as a sum of numbers $4, 3, 2$. Hence $t \in E(x)$ and $E(x)$ has non-empty interior. So $x \in MC$. \qed

3. The topological size and Borel classification of $C, I$ and $MC$.

Let us observe that all the sets $c_{00}, C, I$ and $MC$ are dense in $\ell_1$. Moreover, $c_{00}$ is an $F_\sigma$-set of the first category. We are interested in studying the topological size and Borel classification of considered sets. To do it, let us consider the hyperspace $H(\mathbb{R})$, that is the space of all non-empty compact subsets of reals, equipped with the Vietoris topology (see [19], 4F, pp.24-28). Recall, that the Vietoris topology is generated by the subbase of sets of the form $\{K \in H(\mathbb{R}) : K \subset U\}$ and $\{K \in H(\mathbb{R}) : K \cap U \neq \emptyset\}$ for all open sets $U$ in $\mathbb{R}$. This topology is metrizable by the Hausdorff metric $d_H$ given by the formula
\[ d_H(A, B) = \max\{\max_{t \in A} d(t, B), \max_{s \in B} d(s, A)\} \]
where $d$ is the natural metric in $\mathbb{R}$. It is known that the set $N$ of all nowhere dense compact sets is a $G_\delta$-set in $H(\mathbb{R})$ and the set $F$ of all compact sets
with finite number of connected components is an $\mathcal{F}_\sigma$-set. To see this, it is enough to observe that

- $K$ is nowhere dense if and only if for any set $U_n$ from a fixed countable base of natural topology in $\mathbb{R}$ there exists a set $U_m$ from this base, such that $\text{cl}(U_m) \subset U_n$ and $K \subset (\text{cl}(U_m))^c$;
- $K$ has more then $k$ components if and only if there exist pairwise disjoint open intervals $J_1, J_2, \ldots, J_{k+1}$, such that $K \subset J_1 \cup J_2 \cup \cdots \cup J_{k+1}$ and $K \cap J_i \neq \emptyset$ for $i = 1, 2, \ldots, k+1$.

Now, let us observe that if we assign the set $E(x)$ to the sequence $x \in \ell_1$, we actually define the function $E : \ell_1 \to H(\mathbb{R})$.

**Lemma 7.** The function $E$ is Lipschitz with Lipschitz constant $L = 1$, hence it is continuous.

**Proof.** Let $t \in E(x)$. Then there exists a subset $A$ of $\mathbb{N}$ such that $t = \sum_{n \in A} x(n)$. We have

$$d(t, E(y)) = d(t, \sum_{n \in A} y(n)) = \left| \sum_{n \in A} (x(n) - y(n)) \right| \leq \sum_{n \in A} |(x(n) - y(n))| = \left\| x - y \right\|_1$$

where $\left\| \cdot \right\|_1$ denotes the norm in $\ell_1$. Hence, $d_H(E(x), E(y)) \leq \left\| x - y \right\|_1$. □

**Theorem 8.** The set $\mathcal{C}$ is a dense $G_\delta$-set (and hence residual), $\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set (i.e. it is $\mathcal{F}_\sigma$ but not $G_\delta$) of the first category, and $\mathcal{MC}$ is in the class $(\mathcal{F}_{\sigma\delta} \cap G_{\delta\sigma}) \setminus G_{\delta}$.

**Proof.** Let us observe that $\mathcal{C} \cup c_{00} = E^{-1}[N]$ and $\mathcal{I} \cup c_{00} = E^{-1}[F]$ where $N$, $F$, $E$ are defined as before. Hence $\mathcal{C} \cup c_{00}$ is $G_\delta$-set and $\mathcal{I} \cup c_{00}$ is $\mathcal{F}_\sigma$-set. Thus $\mathcal{C}$ is $G_\delta$-set (because $c_{00}$ is $\mathcal{F}_\sigma$-set) and $\mathcal{I} \cup c_{00}$ is $\mathcal{F}_\sigma$-set. Thus $\mathcal{C}$ is $G_\delta$-set and $\mathcal{I} \cup \mathcal{MC}$ is $\mathcal{F}_\sigma$. Moreover, $\mathcal{I} = (\mathcal{I} \cup c_{00}) \cap (\mathcal{I} \cup \mathcal{MC})$ is $\mathcal{F}_\sigma$-set, too. By the density of $\mathcal{C}$, $\mathcal{C}$ is residual. Since $\mathcal{I}$ is dense of the first category, it cannot be $G_\delta$-set. For the same reason, $\mathcal{MC}$ also cannot be $G_\delta$-set. Since $\mathcal{MC}$ is a difference of two $\mathcal{F}_\sigma$-sets, it is in the class $\mathcal{F}_{\sigma\delta} \cap G_{\delta\sigma}$. □
Remark 9. In [7] it was shown the following similar result by the use of quite different methods: the set of bounded sequences, with the set of limit points homeomorphic to the Cantor set, is strongly $c$-algebrable and residual in $l^\infty$.

