RECOVERING A PURELY ATOMIC FINITE MEASURE FROM ITS RANGE

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Abstract. Let μ be a purely atomic finite measure. By the range of μ we understand the set $\operatorname{rng}(\mu) = \{\mu(E) : E \subset \mathbb{N}\}$. We are interested in the two following questions. Which set can be a range of some measure μ? Can the purely atomic measure μ be uniquely recovered from its range?

1. Introduction

Assume that μ is a purely atomic finite measure. We may assume that μ is defined on $\mathbb{N}$ and $\mu(\{n\}) \geq \mu(\{n+1\})$. Throughout the paper we assume that measures are always purely atomic, finite and they are defined on $\mathbb{N}$ such that their $n+1$-st atoms have measures not greater than their n-th atoms. We are interested in the following questions:

- For which subsets R of $\mathbb{R}$ there is a measure μ such that R is its range (i.e. $R = \operatorname{rng}(\mu) := \{\mu(E) : E \subset \mathbb{N}\}$)?
- For which subsets R of $\mathbb{R}$ there is exactly one measure μ with $R = \operatorname{rng}(\mu)$?

To simplify the notation let $x_n = \mu(\{n\})$ be a measure of the n-th largest atom of μ. Note that

$$\operatorname{rng}(\mu) = \{\mu(E) : E \subset \mathbb{N}\} = \{\sum_{n\in E} \mu(\{n\}) : E \subset \mathbb{N}\} = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : \varepsilon_n = \{0,1\}^{\mathbb{N}}\}.$$

The latter set is also denoted by $A(x_n)$ and it is called the achievement set of (x_n) (see [16]). Let us present here two simple examples.

Example 1.1. Consider the procedure of rolling dice until the value on the dice is less than 5. For $E \subset \mathbb{N}$ let $\mu_1(E)$ be the probability that the procedure stops for some n from E. Then $\mu_1(\{n\}) = \frac{2}{3^n}$. It is easy to see that for $x_n = \mu_1(\{n\})$ the set $A(x_n)$, or $\operatorname{rng}(\mu_1)$, is equal to the classical Cantor ternary set C.

Example 1.2. Consider the procedure of tossing a fair coin until the head appears. For $E \subset \mathbb{N}$ let $\mu_2(E)$ be the probability that the procedure stops for some n from E. Then $\mu_2(\{n\}) = \frac{1}{2^n}$ and $\operatorname{rng}(\mu_2) = [0,1]$.

Achievement sets of sequences, defined for all summable sequences (x_n), have been considered by many authors; some results have been rediscovered several times. Let us list basic properties of $A(x_n)$ (some of them were observed by Kakeya in [17] in 1914):

(i) $A(x_n)$ is a compact perfect or finite set,
(ii) If $|x_n| > \sum_{i>n} |x_i|$ for all sufficiently large n's, then $A(x_n)$ is homeomorphic to the ternary Cantor set C,
(iii) If $|x_n| \leq \sum_{i>n} |x_i|$ for all sufficiently large n's, then $A(x_n)$ is a finite union of closed intervals. Moreover, if $|x_n| \geq |x_{n+1}|$ for all but finitely many n's and $A(x_n)$ is a finite union of closed intervals, then $|x_n| \leq \sum_{i>n} |x_i|$ for all but finitely many n's.

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In particular, for decreasing sequence (x_n) the inequality $x_n \leq \sum_{i>n} x_i$ for all n is equivalent to $A(x_n)$ being an interval.

One can see that $A(x_n)$ is finite if and only if $x_n = 0$ for all but finite number of n's, i.e. $(x_n) \in c_{00}$. Kakeya conjectured that if $(x_n) \in \ell_1 \setminus c_{00}$, then $A(x_n)$ is always a Cantor set C or it is a finite union of intervals.

