ALGEBRABILITY OF CONDITIONALLY CONVERGENT SERIES WITH CAUCHY PRODUCT

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Abstract. We show that the set of conditionally convergent real series considered with Cauchy product is $(\omega, 1)$-algebrable.

By FS we denote the linear space of all formal series over $\mathbb{R}$. We can consider FS as a linear algebra with two different products, namely for $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ let

$$\left(\sum_{n=0}^{\infty} x_n\right) \cdot \left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty} x_n y_n$$

(point-wise product) and

$$\left(\sum_{n=0}^{\infty} x_n\right) \times \left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} x_k y_{n-k}$$

(Cauchy product). By CCS we denote the set of all conditionally convergent series. In [APS] Aizpuru et al. proved $\mathfrak{c}$–lineability of CCS and they consider the algebras in $(\mathrm{FS}, \cdot)$ consisting of elements from CCS and c_{00}.

We say that subset E of some linear algebra is (α, β)–algebrable if there is a β–generated algebra A such that $A \subset E \setminus \{0\}$ such that A is not τ–generated for any $\tau < \beta$ and linear dimension of A is equal to α. The notion of algebrability was considered by many authors [ACPS], [APS1], [AS], [GPS], [GS1], [BG].

It is easy to see that CCS is not algebrable in $(\mathrm{FS}, \cdot)$. However if we consider the series of complex numbers, it appears that the set of all conditionally convergent series with point-wise product is $(\mathfrak{c}, \mathfrak{c})$-algebrable [BGP]. This note is devoted to show that CCS is $(\omega, 1)$–algebrable in $(\mathrm{FS}, \times)$.

1991 *Mathematics Subject Classification.* Primary: 40A05; Secondary: 15A03.

Key words and phrases. algebrability, Cauchy product of series.

Our main tool will be the following classical result by Pringsheim. A family $\{\sum_{n=0}^\infty x_n^s : s \in S\}$ of series is absolutely equi-convergent if for any $\varepsilon > 0$ there is N such that $\sum_{k=n}^\infty |x_n^s| < \varepsilon$ for any $n \geq N$ and $s \in S$.

Theorem 1. [P] *Let* $\sum_{n=0}^\infty a_n$ *and* $\sum_{n=0}^\infty b_n$ *be convergent series. Assume that the series*

$$(a_0 + a_1) + (a_2 + a_3) + (a_4 + a_5) + ...$$

is absolutely convergent. Suppose moreover that the family of series

$$\mathcal{F} = \left\{ \sum_{n=0}^\infty a_{\varphi(n)} b_{\psi(n)} : \varphi, \psi : \mathbb{N} \to \mathbb{N} \text{ with } \varphi(n), \psi(n) \geq n \right\}$$

is absolutely equi-convergent. Then $\sum_{n=0}^\infty c_n = (\sum_{n=0}^\infty a_n) \times (\sum_{n=0}^\infty b_n)$ *is convergent.*

1. CONDITIONALLY CONVERGENT SERIES

We say that series $\sum_{n=0}^\infty a_n$ is *alternating* if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ for any $n = 0, 1, 2, ...$ It is an easy observation that the Cauchy product of two alternating series is alternating.

Theorem 2. CCS *is* $(\omega, 1)$*–algebrable in* (FS, $\times$).

Proof. Put $a_n = \frac{(-1)^n}{n+1}$ for any $n \in \mathbb{N}$. Note that the series $\sum_{n=0}^\infty a_n$ is alternating. Define numbers $a_n^{(k)}$ inductively: $a_n^{(1)} = a_n$ for any $n \in \mathbb{N}$ and

$$a_n^{(k+1)} = \sum_{m=0}^n a_m^{(k)} a_{n-m}^{(1)}$$

for any $k \geq 1$. We will use the well-known fact that

$$\lim_{n\to\infty} \left(\sum_{m=0}^n \frac{1}{m+1} - \ln(n+1) \right) = \gamma \text{ (Euler–Mascheroni constant).}$$

Then there are $0 < C_1 < C_2 < \infty$ such that

$$C_1 \ln(n+1) \leq \sum_{m=0}^n \frac{1}{m+1} \leq C_2 \ln(n+1)$$

for any $n \geq 1$. Having this we will show inductively that

$$\frac{C_1^{(k)} \ln^{k-1}(n+1)}{n+1} \leq |a_n^{(k)}| \leq \frac{C_2^{(k)} \ln^{k-1}(n+1)}{n+1}$$

for any $n, k \geq 1$ and certain positive constants $C_1^{(k)}$ and $C_2^{(k)}$. This is obvious for $k = 1$. Assume that this is true for some k. Then