4. Spaceability

In this section we will show that $I$ is spaceable while $C$ is not spaceable. This shows that there is a subset $M$ of $\ell_1$ containing a dense $G_\delta$ subset and such that it contains a linear subspace of dimension $\mathfrak{c}$, but $Y \setminus M \neq \emptyset$ for any infinitely dimensional closed subspace $Y$ of $\ell_1$.

Theorem 10. Let $I_1$ be a subset of $I$ which consists of those $x \in \ell_1$ for which $E(x)$ is an interval. Then $I_1$ is spaceable.

Proof. Let $A_1, A_2, \ldots$ be a partition of $\mathbb{N}$ into infinitely many infinite subsets. Let $A_n = \{k_n^1 < k_n^2 < k_n^3 < \ldots \}$. Define $x_n \in \ell_1$ in the following way. Let $x_n(k_n^j) = 2^{-j}$ and $x_n(i) = 0$ if $i \notin A_n$. Then $\|x_n\|_1 = 1$ and $\{x_n : x \in \mathbb{N}\}$ forms a normalised basic sequence. Let $Y$ be a closed linear space generated by $\{x_n : x \in \mathbb{N}\}$. Then

$$y \in Y \iff \exists t \in \ell_1 \left( y = \sum_{n=1}^{\infty} t(n)x_n \right).$$

Since $E(x_n) = [0, 1]$, then $E\left(\sum_{n=1}^{\infty} t(n)x_n\right) = \bigcup_{n=1}^{\infty} I_n$ where $I_n$ is an interval with endpoints 0 and $t(n)$. Put $t^+(n) = \max\{t(n), 0\}$ and $t^-(n) = \min\{-t(n), 0\}$. Then $E\left(\sum_{n=1}^{\infty} t(n)x_n\right) = [\sum_{n=1}^{\infty} t^-(n), \sum_{n=1}^{\infty} t^+(n)]$ and the result follows.

Let us remark the very recent result by Bernal-González and Ordóñez Cabrera [10, Theorem 2.2]. The authors gave sufficient conditions for spaceability of sets in Banach spaces. Using that result, one can prove spaceability of $I$ but it cannot be used to prove Theorem 10, since the assumptions are not fulfilled.
However we do not know more results giving the sufficient conditions for a set in Banach space to not be spaceable. An interesting example of a non-spaceable set was given in the classical paper [14] by Gurarii where it was proved that the set of all differentiable functions from $C[0, 1]$ is not spaceable. It is well known that the set of all differentiable functions in $C[0, 1]$ is dense but meager. We will prove that even dense $G_δ$-sets in Banach spaces may not be spaceable.

**Theorem 11.** Let $Y$ be an infinitely dimensional closed subspace of $ℓ_1$. Then there is $y \in Y$ such that $E(y)$ contains an interval.

**Proof.** Let $Y$ be an infinitely dimensional closed subspace of $ℓ_1$. Let $ε_n ↘ 0$.

Let $x_1$ be any nonzero element of $Y$ with $\|x_1\|_1 = 1 + ε_1$. Since $x_1 ∈ ℓ_1$, there is $n_1$ with $\sum_{n=n_1+1}^{∞} |x_1(n)| ≤ ε_1$. Let $E_1$ consist of finite sums $\sum_{n=1}^{n_1} δ_n x_1(n)$ where $δ_i ∈ \{0, 1\}$. Then $E_1$ is a finite set with $\min E_1 = ∑_{n=1}^{n_1} x_1^−(n)$, $\max E_1 = ∑_{n=1}^{n_1} x_1^+(n)$ and $1 ≤ \max E_1 - \min E_1 ≤ 1 + ε_1$.