On the other hand, in 1970 Renyi in [25] repeated the results of Kakeya in terms of purely atomic measures and he asked if the Cantor sets and finite unions of closed intervals are the only possible sets being the ranges of finite measures. Geometric properties of achievement sets of sequences and ranges of purely atomic finite measures are the same. This follows from the simple observation, that the set of sums of subseries for the series $\sum_{n=1}^{\infty} x_n$ is isometric to the analogous set for the series of their absolute values $\sum_{n=1}^{\infty} |x_n|$. Therefore a positive answer for the Renyi's question is equivalent to the Kakeya's conjecture.

In 1980 Weinstein and Shapiro in [26] gave an example which showed that the Kakeya conjecture is false. It follows from the references of their paper that they did not know the Renyi's problem. On the other hand, Ferens in [12] has given the example similar to that of Weinstein and Shapiro, solving the problem of Renyi. In this case, the author did not know the conjecture of Kakeya.

In [13] Guthrie and Nymann gave a very simple example of a sequence whose achievement set is not a finite union of closed intervals but it has a nonempty interior. They used the sequence $(t_n) = (\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \ldots)$. Moreover, they formulated the following:

Theorem 1.3. *For any $(x_n) \in \ell_1 \setminus c_{00}$, the set $A(x_n)$ is one of the following types:*

(i) *a finite union of closed intervals,*
(ii) *a Cantor set C,*
(iii) *homeomorphic to the set $\mathbf{T} = A(t_n) = A(\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \frac{3}{64}, \ldots)$.*

Although their proof had a gap, the theorem is true and the correct proof was given by Nymann and Saenz in [24]. Guthrie, Nymann and Saenz have observed that the set $\mathbf{T}$ is homeomorphic to the set N described by the formula

$$N = [0,1] \setminus \bigcup_{n \in \mathbb{N}} U_{2n},$$

where U_n denotes the union of 2^{n-1} open middle thirds which are removed from the interval $[0,1]$ at the n-th step in the construction of the classic Cantor ternary set C. Such sets are called Cantorvals in the literature (to emphasize the similarity to the interval and to the Cantor set simultaneously). It is known that a Cantorval is just a nonempty compact set in $\mathbb{R}$, that it is the closure of its interior and both endpoints of any nontrivial component are accumulation points of its trivial components. Other topological characterizations of Cantorvals can be found in [6] and [20].

All known examples of sequences whose achievement sets are Cantorvals belong to the class of multigeometric sequences or are linear combinations of such sequences, see [2],[3]. This class was deeply investigated in [16], [7], [4] and [1]. In particular, the achievement sets of multigeometric series and similar sets obtained in more general case are the attractors of affine iterated function systems, see [1]. More information on achievement sets can be found in the surveys [6], [21] and [22].

It is almost obvious that any achievement set E of a summable sequence contains zero and is symmetric in the sense that there exists a number t such that if $t - x \in E$ then $t + x \in E$ too. It is a natural question if every compact, perfect set with these properties is an achievement set for some sequence. This question was posted by W. Kubiś in Łódź in 2015. In particular, in [4] the authors ask if the Cantorval N is an achievement set of any sequence.

The negative answer to the last question was recently given in [5]. Independently the authors of [8] have showed that the Cantorval $\widetilde{T}$ for which the gaps are the intervals of the Guthrie-Nymann-Cantorval T and vice-versa, is not an achievement set for any sequence.

On the other hand, T. Banakh in Lviv in 2016 asked if Cantor achievement sets are uniquely defined, i. e. they are achievement sets of only one sequence.