$$|a_n^{(k+1)}| = \left|\sum_{m=0}^{n} a_m^{(k)} a_{n-m}^{(1)}\right| = \sum_{m=0}^{n} |a_m^{(k)}||a_{n-m}^{(1)}| \leq \sum_{m=0}^{n} \frac{C_2^{(k)} \ln^{k-1}(m+1)}{m+1} \cdot \frac{1}{n+1-m} \leq$$

$$\leq C_2^{(k)} \ln^{k-1}(n+1) \sum_{m=0}^{n} \frac{1}{m+1} \cdot \frac{1}{n+1-m} = \frac{C_2^{(k)} \ln^{k-1}(n+1)}{n+2} \sum_{m=0}^{n} \left(\frac{1}{m+1} + \frac{1}{n+1-m}\right) \leq$$

$$\leq \frac{2C_2^{(k)} \ln^{k-1}(n+1)}{n+2} \sum_{m=0}^{n} \frac{1}{m+1} \leq \frac{2C_2^{(k)} C_2 \ln^{k}(n+1)}{n+2} \leq \frac{2C_2^{(k)} C_2 \ln^{k}(n+1)}{n+1}.$$

Put $C_2^{(k+1)} = 2C_2^{(k)} C_2$. We also have

$$|a_n^{(k+1)}| = \left|\sum_{m=0}^{n} a_m^{(k)} a_{n-m}^{(1)}\right| = \sum_{m=0}^{n} |a_m^{(k)}||a_{n-m}^{(1)}| \geq \sum_{m=0}^{n} \frac{C_1^{(k)} \ln^{k-1}(m+1)}{m+1} \cdot \frac{1}{n+1-m} \geq$$

$$\sum_{m=n/2}^{n} \frac{C_1^{(k)} \ln^{k-1}(m+1)}{m+1} \cdot \frac{1}{n+1-m} \geq C_1^{(k)} \ln^{k-1}\left(\frac{n}{2}\right) \sum_{m \geq (n-1)/2}^{n} \frac{1}{m+1} \cdot \frac{1}{n+1-m} \geq$$

$$\geq \frac{C_1^{(k)} \ln^{k-1}(\sqrt{n-1})}{n+2} \sum_{m \geq (n-1)/2}^{n} \left(\frac{1}{m+1} + \frac{1}{n+1-m}\right) \geq$$

$$\geq \frac{C_1^{(k)} \ln^{k-1}(n-1)}{2^{k-1}(n+2)} \sum_{m=0}^{n} \frac{1}{m+1} \geq \frac{C_1^{(k)} C_1 \ln^{k}(n-1)}{2^{k-1}(n+2)} =$$

$$= \frac{C_1^{(k)} C_1 \ln^{k}(n+1)}{2^{k-1}(n+1)} \cdot \frac{(n+1)\ln^{k}(n-1)}{(n+2)\ln^{k}(n+1)} \geq \frac{C_1^{(k)} C_1 \tilde{C} \ln^{k}(n+1)}{2^{k-1}(n+1)},$$

where

$$0 < \tilde{C} \leq \frac{(n+1)\ln^{k}(n-1)}{(n+2)\ln^{k}(n+1)}$$

for any $n \in \mathbb{N}$. Put $C_1^{(k+1)} = C_1^{(k)} C_1 / 2^k \tilde{C}$.

Let N be such that the map $n \mapsto \frac{\ln^{k-1}(n+1)}{n+1}$ is decreasing on $\{N, N+1, N+2, ...\}$ Note that for $m \geq N$

$$\sum_{n=m}^{\infty} |a_{\varphi(n)}^{(1)}||a_{\psi(n)}^{(k)}| \leq \sum_{n=m}^{\infty} \frac{1}{\varphi(n)+1} \cdot \frac{C_2^{(k)} \ln^{k-1}(\psi(n)+1)}{\psi(n)+1} \leq$$

$$\leq \sum_{n=m}^{\infty} \frac{1}{n+1} \cdot \frac{C_2^{(k)} \ln^{k-1}(n+1)}{n+1} = \sum_{n=m}^{\infty} \frac{C_2^{(k)} \ln^{k-1}(n+1)}{(n+1)^2} < \infty$$

for any $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $\varphi(n), \psi(n) \geq n$. See that $|a_n^{(1)} + a_{n+1}^{(1)}| \leq \frac{1}{n+1} - \frac{1}{n+2} \leq \frac{1}{(n+1)^2}$. Hence using Theorem 1 we obtain that $\sum_{n=0}^{\infty} a_n^{(k+1)}$ is convergent.

Let A be a sub-algebra of $(\mathrm{FS}, \times)$ generated by $\sum_{n=0}^{\infty} a_n$. To end the proof it is enough to show that the series

$$\sum_{n=0}^{\infty} \left(c_1 a_n^{(1)} + c_2 a_n^{(2)} + ... + c_k a_n^{(k)} \right)$$

is conditionally convergent for any natural number k and any reals $c_1, ..., c_k$ with $c_k \neq 0$. This follows from the fact that $(\sum_{n=0}^{\infty} a_n)^k = \sum_{n=0}^{\infty} a_n^{(k)}$. This series is clearly convergent as a linear combination of convergent series. We will show that it is not absolutely convergent.