Let $Y_1 = Y \cap \{x ∈ ℓ_1 : x(n) = 0 \text{ for every } n ≤ n_1\}$. Since $\{x ∈ ℓ_1 : x(n) = 0 \text{ for every } n ≤ n_1\}$ has a finite co-dimension, then $Y_1$ is infinitely dimensional. Let $x_2$ be any nonzero element of $Y_1$ with $\|x_2\|_1 = 1 + ε_2$. Since $x_2 ∈ ℓ_1$, there is $n_2 > n_1$ with $\sum_{n=n_2+1}^{∞} |x_2(n)| ≤ ε_2$, $i = 1, 2$. Let $E_2$ consist of finite sums $\sum_{n=n_1+1}^{n_2} δ_n x_2(n)$, where $δ_i ∈ \{0, 1\}$. Then $E_2$ is a finite set with $\min E_2 = ∑_{n=n_1+1}^{n_2} x_2^−(n)$, $\max E_2 = ∑_{n=n_1+1}^{n_2} x_2^+(n)$ and $1 ≤ \max E_2 - \min E_2 ≤ 1 + ε_2$.

Proceeding inductively, we define natural numbers $n_1 < n_2 < n_3 < \ldots$, infinitely dimensional closed spaces $Y ⊃ Y_1 ⊃ Y_2 ⊃ \ldots$ such that $Y_k = \{x ∈ Y : x(n) = 0 \text{ for every } n ≤ n_k\}$, nonzero elements $x_k ∈ Y_{k-1}$ with $\|x_k\|_1 = 1 + ε_k$ and $\sum_{n=n_k+1}^{∞} |x_k(n)| ≤ ε_k$, $i = 1, 2, \ldots, k$, and finite sets $E_k$ consisting of sums $\sum_{n=n_k+1}^{n_{k+1}} δ_n x_k(n)$ where $δ_i ∈ \{0, 1\}$. Note that $1 ≤ \text{diam}(E_k) ≤ 1 + ε_k$. Consider $y = ∑_{k=1}^{∞} x_k/2^k$. We claim that $E(y)$ contains an interval $I := [\min E_1, \max E_1]$. 
Note that for any \( t \in I \) there is \( t_1 \in E_1 \) with \( |t - t_1| \leq (1 + \varepsilon_1)/2 \). Since \( 1 \leq \text{diam}(E_2) \leq 1 + \varepsilon_2 \), there is \( t_2 \in E_1 + \frac{1}{2}E_2 \) with \( |t - t_2| \leq (1 + \varepsilon_2)/2^2 \). Hence, there is \( \tilde{t} \in E(x_1 + x_2/2) \) with \( |t - \tilde{t}| \leq (1 + \varepsilon_2)/2^2 + \varepsilon_1 \). Since \( 1 \leq \text{diam}(E_k) \leq 1 + \varepsilon_k \), then inductively we can find \( t_k \in E_1 + \frac{1}{2}E_2 + \cdots + \frac{1}{2^{k-1}}E_k \) with \( |t - t_k| \leq (1 + \varepsilon_k)/2^k \). Hence, there is \( \tilde{t} \in E(x_1 + x_2/2 + \cdots + x_k/2^{k-1}) \) with \( |t - \tilde{t}| \leq (1 + \varepsilon_k)/2^k + \varepsilon_k - 1/2 + \cdots + \varepsilon_k - 1/2^{k-1} \leq (1 + \varepsilon_k)/2^k + 2\varepsilon_k - 1 \). Since \( E(y) \) is closed and it contains \( E(x_1 + x_2/2 + \cdots + x_k/2^{k-1}) \), then \( t \in E(y) \) and consequently \( I \subset E(y) \). □

Immediately we get the following.

**Corollary 12.** The set \( C \) is not spaceable.

We end the paper with the list of open questions on the set \( \mathcal{M}C \).

**Problem 13.**

(i) Is \( \mathcal{M}C \) \( c \)-algebra-ble?

(ii) Is \( \mathcal{M}C \) an \( \mathcal{F}_\sigma \) subset of \( \ell_1 \)?

(iii) Is \( \mathcal{M}C \) spaceable?

**Acknowledgment.** The second and the third authors have been supported by the Polish Ministry of Science and Higher Education Grant No. N N201 414939 (2010-2013). We want to thank F. Prus-Wiśniowski who has informed us about the trichotomy of Guthrie and Nymann, and other references on subsums of series.

**References**


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