The paper is organized as follows. In Section 2 we present gap lemmas and the center of distances notion which are useful tools in succeeding sections. In Section 3 we show that if the range of a mesure μ is an interval, in other words μ is interval filling, then there is a measure ν such that the sets $\{\mu(n) : n \in \mathbb{N}\}$ and $\{\nu(n) : n \in \mathbb{N}\}$ are pairwise disjoint. We also give an example of a symmetric set which is a finite union of intervals but is not the range of any measure. In Section 4 we give sufficient conditions on a Cantor set which is the range of some measure to be the range of no other measure. We present also sufficient conditions for a set R to be a Cantor set achieved by a unique measure μ. In Section 5 there is given a connection between achievement sets of multigeometric sequences and IFS fractals. We show that the Guthrie-Nymann Cantorval is uniquely achieved. In Section 6 we show that some Ferens fractals which are symmetric Cantors or Cantorvals are not ranges of any measure. In Section 7 we briefly discus the Guthrie-Nymann-Jones Cantorvals $A(r)$ of one parameter $r = 1, 2, \dots$ which generalize the Guthrie-Nymann Cantorval. For some r, $A(r)$ is not a range of any measure; for some r, $A(r)$ can be achieved in continuum many ways by measure range; $A(1)$ is a Guthrie-Nymann Cantorval which is uniquely achieved.

2. The Gaps Lemmas and Center of Distances

In the whole paper let us assume that (x_n) is a **nonincreasing summable sequence of positive real numbers** – the measures $\mu(\{n\})$ of μ-atoms. Denote (as in [13], [24], [6]):

$$R = A(x_n) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}} \right\};\ F_k = \left\{ \sum_{n=1}^{k} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{k} \right\}.$$

So F_k is a finite approximation of the range R. Let $r_k := \sum_{n=k+1}^{\infty} x_n$. By a gap in the range R we understand any interval (a, b) such that $a \in R$, $b \in R$ and $(a, b) \cap R = \emptyset$. The following two lemmas can be found in [6]. The first is obvious.

Lemma 2.1. *(First Gap Lemma) If $x_k > r_k$ then (r_k, x_k) is a gap in the range R.*

The next observation is extracted from the proof of the crucial Lemma 4 of [24], where it was formulated as not a quite correct claim (however the Lemma and the main result of [24] are true). It can be found in [5] or in [6].

Lemma 2.2. *(Second Gap Lemma) Let (a, b) be a gap in the range R, and let p be defined by the formula $p := \max\{n : x_n \geq b - a\}$. Then:*

(i) $b \in F_p$,
(ii) *If $F_p = \{f_1^{(p)} < f_2^{(p)} < \ldots < f_{m(p)}^{(p)}\}$ and $b = f_j^{(p)}$, then $a = f_{j-1}^{(p)} + r_p$.*

The next Lemma has recently been proved in [5]. Since it will be used several times and for the reader's convenience, we present it with the proof.

Lemma 2.3. *(Third Gap Lemma) Suppose that (a, b) is a gap in the range R such that for any gap (a_1, b_1) with $b_1 < a$ we have $b - a > b_1 - a_1$ (in other words (a, b) is the longest gap from the left). Then for some $k \in \mathbb{N}$ we have $b = x_k$ and $a = r_k$.*

Proof. By the Second Gap Lemma b is a finite sum of terms of (x_n). Let $b = x_{n_1}+...+x_{n_m}$ with $x_{n_1} \geq ... \geq x_{n_m}$. Suppose that $m \geq 2$. Firstly observe that $x_{n_m} \geq b-a$ (indeed, if $x_{n_m} < b-a$ then $b - x_{n_m} \in (a,b)\cap R$ which is impossible). Of course $x_{n_m} < b$ and, since (a,b) is a gap, $x_{n_m} \leq a$. Any gap in the set $X := R\cap[0,x_{n_m}]$ is shorter than $b-a$. On the other hand, $b \in X+(b-x_{n_m})$ and $X+(b-x_{n_m}) \subset R$, so $(a,b)\cap(X+(b-x_{n_m})) = \emptyset$, and hence $(a-b+x_{n_m}, x_{n_m})$ is the gap in X which gives a contradiction. Thus $m = 1$ which means that $b = x_k$ for some $k \in \mathbb{N}$.

