CONTINUOUS DEPENDENCE ON PARAMETERS FOR SECOND ORDER DISCRETE BVP'S

MAREK GALEWSKI AND SZYMON GŁĄB

Abstract. Using Fan's Min–Max Theorem we investigate the existence of solutions and thier dependence on parameters for some second order discrete boundary value problem. The approach is based on variational methods and solutions are obtained as saddle points to the relevant Euler action functional.

1. Introduction

Boundary value problems governed by discrete equations have received some attention lately by both variational and topological approach. The variational techniques applied for discrete problems include, among others, the mountain pass methodology, the linking theorem, the Morse theory, the three critical point, compare with [2], [3], [8], [11], [12], [13]. Moreover, the fixed point approach is in fact much more prolific in the case of discrete problem and covers the techniques already applied for continuous problems, see for example [1], [5], with both list of references far from being exhaustive.

While in the literature mainly the problem of the existence of solutions and their multiplicity is considered, we are going to go a bit further and investigate also the dependence on a functional parameter u for the following discrete boundary value problem which is a saddle -point type system. Let $D > 0$ be fixed. The problem which we consider reads

$$\left\{\begin{array}{l} \Delta^2 x(k-1) = F_x(k, x(k), y(k), u(k)), \\ \Delta^2 y(k-1) = -F_y(k, x(k), y(k), u(k)), \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{array}\right. \tag{1}$$

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where $F : [1,T] \times \mathbb{R} \times \mathbb{R} \times [-D,D] \to \mathbb{R}$ is a continuous function differentiable with respect to the second and the third variable,

$$u \in L_D = \{u \in C([1,T],\mathbb{R}) : ||u||_C \leq D\},$$

where $||u||_C$ denotes the classical maximum norm $||u||_C = \max_{k\in[1,T]} |u(k)|$ and $[a,b]$ for $a < b$, $a, b \in \mathbb{Z}$ denotes a discrete interval $\{a, a+1, ..., b\}$. By a solution to (1) we mean a function $x : [0, T+1] \to \mathbb{R}$ which satisfies the given equation and the associated boundary conditions.

Such type of a difference equation as (1) may arise from evaluating the Dirichlet boundary value problem

$$\frac{d^2}{dt^2}x = G_x\left(t,x,y,u\right), \; \frac{d^2}{dt^2}y = -G_y\left(t,x,y,u\right),$$

$$0 < t < 1, \, x\left(0\right) = x(1) = 0, y\left(0\right) = y(1) = 0$$

where $G : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and subject to some growth conditions. Such a continuous problem subject to a functional parameter has been considered in [6].

The question whether the system depends continuously on a parameter is vital in context of the applications, where the measurements are known with some accuracy. This question is even more important when the solution to the problem under consideration is not unique as is the case of the present note. In the boundary value problems for differential equations there are some results towards the dependence of a solution on a functional parameter, see [7], [6] with references therein. This is not the case with discrete equations where we have only some results which use the critical point theory, see [4]. The approach of this note is different from this of [4] since it does not relay on coercivity arguments but on a min-max inequality due to Ky Fan, see [10]. In our approach we use some ideas developed in [6] suitable modified due to the finite dimensionality of the space under consideration. Namely, we need less restrictive assumptions as far as the nonlinear terms are concerned. We believe that such assumptions can also be employed in the continuous case, thus advancing somehow the results from [6] by making use of the convexity and concavity notions.

The following results will be used in the sequel, see [10].