We may assume that $c_k = 1$ and $k \geq 2$. Let $M_1 = \max_{i=1,2,...,k-1} \frac{|c_i|}{|c_k|}$, $M_2 = \max_{i=1,2,...,k} |C_2^{(i)}|$. Let $m_0 \in \mathbb{N}$ be such that

$$\ln(n+1) > \frac{2(k-1)M_1 M_2}{C_1^{(k)}}$$

for any $n \geq m_0$. Then

$$|a_n^{(k)}| \geq \frac{C_1^{(k)} \ln^{k-1}(n+1)}{n+1} = \ln(n+1) \cdot \frac{C_1^{(k)} \ln^{k-2}(n+1)}{n+1} > \frac{2(k-1)M_1 M_2}{C_1^{(k)}} \cdot \frac{C_1^{(k)} \ln^{k-2}(n+1)}{n+1} \geq$$

$$\geq 2M_1 M_2 \left(\frac{\ln^{k-2}(n+1)}{n+1} + \frac{\ln^{k-1}(n+1)}{n+1} + ... + \frac{\ln(n+1)}{n+1} + \frac{1}{n+1} \right) \geq$$

$$\geq 2M_1 \left(|a_n^{(k-1)}| + |a_n^{(k-2)}| + ... + |a_n^{(2)}| + |a_n^{(1)}| \right) \geq$$

$$\geq \frac{2}{|c_k|} \left(|c_{k-1} a_n^{(k-1)}| + |c_{k-2} a_n^{(k-2)}| + ... + |c_2 a_n^{(2)}| + |c_1 a_n^{(1)}| \right) \geq$$

$$\geq \frac{2}{|c_k|} \left| c_{k-1} a_n^{(k-1)} + c_{k-2} a_n^{(k-2)} + ... + c_2 a_n^{(2)} + c_1 a_n^{(1)} \right|.$$

Therefore

$$|c_k a_n^{(k)}| - |c_1 a_n^{(1)} + c_2 a_n^{(2)} + ... + c_{k-1} a_n^{(k-1)}| \geq |c_k a_n^{(k)}| - \frac{1}{2}|c_k a_n^{(k)}| = \frac{1}{2}|c_k a_n^{(k)}|.$$

Hence

$$\sum_{n=0}^{\infty}\left|c_1a_n^{(1)}+c_2a_n^{(2)}+...+c_ka_n^{(k)}\right|\geq\sum_{n=m_0}^{\infty}\left|c_1a_n^{(1)}+c_2a_n^{(2)}+...+c_ka_n^{(k)}\right|\geq$$

$$\sum_{n=m_0}^{\infty}\left||c_ka_n^{(k)}|-|c_1a_n^{(1)}+c_2a_n^{(2)}+...+c_{k-1}a_n^{(k-1)}|\right|\geq\sum_{n=m_0}^{\infty}\frac{|c_ka_n^{(k)}|}{2}=\infty.$$

Note that in particular we have proved that the set $\{(\sum_{n=0}^{\infty}a_n)^k : k\geq 1\}$ is linearly independent. □

2. Appendix

Since Prinsheim's paper [P] is not readily accessible, we reproduce here the proof of Theorem 1 for the sake of completeness of this note.

Proof. First we will show that $c_n\to 0$. We have

$$c_{2m}=\sum_{k=0}^{m}a_kb_{2m-k}+\sum_{k=0}^{m}a_{2m-k}b_k-a_mb_m,$$

$$c_{2m+1}=\sum_{k=0}^{m}a_kb_{2m+1-k}+\sum_{k=0}^{m}a_{2m+1-k}b_k.$$

Hence

$$|c_n|\leq\sum_{k=0}^{m}|a_kb_{n-k}|+\sum_{k=0}^{m}|a_{n-k}b_k|+|a_mb_m|$$

where $m=\max\{k\in\mathbb{Z}:k\leq n/2\}$. Since $\mathcal{F}$ is absolutely equi-convergent, we find $N\in\mathbb{N}$ with

$$\sum_{k=N}^{m}|a_kb_{n-k}|<\frac{\varepsilon}{5}\quad\text{and}\quad\sum_{k=N}^{m}|a_{n-k}b_k|<\frac{\varepsilon}{5}.$$