Since $a \in R$, $r_k \geq a$. Suppose that $r_k > a$. Let m be the smallest number satisfying $\sum_{n=k+1}^{m} x_n > a$. Hence $\sum_{n=k+1}^{m} x_n > b$, because (a,b) is a gap. Let now $X := R\cap[0,x_m]$. Then the set $X + \sum_{n=k+1}^{m-1} x_n$ is included in E and it has all gaps shorter than $b-a$, which gives a contradiction again. □

In [8] the authors have introduced the notion of the center of distances of a metric space X, defined as $S(X) = \{\alpha : \forall_{x\in X}\exists_{y\in X} d(x,y) = \alpha\}$. They especially consider the case when X is the achievement set of a sequence (x_n) and observe the following.

Lemma 2.4. *([8])* $\{x_n : n\in\mathbb{N}\} \subset S(A(x_n)) \subset A(x_n)$.

We present a short proof of this for the readers' convenience.

Proof. Let $n\in\mathbb{N}$. Fix $t\in A(x_n)$. Then there is $E\subset\mathbb{N}$ with $t = \sum_{m\in E} x_m$. If $n\in E$, then $t - x_n \in A(x_n)$. If $n\notin E$, then $t + x_n \in A(x_n)$. Therefore for any $t\in A(x_n)$ there is $s\in A(x_n)$ with $|t-s| = x_n$, which means that $x_n \in S(A(x_n))$. Since $0\in A(x_n)$, then for any $t\in S(A(x_n))$ by the definition of the center of distances there is $s\in A(x_n)$ with $|s-0| = t$. Since $A(x_n)$ consists of nonnegative real numbers, $s = t$ and consequently $S(A(x_n)) \subset A(x_n)$. □

The authors of [8] have given a variety of examples of sequences for which the equality $S(X) = \{x_n\}\cup\{0\}$ holds. Some of them are geometric sequences $(aq^n)_{n=1}^{\infty}$ with $q < \frac{1}{2}, a \geq 0$. The authors also proved that for the Guthrie-Nymann-Cantorval $\mathbf{T} = A(x_n)$, where $x_{2n-1} = \frac{3}{4^n}, x_{2n} = \frac{2}{4^n}$ we also get $S(X) = \{x_n\}\cup\{0\}$. For more details see [13].

The previous Lemma can be completed as follows.

Lemma 2.5. *If $x_k = x_{k+1} = \cdots = x_{k+2j-2}$ for some k and j, then jx_k belongs to $S(A(x_n))$.*

Proof. Let us observe that if we replace the terms $x_k, x_{k+1}, \ldots, x_{k+j-1}$ in the sequence (x_n) by one term jx_k, then in the modified sequence we can obtain any number mx_k where $m = 1,2,\ldots,k+2j-2$ by summing up some of the new terms $jx_k, x_{k+1}, \ldots, x_{k+2j-2}$. Consequently $A(x_n)$ equals the achievement set of the modified sequence. Therefore by Lemma 2.4 we obtain that $jx_k \in S(A(x_n))$. □

3. Interval filling sequences

We say that a purely atomic finite measure is interval filling if its range is an interval. A sequence of values of such a measure on its atoms is called an interval filling sequence. This notion was introduced in [9] and intensively studied f.e. in [10], [11] and in many other papers. By the Kakeya Theorem, a nonincreasing, summable sequence (x_n) of positive numbers is interval filling if and only if it is slowly convergent, i.e. if for every n the term x_n is no greater than the rest $r_n = \sum_{k=n+1}^{\infty} x_k$ (the authors of [9] have rediscovered this result). It is almost obvious that we cannot uniquely recover a sequence if its achievement set is an interval.

Example 3.1. $[0,1] = A(x_n) = A(y_n)$, where $x_n = \frac{1}{2^n}$, $y_{2n-1} = \frac{1}{3^n} = y_{2n}$. One can easily observe, that the both sequences (x_n) and (y_n) are slowly convergent and $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n = 1$.