Theorem 1 (Fan's Min–Max Theorem). *Let X and Y be Hausdorff topological vector spaces, $A \subset X$ and $B \subset Y$ be convex sets, and $J : A \times B \to \mathbb{R}$ be a functional which satisfies the following conditions:*

(i) *for each fixed $y \in B$, the functional $x \to J(x,y) \in \mathbb{R}$ is convex and lower semi-continuous on A;*

(ii) *for each fixed* $x \in A$, *the functional* $y \to J(x,y) \in \mathbb{R}$ *is concave and upper semi-continuous on* B;

(iii) *for some* $x_0 \in A$ *and some* $\delta_0 < \inf_{x\in A}\sup_{y\in B} J(x,y)$, *the set* $\{y \in B : J(x_0, y)\}$ *is compact.*

Then

$$\sup_y \inf_x J(x,y) = \inf_x \sup_y J(x,y).$$

Definition 2. *Let* (X,τ) *be a Hausdorff topological space and let* $(A_n)_{n=1}^{\infty}$ *be a sequence of nonempty subsets of* X. *The set of accumulation points of sequences* $(a_n)_{n=1}^{\infty}$ *with* $a_n \in A_n$ *for* $n = 1, 2, 3, \ldots$ *is called the upper limit of* $(A_n)_{n=1}^{\infty}$ *and denoted by* $\limsup A_n$.

2. Variational framework for problem (1) and the assumptions

Solutions to (1) will be investigated in the space

$$H = \{x : [0, T+1] \to \mathbb{R} : x(0) = x(T+1) = 0\}$$

considered with the norm

$$||x|| = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^2\right)^{1/2}.$$

Then $(H, ||\cdot||)$ becomes a T dimensional Hilbert space. Let c be the smallest positive constant such that

$$\sum_{k=1}^{T} |x(k)|^2 \le c \cdot \sum_{k=1}^{T+1} |\Delta x(k-1)|^2$$

for any $x \in H$; see [9, Lemma 1].

Since the approach of present note is a variational one, we investigate the action functional $J_u : H \times H \to \mathbb{R}$, corresponding to problem (1). For a fixed parameter $u \in L_D$, J_u is of the form

$$J_u(x,y) = \sum_{k=1}^{T+1} \left(\frac{|\Delta x(k-1)|^2}{2} - \frac{|\Delta y(k-1)|^2}{2}\right) +$$
$$\sum_{k=1}^{T} F(k, x(k), y(k), u(k)).$$

We assume that F has the following properties:

H1 $F : [1,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is differentiable with respect to the second and the third variable; $F_x, F_y : [1,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

H2 For any fixed $y \in \mathbb{R}$ there are constants $\beta_1(y), \alpha_1(y) > 0$ such that

$$|F_x(k, 0, y, u)| \leq \beta_1(y), |F(k, 0, y, u)| \leq \alpha_1(y)$$

for all $u \in \mathbb{R}$, $|u| \leq D$ and all $k \in [1, T]$.

H3 There are constants $\beta_2, \alpha_2 > 0$ such that

$$|F_y(k, 0, 0, u)| \leq \beta_2, |F(k, 0, 0, u)| \leq \alpha_2$$

for all $u \in \mathbb{R}$, $|u| \leq D$ and all $k \in [1, T]$.

H4 Function $x \to F(k, x, y, u)$ is convex on $\mathbb{R}$ for all $y \in \mathbb{R}$, $u \in [-D, D]$, $k \in [1, T]$.

H5 Function $y \to F(k, x, y, u)$ is concave on $\mathbb{R}$ for all $x \in \mathbb{R}$, $u \in [-D, D]$, $k \in [1, T]$.

Example 1. Let

$$F(k, x, y, u) = f_1(k)x^4 - f_2(k)y^4 + f_3(k)yxu + f_4(k)x^2 - f_5(k)y^2$$

where $f_i(k) > 0$ for $k \in [1, T]$. Note that **H2** and **H3** are fulfilled for $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1(y) = yD \max_{k \in [1,T]} f_3(k)$. Since each f_i is positive, then $F''_x > 0$ and $F''_y < 0$. Hence **H4** and **H5** hold.

