ALGEBRABILITY OF CONDITIONALLY CONVERGENT SERIES WITH CAUCHY PRODUCT

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Abstract. We show that the set of conditionally convergent real series considered with Cauchy product is \((\omega, 1)\)-algebrable.

By FS we denote the linear space of all formal series over \(\mathbb{R}\). We can consider FS as a linear algebra with two different products, namely for \(\sum_{n=0}^{\infty} x_n\) and \(\sum_{n=0}^{\infty} y_n\) let

\[
\left( \sum_{n=0}^{\infty} x_n \right) \cdot \left( \sum_{n=0}^{\infty} y_n \right) = \sum_{n=0}^{\infty} x_n y_n
\]

(point-wise product) and

\[
\left( \sum_{n=0}^{\infty} x_n \right) \times \left( \sum_{n=0}^{\infty} y_n \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x_k y_{n-k}
\]

(Cauchy product). By CCS we denote the set of all conditionally convergent series. In [APS] Aizpuru et al. proved \(c\)-lineability of CCS and they consider the algebras in \((FS, \cdot)\) consisting of elements from CCS and \(c_{00}\).

We say that subset \(E\) of some linear algebra is \((\alpha, \beta)\)-algebrable if there is a \(\beta\)-generated algebra \(A\) such that \(A \subset E \setminus \{0\}\) such that \(A\) is not \(\tau\)-generated for any \(\tau < \beta\) and linear dimension of \(A\) is equal to \(\alpha\). The notion of algebrability was considered by many authors [ACPS], [APS1], [AS], [GPS], [GS1], [BG].

It is easy to see that CCS is not algebrable in \((FS, \cdot)\). However if we consider the series of complex numbers, it appears that the set of all conditionally convergent series with point-wise product is \((c, c)\)-algebrable [BGP]. This note is devoted to show that CCS is \((\omega, 1)\)-algebrable in \((FS, \times)\).

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Our main tool will be the following classical result by Pringsheim. A family \( \{ \sum_{n=0}^{\infty} x_n^s : s \in S \} \) of series is absolutely equi-convergent if for any \( \varepsilon > 0 \) there is \( N \) such that \( \sum_{k=n}^{\infty} |x_n^s| < \varepsilon \) for any \( n \geq N \) and \( s \in S \).

**Theorem 1.** [P] Let \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) be convergent series. Assume that the series \((a_0 + a_1) + (a_2 + a_3) + (a_4 + a_5) + \ldots\) is absolutely convergent. Suppose moreover that the family of series \( F = \{ \sum_{n=0}^{\infty} a_{\varphi(n)} b_{\psi(n)} : \varphi, \psi : \mathbb{N} \to \mathbb{N} \text{ with } \varphi(n), \psi(n) \geq n \} \) is absolutely equi-convergent. Then \( \sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n) \times (\sum_{n=0}^{\infty} b_n) \) is convergent.

1. **Conditionally Convergent Series**

We say that series \( \sum_{n=0}^{\infty} a_n \) is alternating if \( a_{2n} \geq 0 \) and \( a_{2n+1} \leq 0 \) for any \( n = 0, 1, 2, \ldots \). It is an easy observation that the Cauchy product of two alternating series is alternating.

**Theorem 2.** CCS is \( (\omega, 1) \)-algebrable in \( (FS, \times) \).

**Proof.** Put \( a_n = \frac{(-1)^n}{n+1} \) for any \( n \in \mathbb{N} \). Note that the series \( \sum_{n=0}^{\infty} a_n \) is alternating. Define numbers \( a_n^{(k)} \) inductively: \( a_n^{(1)} = a_n \) for any \( n \in \mathbb{N} \) and \( a_n^{(k+1)} = \sum_{m=0}^{n} a_m^{(k)} a_{n-m} \) for any \( k \geq 1 \). We will use the well-known fact that

\[
\lim_{n \to \infty} \left( \sum_{m=0}^{n} \frac{1}{m+1} - \ln(n+1) \right) = \gamma \quad \text{(Euler–Mascheroni constant)}.
\]