It is worth noticing that it follows from the above example that an algebraic sum of two copies of the Cantor ternary set is an interval, what was proved by Steinhaus [14] (see [15]) about three years later than Kakeya has published his results. It is also interesting that the sets of values of (x_n) and (y_n) are not only different but even disjoint. We denote this by $(x_n) \cap (y_n) = \emptyset$. However, with additional assumptions, the authors of [18], [19], [23] have obtained some uniqueness results for interval filling finite measures. The following theorem is an improvement of Example 3.1.

Theorem 3.2. *For a given set R which is the range of some measure, the following conditions are equivalent:*

(i) *R is an interval,*
(ii) *there are two purely atomic measures μ and ν, both with range R, such that the μ-measures of atoms are all distinct from the ν-measures of atoms,*
(iii) *for any purely atomic measure μ with range R, there is another purely atomic measure ν whose values on atoms are distinct from those of μ,*
(iv) *for any purely atomic measure μ with range R, there is another purely atomic measure ν whose values on finite nonempty sets are distinct from those of μ.*

Proof. Evidently $(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.
$(ii) \Rightarrow (i)$. Let us assume that $R = A(x_n) = A(y_n)$, where $x_n = \mu(\{n\})$ and $y_n = \nu(\{n\})$, and suppose that R is not an interval. Then R has a gap. Let (a, b) be the longest gap in R (there may be finitely many longest gaps and we choose the one from the left side). By Lemma 2.3 there exist natural numbers k and l for which $x_k = y_l = b$. Thus $\mu(\{k\}) = \nu(\{l\})$ which yields a contradiction with (ii).
$(i) \Rightarrow (iv)$. Without loss of generality we may assume that the range R of μ equals $[0, 1]$. Let us construct inductively (y_n) such that

(a) y_1 is any number in $(\frac{1}{3}, \frac{1}{2}) \setminus \{\mu(F) : F \text{ is finite}\}$;
(b) $y_{n+1} > \frac{1}{3}(1 - \sum_{i=1}^{n} y_i)$;
(c) $y_{n+1} < \frac{1}{2}(1 - \sum_{i=1}^{n} y_i)$;
(d) $y_{n+1} \neq \mu(F) - \sum_{i=1}^{n} \varepsilon_i y_i$ for any finite $F \subset \mathbb{N}$ and any $(\varepsilon_i)_{i=1}^{n} \in \{0, 1\}^n$.

Since the set of forbidden numbers $\mu(F) - \sum_{i=1}^{n} \varepsilon_i y_i$ for y_{n+1} prescribed in (d) is countable, the choice of such sequence (y_n) is possible.

We will show inductively that $1 - \sum_{i=1}^{n+1} y_i < (\frac{2}{3})^{n+1}$. By (a) we have $y_1 > \frac{1}{3}$ which implies $1 - y_1 < \frac{2}{3}$. Using (b) and the inductive assumption we obtain

$$1 - \sum_{i=1}^{n+1} y_i = 1 - \sum_{i=1}^{n} y_i - y_{n+1} < 1 - \sum_{i=1}^{n} y_i - \frac{1}{3}(1 - \sum_{i=1}^{n} y_i) = \frac{2}{3}(1 - \sum_{i=1}^{n} y_i) < \frac{2}{3} \cdot (\frac{2}{3})^n = (\frac{2}{3})^{n+1}.$$

Hence $\sum_{n=1}^{\infty} y_n = 1$. By (a) we have $y_1 < \frac{1}{2}$, which implies $\sum_{n=2}^{\infty} y_n = 1 - y_1 > \frac{1}{2} > y_1$. Using (c) we obtain $y_{n+1} < \frac{1}{2}(1 - \sum_{i=1}^{n} y_i) = \frac{1}{2} \sum_{i=n+1}^{\infty} y_i = \frac{1}{2} y_{n+1} + \frac{1}{2} \sum_{i=n+2}^{\infty} y_i$. Thus $y_n < \sum_{i=n+1}^{\infty} y_i$ for every n. Therefore $A(y_n) = R$.