Example 2. Let

$$F(k, x, y, u) = f_1(k)x^2 + f_2(k)\sin x - f_3(k)y^4 - f_4(k)(y \sin y + 2\cos y) + f_5(k)xyu$$

where $f_i(k) > 0$, $2f_1(k) > f_2(k)$ and $12f_3(k) > f_4(k)$ for $k \in [1, T]$. Note that **H2** and **H3** are fulfilled for $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1(y) = \max_{k \in [1,T]} f_2(k) + yD \max_{k \in [1,T]} f_5(k)$. As in the previous example we easily obtain that $F''_x > 0$ and $F''_y < 0$, which implies **H4** and **H5**.

We list some properties of functional J_u in the following lemma.

Lemma 3. *Assume **H1**. Let $u \in L_D$ be fixed. Functional J_u is continuous and continuously differentiable in the sense of Gâteaux on $H \times H$. Moreover, $(x, y) \in H \times H$ is a critical point to J_u if and only if it satisfies* (1).

Proof. Continuity of J_u follows by continuity of the norm and of a functional F. Let us show that J_u has continuous partial Gâteaux derivatives with respect to x and y.

Let us fix $y \in H$. Let $x \in H$ be arbitrary. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by the formula $\varphi(\varepsilon) = J_u(x + \varepsilon h, y)$, where $h \in H$ is a fixed direction. Then

$$\varphi(\varepsilon) = \sum_{k=1}^{T+1} \left(\frac{|\Delta x(k-1)+\varepsilon \Delta h(k-1)|^2}{2} - \frac{|\Delta y(k-1)|^2}{2} \right) +$$

$$\sum_{k=1}^{T} F(k, x(k) + \varepsilon h(k), y(k), u(k)).$$

Since φ is continuously differentiable we get what follows

$$\varphi'(0) = \sum_{k=1}^{T+1} \Delta x(k-1)\Delta h(k-1) + \sum_{k=1}^{T} F_x(k, x(k), y(k), u(k))h(k) =$$

$$\sum_{k=1}^{T} \Delta x(k-1)\Delta h(k-1) + \Delta x(T)\Delta h(T) + \sum_{k=1}^{T} F_x(k, x(k), y(k), u(k))h(k).$$

Note that

$$\Delta x(T)h(T) + \Delta x(T)\Delta h(T) =$$

$$(x\,(T+1) - x\,(T))\,h\,(T) + (x\,(T+1) - x\,(T))\,(h\,(T+1) - h\,(T)) = 0.$$

Now, summing by parts, we see that

$$\varphi'(0) = -\sum_{k=1}^{T} \Delta^2 x(k-1)h(k) + \Delta x(T)h(T) + \Delta x(T)\Delta h(T)+$$

$$\sum_{k=1}^{T} F_x(k, x(k), y(k), u(k))h(k) =$$

$$\sum_{k=1}^{T} \left(-\Delta^2 x(k-1) + F_x(k, x(k), y(k), u(k))\right) h(k).$$

Thus J_u has a continuous partial Gâteaux derivative with respect to x.

Let us now fix $x \in H$. Let $y \in H$ be arbitrary. We put $\psi(\varepsilon) = J_u(x, y + \varepsilon h)$ for a fixed direction $h \in H$. Then reasoning as in the above we show that

$$\psi'(0) = \sum_{k=1}^{T} \left(\Delta^2 y(k-1) + F_y(k, x(k), y(k), u(k))\right) h(k).$$

Letting $\varphi'(0) = 0$ and $\psi'(0) = 0$ we see that (x, y) is a critical point to J_u if and only it satisfies system (1). $\square$

With the aid of Theorem 1 we are able to find saddle points for functional J_u. Since J_u is differentiable in the sense of Gâteaux, it is apparent that such points are the critical points to J_u. Since in turn critical points to J_u constitute solutions to (1), we arrive at existence result once we get the existence of saddle points. Moreover, since the spaces in which we work are finite dimensional one, there is no need to distinguish between the weak and the strong solutions. In fact in this case the weak solution appears to be a strong one.