Then there are \( 0 < C_1 < C_2 < \infty \) such that

\[
C_1 \ln(n+1) \leq \sum_{m=0}^{n} \frac{1}{m+1} \leq C_2 \ln(n+1)
\]
for any \( n \geq 1 \). Having this we will show inductively that
\[
\frac{C_1^{(k)} \ln^{k-1}(n+1)}{n+1} \leq |a_n^{(k)}| \leq \frac{C_2^{(k)} \ln^{k-1}(n+1)}{n+1}
\]
for any \( n, k \geq 1 \) and certain positive constants \( C_1^{(k)} \) and \( C_2^{(k)} \). This is obvious for \( k = 1 \). Assume that this is true for some \( k \). Then
\[
|a_n^{(k+1)}| = \sum_{m=0}^{n} a_m^{(k)} a_{n-m} \leq \sum_{m=0}^{n} C_2^{(k)} \ln^{k-1}(m+1) \cdot \frac{1}{m+1} \leq \frac{C_2^{(k)} \ln^{k-1}(n+1)}{n+2} \leq \frac{C_2^{(k)} \ln^{k}(n+1)}{n+1}.
\]
Put \( C_2^{(k+1)} = 2C_2^{(k)}C_2 \). We also have
\[
|a_n^{(k+1)}| = \sum_{m=0}^{n} a_m^{(k)} a_{n-m} \leq \sum_{m=0}^{n} C_1^{(k)} \ln^{k-1}(m+1) \cdot \frac{1}{m+1} \geq \frac{C_1^{(k)} \ln^{k-1}(n+1)}{n+2} \leq \frac{C_1^{(k)} \ln^{k}(n+1)}{n+1},
\]
where
\[
0 < \tilde{C} \leq \frac{(n+1) \ln^k(n+1)}{(n+2) \ln^k(n+1)}
\]
for any \( n \in \mathbb{N} \). Put \( C_1^{(k+1)} = C_1^{(k)}C_1/2^k \tilde{C} \).

Let \( N \) be such that the map \( n \rightarrow \frac{\ln^{k-1}(n+1)}{n+1} \) is decreasing on \( \{ N, N + 1, N + 2, \ldots \} \). Note that for \( m \geq N \)
\[
\sum_{n=m}^{\infty} |a_{\varphi(n)}^{(1)}||a_{\psi(n)}^{(k)}| \leq \sum_{n=m}^{\infty} \frac{C_2^{(k)} \ln^{k-1}(\psi(n)+1)}{\psi(n)+1} \leq
\]
for any $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $\varphi(n), \psi(n) \geq n$. See that $|a_n^{(1)} + a_{n+1}^{(1)}| \leq \frac{1}{n+1} - \frac{1}{n+2} \leq \frac{1}{(n+1)^2}$. Hence using Theorem 1 we obtain that $\sum_{n=0}^{\infty} a_n^{(k+1)}$ is convergent.

Let $A$ be a sub-algebra of $(FS, \times)$ generated by $\sum_{n=0}^{\infty} a_n$. To end the proof it is enough to show that the series

$$\sum_{n=0}^{\infty} \left( c_1 a_n^{(1)} + c_2 a_n^{(2)} + \ldots + c_k a_n^{(k)} \right)$$

is conditionally convergent for any natural number $k$ and any reals $c_1, \ldots, c_k$ with $c_k \neq 0$. This follows from the fact that $(\sum_{n=0}^{\infty} a_n)^k = \sum_{n=0}^{\infty} a_n^{(k)}$. This series is clearly convergent as a linear combination of convergent series. We will show that it is not absolutely convergent.

We may assume that $c_k = 1$ and $k \geq 2$. Let $M_1 = \max_{i=1,2,\ldots,k-1} \frac{|c_i|}{|c_k|}$, $M_2 = \max_{i=1,2,\ldots,k} |C_2^{(i)}|$. Let $m_0 \in \mathbb{N}$ be such that

$$\ln(n+1) > \frac{2(k-1)M_1 M_2}{C_1^{(k)}}$$

for any $n \geq m_0$. Then

$$|a_n^{(k)}| \geq \frac{C_1^{(k)} n^{k-1}(n+1)}{n+1} = \ln(n+1) \cdot \frac{C_1^{(k)} n^{k-2}(n+1)}{n+1} > \frac{2(k-1)M_1 M_2 \cdot C_1^{(k)} n^{k-2}(n+1)}{C_1^{(k)}} n+1$$