Let ν be a measure such that $\nu(\{n\}) = y_n$. Finally we will show that $\nu(G) \notin \{\mu(F) : F \text{ is finite and nonempty}\}$ for every nonempty $G \subset \mathbb{N}$. If $\max G = 1$, then $G = \{1\}$ and $\nu(G) = y_1$. By (a) we obtain that $\nu(G) \notin \{\mu(F) : F \text{ is finite}\}$. Assume now that that $\max G = n + 1$ for some $n \in \mathbb{N}$. Then $\nu(G) = y_{n+1} + \sum_{i=1}^{n} \varepsilon_i y_i$ for some $(\varepsilon_i)_{i=1}^{n} \in \{0, 1\}^n$. By (d) we obtain that $\nu(G) \notin \{\mu(F) : F \text{ is finite}\}$ as well. □

Recall that if the inequality $x_n \leq \sum_{i=n+1}^{\infty} x_i$ holds for all $n > k$, then $A(x_n)$ is a finite union of closed intervals. More precisely $A(x_n) = \{\sum_{i=1}^{n} \varepsilon_i x_i : (\varepsilon_i)_{i=1}^{k} \in \{0, 1\}^k\} + A((x_n)_{n>k})$. So, we have:

Proposition 3.3. *The range R of a measure is a finite union of intervals if and only if there exist two measures μ and ν with $R = \operatorname{rng}(\mu) = \operatorname{rng}(\nu)$ and the set $\{\mu(\{n\}) : n \in \mathbb{N}\} \cap \{\nu(\{n\}) : n \in \mathbb{N}\}$ is finite.*

Proof. It follows from Lemma 2.3 and Theorem 3.2. □

We already know that if the range R of a measure is a finite union of intervals, then the measure μ with $R = \operatorname{rng}(\mu)$ is not unique. Let us consider the opposite question – for which sets X being a finite union of intervals is there a measure with $\operatorname{rng}(\mu) = X$? As it was mentioned in the Introduction the range $\operatorname{rng}(\mu)$ of a measure μ, or achievement set $A(x_n)$, contains zero and is symmetric. More precisely, $\frac{1}{2}\mu(\mathbb{N})$ is a point of reflection of $\operatorname{rng}(\mu)$. To see it, fix $E \subset \mathbb{N}$ and note that $\mu(E) + \mu(\mathbb{N} \setminus E) = \mu(\mathbb{N})$.

Note that if achievement set is a union of two closed intervals, then both of them have the same length by symmetry. It is clear that $A(x_n) = [0, a] \cup [b, b + a]$, where $b > a$, holds for $x_1 = b$ and $x_{n+1} = \frac{a}{2^n}$ for $n \in \mathbb{N}$, so we may obtain any union of two closed intervals having the same length as an achievement set. Moreover (a, b) is the only gap, so by Lemma 2.3 we get $y_1 = b$ for any (y_n) such that $A(y_n) = [0, a] \cup [b, b+a]$. The case become more complicated when we consider the union of three closed intervals, that is $[0, a] \cup [b, b+c] \cup [2b-a+c, 2b+c]$ – this is a general form of symmetric union of three disjoint intervals which contains zero. The question is whether there exists a sequence (x_n) such that $A(x_n) = [0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$. It turns out that some sets of the form $[0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$ are not ranges of measures, while some others are. We are far from the full characterization of finite unions of intervals (or even unions of three intervals) which are ranges of measures, but we present some partial results which suggest that such characterization will be complicated.