3. Existence of saddle point solutions

Theorem 4 (Existence of saddle points)**.** *Assume that conditions **H1**-**H5** hold. Let $u \in L_D$ be fixed. Then it follows that*
(A) There is a saddle point (x_u, y_u) for the functional J;
(B) There are balls $B_1 = \{x : ||x|| \leq r_1\}$ and $B_2 = \{y : ||y|| \leq r_2\}$ such that $(x_u, y_u) \in B_1 \times B_2$;
(C) The set of all saddle points of J_u is compact.

Proof. By **H4** since F is convex with respect to x it follows that

$$F(k,x,y,u) - F(k,0,y,u) \geq F_x(k,0,y,u)\, x \geq -\left|F_x(k,0,y,u)\right| |x|$$

Since H is finite-dimensional there exists a constant c_1 such that

$$\sum_{k=1}^{T} |x(k)| \leq c_1 ||x||.$$

For a fixed $y \in H$ using the above observations and **H2** we see that

$$J_u(x,y) \geq \sum_{k=1}^{T+1} \left(\frac{|\Delta x(k-1)|^2}{2} - \frac{|\Delta y(k-1)|^2}{2} - \left|F_x(k,0,y(k),u(k))\right| |x(k)| \right)$$

$$- \sum_{k=1}^{T+1} \left|F(k,0,y(k),u(k))\right| \geq$$

$$\frac{1}{2}||x||^2 - c_1 \beta_1(y) ||x|| + \gamma_1(y),$$

where $\gamma_1(y) = -\sum_{k=1}^{T+1} \left(\frac{|\Delta y(k-1)|^2}{2} + \alpha_1(y(k)) \right)$. Both constants do not depend on u. Thus $x \to J_u(x,y)$ is coercive on H. By **H1** and **H4** it is continuous and convex for each u. Hence it makes sense to define

$$J_u^-(y) = \min_x J_u(x,y).$$

By **H5** the functional J_u^- is concave. By **H3** and concavity of F with respect to y we obtain that
(2)

$$J_u^-(y) \leq J_u(0,y) \leq \sum_{k=1}^{T+1} \left(-\frac{|\Delta y(k-1)|^2}{2} + \left|F_y(k,0,0,u(k))\right| |y(k)| \right)$$

$$+ \sum_{k=1}^{T+1} \left|F(k,0,0,u(k))\right| \leq$$

$$-\frac{1}{2}||y||^2 + c_1 \beta_2 ||y|| + (T+1)\,\alpha_2.$$

Hence J_u^- is anti-coercive and it attains its supremum at some point y_u. By **H2** we have

$$J_u^-(y_u) \geq J_u^-(0) = \min_x J_u(x,0) \geq$$

$$\min_x \left(\frac{1}{2}||x||^2 - c_1 \beta_1(0) ||x|| + \gamma_1(0) \right) = \nu,$$

where constant ν does not depend neither on u nor on y or x. Since J_u^- is anti-coercive, there is $r_2 > 0$ such that $J_u^-(y) < \gamma_1$ for every $||y|| > r_2$. Since J_u^- is continuous the set $\{y : J_u^-(y) \geq \nu\}$ is compact and is contained in some ball B_2. Hence each y_u is in B_2.

Analogously one can show that there is x_u with

$$J_u^+(x_u) = \min_x J_u^+ = \min_x \max_y J_u(x, y).$$

Furthermore, there is a ball B_1 with $x_u \in B_1$ for each such x_u.

We have already showed that for each x there exists $\max_y J_u(x, y)$. Hence for some δ_0 we have

$$\delta_0 < \min_x J_u(x, 0) \leq \min_x \max_y J_u(x, y).$$

By (2) we obtain

$$\{y : J_u(0, y) \geq \delta_0\} \subset \{y : -\frac{1}{2}||y||^2 + c_1\beta_2||y|| + (T+1)\,\alpha_2 \geq \delta_0\}.$$

Since the set of right hand of inclusion is compact, so is the set $\{y : J_u(0, y) \geq \delta_0\}$. Thus, the assumptions **H4** and **H5** and Fan's minimax Theorem 1, give the existence of a saddle point of J_u. Moreover the set of all saddle points of J_u is compact. □