$$\geq 2M_1 M_2 \left( \frac{\ln^{k-2}(n+1)}{n+1} + \frac{\ln^{k-1}(n+1)}{n+1} + \ldots + \frac{\ln(n+1) + 1}{n+1} \right) \geq$$

$$\geq 2M_1 \left( |a_n^{(k-1)}| + |a_n^{(k-2)}| + \ldots + |a_n^{(2)}| + |a_n^{(1)}| \right) \geq$$

$$\geq \frac{2}{|c_k|} \left( |c_{k-1} a_n^{(k-1)}| + |c_{k-2} a_n^{(k-2)}| + \ldots + |c_2 a_n^{(2)}| + |c_1 a_n^{(1)}| \right) \geq$$

$$\geq \frac{2}{|c_k|} \left| c_{k-1} a_n^{(k-1)} + c_{k-2} a_n^{(k-2)} + \ldots + c_2 a_n^{(2)} + c_1 a_n^{(1)} \right|.$$

Therefore

$$|c_k a_n^{(k)}| - |c_1 a_n^{(1)} + c_2 a_n^{(2)} + \ldots + c_{k-1} a_n^{(k-1)}| \geq |c_k a_n^{(k)}| - \frac{1}{2} |c_k a_n^{(k)}| = \frac{1}{2} |c_k a_n^{(k)}|. $$
Hence
\[
\sum_{n=0}^{\infty} \left| c_1a_n^{(1)} + c_2a_n^{(2)} + \ldots + c_ka_n^{(k)} \right| \geq \sum_{n=m_0}^{\infty} \left| c_1a_n^{(1)} + c_2a_n^{(2)} + \ldots + c_{k-1}a_n^{(k-1)} \right| \geq \sum_{n=m_0}^{\infty} \frac{|c_k a_n^{(k)}|}{2} = \infty.
\]
Note that in particular we have proved that the set \(((\sum_{n=0}^{\infty} a_n)^k : k \geq 1\) is linearly independent.

\[\square\]

2. Appendix

Since Prinsheim’s paper [P] is not readily accessible, we reproduce here the proof of Theorem 1 for the sake of completeness of this note.

\textbf{Proof.} First we will show that \(c_n \to 0\). We have
\[
c_{2m} = \sum_{k=0}^{m} a_k b_{2m-k} + \sum_{k=0}^{m} a_{2m-k} b_k - a_m b_m,
\]
\[
c_{2m+1} = \sum_{k=0}^{m} a_k b_{2m+1-k} + \sum_{k=0}^{m} a_{2m+1-k} b_k.
\]
Hence
\[
|c_n| \leq \sum_{k=0}^{m} |a_k b_{n-k}| + \sum_{k=0}^{m} |a_{n-k} b_k| + |a_m b_m|
\]
where \(m = \max\{k \in \mathbb{Z} : k \leq n/2\}\). Since \(F\) is absolutely equi-convergent, we find \(N \in \mathbb{N}\) with
\[
\sum_{k=N}^{m} |a_k b_{n-k}| < \frac{\varepsilon}{5} \quad \text{and} \quad \sum_{k=N}^{m} |a_{n-k} b_k| < \frac{\varepsilon}{5}.
\]
Let \(n\) be such that
\[
|b_n|, |b_{n-1}|, \ldots, |b_{n-N+1}| < \frac{\varepsilon}{\max_{i \in \mathbb{N}} |a_i| 4N},
\]
\[
|a_n|, |a_{n-1}|, \ldots, |a_{n-N+1}| < \frac{\varepsilon}{\max_{i \in \mathbb{N}} |b_i| 4N}.
\]
and \(|a_m b_m| < \varepsilon/5\). Then
\[
\sum_{k=0}^{N-1} |a_k b_{n-k}| < \frac{\varepsilon}{5} \quad \text{and} \quad \sum_{k=0}^{N-1} |a_{n-k} b_k| < \frac{\varepsilon}{5}.
\]
Hence
\[ |c_n| \leq \sum_{k=0}^{m} |a_k b_{n-k}| + \sum_{k=0}^{m} |a_{n-k} b_k| + |a_n b_m| = \]
\[
\sum_{k=0}^{N-1} |a_k b_{n-k}| + \sum_{k=0}^{m} |a_{n-k} b_k| + \sum_{k=0}^{m} |a_{n-k} b_k| + |a_n b_m| < \varepsilon.
\]
Therefore \( c_n \to 0 \).