Proposition 3.4. *If $2a < c < 2b$, then $[0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$ is not a range of purely atomic measure.*

Proof. Suppose that $A(x_n) = [0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$ for some (x_n). By Lemma 2.3 there exists $l \in \mathbb{N}$ such that $x_l = b$. By Lemma 2.4 we obtain $b \in S(A(x_n))$. Let $x := b + \frac{c}{2}$. Then $x \in A(x_n)$, and consequently $x + b \in A(x_n)$ or $x - b \in A(x_n)$. But $x + b = 2b + \frac{c}{2} \in (b + c, 2b - a + c)$ and $x - b = \frac{c}{2} \in (a, b)$, which are the gaps of $A(x_n)$. A contradiction. □

Proposition 3.5. *If $a \leq c \leq 2a$, then there exists a sequence (x_n) such that $A(x_n) = [0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$.*

Proof. Define $x_1 = b + c - a$, $x_2 = b$, $x_{n+2} = \frac{a}{2^n}$ for $n \in \mathbb{N}$. It is clear that $A(x_n) = [0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c]$. □

Proposition 3.6. *If $b = 2a$ and $c \geq 2b$, then there exists a sequence (x_n) such that $A(x_n) = [0, a] \cup [b, b+c] \cup [2b - a + c, 2b + c]$.*

Proof. Let $c \geq 2b$. Then there exist unique $k \geq 2$ and $c \in [0, b)$ such that $c = kb + c$. Define $x_1 = 3a + \frac{c}{2}$, $x_2 = 2a + \frac{c}{2}$, $x_n = 2a$ for $n \in \{3, \dots, k + 1\}$, $x_n = \frac{a}{2^{n-k-1}}$ for $n \geq k + 2$ (or any other slowly convergent series with sum a). Thus

$$A(x_n) = \{\sum_{i=1}^{k+1} \varepsilon_i x_i : (\varepsilon_i)_{i=1}^{k+1} \in \{0, 1\}^{k+1}\} + A((x_n)_{n \geq k+2})$$

$$= \bigcup_{m=0}^{k-1} \{2ma, 2a + \frac{c}{2} + 2ma, 3a + \frac{c}{2} + 2ma, 5a + c + 2ma\} + [0, a].$$

Hence

$$A(x_n) = [0, a] \cup [2a, c + 2ka] \cup [a + c + 2ka, 2a + c + 2ka] = [0, a] \cup [b, b + c] \cup [2b - a + c, 2b + c].$$

□

Now we present a characterization of finite unions of intervals which are ranges of purely atomic measures. However, this characterization will not be very informative. It is hard to prove using it that some finite union of intervals *is not* a range of any measure.

Proposition 3.7. *Let X be a finite union of intervals. Then X is a range of a finite measure if and only if there is a measure ν on a finite set such that $X = \operatorname{rng}\nu + [0,a]$ for some $a > 0$.*

Proof. Assume that $X = \operatorname{rng}(\mu)$ for some measure μ on $\mathbb{N}$. Let $x_n = \mu(\{n\})$. Then there is $m \in \mathbb{N}$ such that $x_n \le r_n$ for every $n \ge m$. Hence the achievement set $A((x_n)_{n\ge m})$ is an interval, say $[0,a]$. Let $\nu(n) = \mu(\{n\})$ for $n < m$. Then ν is a finite measure defined on $\{1,2,\dots,m-1\}$. Thus

$$\operatorname{rng}(\mu) = \{\sum_{n=1}^{\infty}\varepsilon_n x_n : \varepsilon_n = 0,1\} = \{\sum_{n=1}^{m-1}\varepsilon_n x_n + \sum_{n=m}^{\infty}\varepsilon_n x_n : \varepsilon_n = 0,1\} =$$

$$\{\sum_{n=1}^{m-1}\varepsilon_n x_n : \varepsilon_n = 0,1\} + \{\sum_{n=m}^{\infty}\varepsilon_n x_n : \varepsilon_n = 0,1\} = \operatorname{rng}(\nu) + [0,a].$$

On the other hand if $X = \operatorname{rng}(\nu) + [0,a]$ for some measure ν on a finite set $F = \{1,2,\dots,n\}$. Let λ be a measure on $\{n+1,n+2,\dots\}$ with $\operatorname{rng}(\lambda) = [0,a]$. Thus the measure μ defined as $\mu(E) = \nu(E\cap\{1,2,\dots,n\}) + \lambda(E\cap\{n+1,n+2,\dots\})$ has the range equal to X. □

4. Uniquely achieved Cantor sets

Let us start from the following example.