Now by Theorem 4 and by Lemma 3 we reach the following result

Theorem 5 (Existence of saddle point solutions)**.** *Assume that conditions **H1**-**H5** hold. Let $u \in L_D$ be fixed. Then, there exists is at least one saddle point $(x_u, y_u) \in H \times H$ for the functional J_u which solves (1).*

In order to obtain existence results we do not need to impose conditions **H2**-**H5** uniformly in u. This is not however the case if one is interested in the dependence on parameters, when assumptions must be placed uniformly with respect to u. Indeed, let us consider a following problem

$$(3) \qquad \begin{cases} \Delta^2 x(k-1) = F_x(k, x(k), y(k)), \\ \Delta^2 y(k-1) = -F_y(k, x(k), y(k)), \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases}$$

where $F : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is differentiable with respect to the second and the third variable. The action functional $J : H \times H \to R$, corresponding to problem (3) is

$$J(x,y) = \sum_{k=1}^{T+1} \left(\frac{|\Delta x(k-1)|^2}{2} - \frac{|\Delta y(k-1)|^2}{2} \right) + \sum_{k=1}^{T} F(k, x(k), y(k)).$$

We assume that

H6 $F : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is differentiable with respect to the second and the third variable; $F_x, F_y : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

H7 For any fixed $y \in \mathbb{R}$ there are constants $\beta_3(y), \alpha_3(y) > 0$ such that

$$|F_x(k,0,y)| \leq \beta_3(y), |F(k,0,y)| \leq \alpha_3(y)$$

for all $k \in [1,T]$.

H8 There are constants $\beta_4, \alpha_4 > 0$ such that

$$|F_y(k,0,0)| \leq \beta_4, |F(k,0,0)| \leq \alpha_4$$

for all $k \in [1,T]$.

H9 Function $x \to F(k,x,y)$ is convex on $\mathbb{R}$ for all $y \in \mathbb{R}$, $k \in [1,T]$.

H10 Function $y \to F(k,x,y)$ is concave on $\mathbb{R}$ for all $x \in \mathbb{R}$, $k \in [1,T]$.

Then we have

Corollary 6. *Assume that conditions **H6-H10** hold. Then, there exists is at least one saddle point* $(x,y) \in H \times H$ *for the functional* J *which further solves* (1).

One remark is in order as concerns the growth assumptions and the proof of Theorem 4. We can in fact start the proof of Theorem 4 with investigating the dependence of F on y instead of x. This would require the obvious change in the assumptions concerning the derivatives.

4. Continuous dependence on parameters

In this section we are interested of the behavior of the sequence of saddle points which correspond to a sequence of parameters. Dependence on parameters in investigated through the convergence of the sequence of action functionals corresponding the sequence of parameters - this approach has already been applied with some success for the continuous and also the discrete problems, see [4], [7]. Let $(u_n)_{n=1}^{\infty} \subset L_D$ be a sequence of parameters. We put $J_n = J_{u_n}$ and we define

$$V_n = \{(\overline{x}, \overline{y}) : J_n(\overline{x}, \overline{y}) = \max_y \min_x J_n(x, y)\} \subset B_1 \times B_2$$

as the set of all saddle points to J_n. Due to Theorem 4 we see that $V_n \neq \emptyset$ for all $n = 1, 2, \dots$.

Theorem 7. *Assume that conditions **H1-H5** hold. Let $(u_n)_{n=1}^{\infty} \subset L_D$ be a convergent sequence of parameters and $u_n \to u_0 \in L_D$ as $n \to \infty$. Then $\emptyset \neq \limsup_{n\to\infty} V_n \subset V_0$.*