Let n be such that

$$|b_n|,|b_{n-1}|,...,|b_{n-N+1}|<\frac{\varepsilon}{\max_{i\in\mathbb{N}}|a_i|4N},$$

$$|a_n|,|a_{n-1}|,...,|a_{n-N+1}|<\frac{\varepsilon}{\max_{i\in\mathbb{N}}|b_i|4N}.$$

and $|a_mb_m|<\varepsilon/5$. Then

$$\sum_{k=0}^{N-1}|a_kb_{n-k}|<\frac{\varepsilon}{5}\quad\text{and}\quad\sum_{k=0}^{N-1}|a_{n-k}b_k|<\frac{\varepsilon}{5}$$

Hence

$$|c_n| \le \sum_{k=0}^{m} |a_k b_{n-k}| + \sum_{k=0}^{m} |a_{n-k} b_k| + |a_m b_m| =$$

$$\sum_{k=0}^{N-1} |a_k b_{n-k}| + \sum_{k=N}^{m} |a_k b_{n-k}| + \sum_{k=0}^{N-1} |a_{n-k} b_k| + \sum_{k=N}^{m} |a_{n-k} b_k| + |a_m b_m| < \varepsilon.$$

Therefore $c_n \to 0$.

Recall that if the series $\sum_{n=0}^{\infty} c_n$ is convergent to some C, then $C = AB$, where $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$. Since $c_n \to 0$, it is enough to show that

$$D_{4m} = C_{4m} - A_{2m}B_{2m} = \sum_{k=0}^{4m} c_k - \sum_{k=0}^{2m} a_k \sum_{l=0}^{2m} b_l$$

tends to zero, if $m \to \infty$.

We have

$$D_{4m} = \sum_{k=0}^{4m} \sum_{l=0}^{k} a_l b_{k-l} - \sum_{k=0}^{2m} a_k \sum_{l=0}^{2m} b_l =$$

$$a_0 \sum_{l=0}^{4m} b_l + a_1 \sum_{l=0}^{4m-1} b_l + ... + a_{4m-1} \sum_{l=0}^{1} b_l + a_{4m} b_0 - (a_0 + a_1 + ... + a_{2m}) \sum_{l=0}^{2m} b_l =$$

$$a_0 \sum_{l=2m+1}^{4m} b_l + a_1 \sum_{l=2m+1}^{4m-1} b_l + ... + a_{2m-1} b_{2m+1} + a_{2m+1} \sum_{l=0}^{2m-1} b_l + a_{2m+2} \sum_{l=0}^{2m-2} b_l + ... + a_{4m} b_0 =$$

$$(a_0 + a_1) \sum_{l=2m+1}^{4m-1} b_l + (a_2 + a_3) \sum_{l=2m+1}^{4m-3} b_l + ... + (a_{2m-2} + a_{2m-1}) b_{2m+1} +$$

$$+ (a_{2m} + a_{2m+1}) \sum_{l=0}^{2m-1} b_l + (a_{2m+2} + a_{2m+3}) \sum_{l=0}^{2m-3} b_l + ... + (a_{4m-2} + a_{4m-1})(b_0 + b_1) +$$

$$a_0 b_{4m} + a_2 b_{4m-2} + ... + a_{2m-2} b_{2m+2} + a_{2m} b_{2m} + a_{2m+2} b_{2m-2} + ... + a_{4m} b_0$$

$$-a_{2m}(b_0 + ... + b_{2m}).$$

Hence

$$|D_{4m}| \le \sum_{k=0}^{m-1} |a_{2k} + a_{2k+1}| \left| \sum_{l=2m+1}^{4m-(2k+1)} b_l \right| + \sum_{k=0}^{m-1} |a_{2m+2k} + a_{2m+2k+1}| \left| \sum_{l=0}^{2m-(2k+1)} b_l \right| +$$

$$+ \sum_{k=0}^{2m} |a_{2k} b_{4m-2k}| + |a_m| \left| \sum_{k=0}^{2m} b_k \right|.$$

Let $G = \max\{\sum_{k=0}^{\infty} |a_{2k} + a_{2k+1}|, |\sum_{k=0}^{\infty} b_k|\} < \infty$. Let $\varepsilon > 0$ Let $M \in \mathbb{N}$ be such that the following inequalities hold for any $m \geq M$

$$\left| \sum_{l=2m+1}^{4m-(2k+1)} b_l \right| < \frac{\varepsilon}{4G},$$

$$\sum_{k=0}^{m-1} |a_{2m+2k} + a_{2m+2k+1}| < \frac{\varepsilon}{4G},$$

$$|a_{2m}| < \frac{\varepsilon}{4G},$$

$$\sum_{k=0}^{2m} |a_{2k} b_{4m-2k}| < \frac{\varepsilon}{4}.$$

To find such m in the last inequality one should repeat the same reasoning as in the first part of the proof where it has been shown that $c_n \to 0$. Now, if $m \geq M$, then $|D_{4m}| < \varepsilon$ and the result follows. □

Acknowledgement. The authors would like to thank Juan Seoane-Sepúlveda for the careful examination of the paper and for valuable suggestions.

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