Recall that if the series \( \sum_{n=0}^{\infty} c_n \) is convergent to some \( C \), then \( C = AB \),
where \( A = \sum_{n=0}^{\infty} a_n \) and \( B = \sum_{n=0}^{\infty} b_n \). Since \( c_n \to 0 \), it is enough to show that
\[ D_{4m} = C_{4m} - A_{2m} B_{2m} = \sum_{k=0}^{4m} c_k - \sum_{k=0}^{2m} a_k \sum_{l=0}^{2m} b_l \]
tends to zero, if \( m \to \infty \).

We have
\[
D_{4m} = \sum_{k=0}^{4m} \sum_{l=0}^{k} a_l b_{k-l} - \sum_{k=0}^{2m} a_k \sum_{l=0}^{2m} b_l =
\]
\[
a_0 \sum_{l=0}^{4m} b_l + a_1 \sum_{l=0}^{4m-1} b_l + \ldots + a_{4m-1} \sum_{l=0}^{1} b_l + a_{4m} b_0 - (a_0 + a_1 + \ldots + a_{2m}) \sum_{l=0}^{2m} b_l =
\]
\[
a_0 \sum_{l=2m+1}^{4m} b_l + a_1 \sum_{l=2m+1}^{4m-1} b_l + \ldots + a_{2m-1} b_{2m+1} + a_{2m} b_{2m+1} + \ldots + a_{4m} b_0 =
\]
\[
(a_0 + a_1) \sum_{l=2m+1}^{4m-1} b_l + (a_2 + a_3) \sum_{l=2m+1}^{4m-3} b_l + \ldots + (a_{2m-2} + a_{2m-1}) b_{2m+1} +
\]
\[
+ (a_{2m} + a_{2m+1}) \sum_{l=2m+1}^{2m-1} b_l + \ldots + (a_{4m-2} + a_{4m-1}) b_0 + b_1 +
\]
\[
a_0 b_{4m} + a_2 b_{4m-2} + \ldots + a_{2m-2} b_{2m+2} + a_{2m} b_{2m} + a_{2m+2} b_{2m} + \ldots + a_{4m} b_0
\]
\[- a_{2m} (b_0 + \ldots + b_{2m}) \]

Hence
\[
|D_{4m}| \leq \sum_{k=0}^{m-1} |a_{2k} + a_{2k+1}| \sum_{l=2m+1}^{4m-(2k+1)} b_l + \sum_{k=0}^{m-1} |a_{2m+2k} + a_{2m+2k+1}| \sum_{l=0}^{2m-(2k+1)} b_l +
\]
\[
+ \sum_{k=0}^{2m} |a_{2k} b_{4m-2k}| + |a_m| \sum_{k=0}^{2m} b_k .
\]
Let $G = \max\{\sum_{k=0}^{\infty} |a_{2k} + a_{2k+1}|, \sum_{k=0}^{\infty} |b_k|\} < \infty$. Let $\varepsilon > 0$ let $M \in \mathbb{N}$ be such that the following inequalities hold for any $m \geq M$

$$\left| \sum_{l=2m+1}^{4m-(2k+1)} b_l \right| < \frac{\varepsilon}{4G},$$

$$\sum_{k=0}^{m-1} |a_{2m+2k} + a_{2m+2k+1}| < \frac{\varepsilon}{4G},$$

$$|a_{2m}| < \frac{\varepsilon}{4G},$$

$$\sum_{k=0}^{2m} |a_{2k}b_{4m-2k}| < \frac{\varepsilon}{4}.$$

To find such $m$ in the last inequality one should repeat the same reasoning as in the first part of the proof where it has been shown that $c_n \to 0$. Now, if $m \geq M$, then $|D_{4m}| < \varepsilon$ and the result follows.

\[\Box\]

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