Proof. At first we observe by continuity of F that J_n tends to J_0 uniformly on $B_1 \times B_2$, where B_1, B_2 are defined in Theorem 4. We will prove that $\emptyset \neq \limsup V_n \subset V_0$. Let $a_n = \max_y \min_x J_n(x, y)$ and let $\varepsilon > 0$. Since J_n tends uniformly to J_0, then $J_n(x, y) \leq J_0(x, y) + \varepsilon$ for each $(x, y) \in B_1 \times B_2$ and every $n \geq n_0$ for some n_0. Then

$$\min_x J_n(x, y) \leq \min_x J_0(x, y) + \varepsilon,$$

$$\max_y \min_x J_n(x, y) \leq \max_y \min_x J_0(x, y) + \varepsilon.$$

Hence $a_k - a_0 \leq \varepsilon$. Similarly one can show that $a_k - a_0 \geq -\varepsilon$. Therefore $a_k \to a_0$.

Let $(x_n, y_n) \in V_n$ for $n = 1, 2, \dots$. Since

$$\{(x_n, y_n)\}_{n=1}^{\infty} \subset B_1 \times B_2$$

we may assume that $(x_n, y_n) \to (x_0, y_0)$. In particular $\limsup V_n \neq \emptyset$. Suppose now that $(x_0, y_0) \notin V_0$. Let $(\overline{x}, \overline{y}) \in V_0$. Then $J_0(\overline{x}, \overline{y}) \neq J_0(x_0, y_0)$. Consider the case

$$J_0(\overline{x}, \overline{y}) - J_0(x_0, y_0) = \eta < 0.$$

Then

$$a_n - a_0 = J_n(x_n, y_n) - J_0(x_0, y_0) =$$
$$\min_x J_n(x, y_n) - J_0(x_0, y_0) \leq$$
$$\leq J_n(\overline{x}, y_n) - J_0(x_0, y_0) =$$
$$J_n(\overline{x}, y_n) - J_0(\overline{x}, y_n) + J_0(\overline{x}, y_n) - J_0(\overline{x}, \overline{y}) + J_0(\overline{x}, \overline{y}) - J_0(x_0, y_0).$$

Since

$$J_0(\overline{x}, \overline{y}) = \max_y J_0(\overline{x}, y) \geq J_0(\overline{x}, y_n),$$

then

$$\limsup_n J_0(\overline{x}, y_n) - J_0(\overline{x}, \overline{y}) \leq 0.$$

By the continuity of F we obtain that $J_n(\overline{x}, y_n) \to J_0(\overline{x}, y_n)$. Therefore

$$\limsup_{n\to\infty}(a_n - a_0) < \eta.$$

A contradiction. Similarly, a contradiction can be obtained when $\eta > 0$. □

Theorem 7 combined with Theorem 5 yield the following main result of our note

Theorem 8. *Assume **H1-H5**. For any fixed $u \in L_D$ there exists at least one solution $y \in V_u$ to problem* (1)*. Let $\{u_n\} \subset L_D$ be a convergent sequence of parameters, where $\lim_{n\to\infty} u_n = u_0 \in L_D$. For any sequence $\{(x_n, y_n)\}$ of solutions $(x_n, y_n) \in V_n$ to the problem* (1) *corresponding to u_n, there exist a subsequence $\{(x_{n_i}, y_{n_i})\} \subset H \times H$ and an element $(x_0, y_0) \subset H \times H$ such that $\lim_{i\to\infty} x_{n_i} = x_0$, $\lim_{i\to\infty} y_{n_i} = y_0$ and $J_0(x_0, y_0) = \max_y \min_x J_0(x, y)$. Moreover $x_0, y_0 \in V_0$, i.e. the pair (x_0, y_0) satisfies* (1) *with $u = u_0$, namely*

$$\begin{cases} \Delta^2 x_0(k-1) = F_x(k, x_0(k), y_0(k), u_0(k)), \\ \Delta^2 y_0(k-1) = -F_y(k, x_0(k), y_0(k), u_0(k)), \\ x_0(0) = x_0(T+1) = y_0(0) = y_0(T+1) = 0. \end{cases}$$

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Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: marek.galewski@p.lodz.pl

Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: szymon.glab@p.lodz.pl