Example 4.1. Let R be the range of the measure from Example 1.1, that is $\mu(\{n\}) = \frac{2}{3^n}$. Observe that the numbers $x_n = \frac{2}{3^n}$ are the right ends of the longest gaps of R from the left. Suppose that $A(y_n) = R$ for some sequence (y_n) with $y_1 \ge y_2 \ge \dots$. Then $\{x_n : n\in\mathbb{N}\} \subset \{y_n : n\in\mathbb{N}\}$. Observe that $\sum_{n=1}^{\infty} x_n = 1$, so $y_n = x_n$ for every $n\in\mathbb{N}$. Hence the ternary Cantor set is obtained in the unique way as achievement set of nonincreasing sequence by the sequence (x_n).

Now, let us consider the question: which sets R are ranges of the uniquely defined measures μ. More precisely, for which sets $R = \operatorname{rng}(\mu)$ for some measure μ, the equality $R = \operatorname{rng}(\nu)$ for some measure ν implies that $\mu = \nu$. A sequence from Example 4.1 satisfies $x_n = 2r_n$ for each $n\in\mathbb{N}$ and $A(x_n)$ is the ternary Cantor set, which is obtained in the unique way. Simple observation shows that the uniqueness of a sequence (x_n) generating the achievement set $A(x_n)$ can be obtained as a direct consequence of Lemma 2.3 if $x_n \ge 2r_n$ for each $n\in\mathbb{N}$. The next theorem improves that result.

Theorem 4.2. *Assume that $\mu(\{n\}) > 2\mu(\{n+1\})$ for $n\in\mathbb{N}$. If $\operatorname{rng}(\mu) = \operatorname{rng}(\nu)$ then $\mu = \nu$.*

Proof. Fix $m\in\mathbb{N}$. As usually $x_m = \mu(\{m\})$. Observe that $x_m > r_m$, where $r_m = \sum_{k=m+1}^{\infty} x_k$. Indeed

$$x_m > 2x_{m+1} > x_{m+1} + 2x_{m+2} > x_{m+1} + x_{m+2} + 2x_{m+3} > \dots.$$

Hence $x_m > r_m - r_{m+k} + x_{m+k}$ for each $k\in\mathbb{N}$. Since $(r_m - r_{m+k} + x_{m+k})_{k=1}^{\infty}$ is a decreasing sequence tending to r_m, we get $x_m > r_m$.

By Lemma 2.1 we obtain that $(r_m, x_m)\cap A(x_n) = \emptyset$. Now we will show that (r_m,x_m) is the longest gap from the left in $A(x_n)$. Indeed for each $m\in\mathbb{N}$ we have

$$x_m - r_m = x_m - \sum_{k=m+1}^{\infty} x_k > 2x_{m+1} - \sum_{k=m+1}^{\infty} x_k = x_{m+1} - \sum_{k=m+2}^{\infty} x_k = x_{m+1} - r_{m+1}.$$

Hence no gap of the form (r_k, x_k) is longer than (r_m, x_m) for $m < k$. Suppose now that (a,b) is the longest gap from the left and $b \notin \{x_n : n\in\mathbb{N}\}$. However by Lemma 2.3 the point b should be a term of any sequence (y_n) for which $A(x_n) = A(y_n)$. This yields a contradiction.

Finally by Lemma 2.3 we get that if $A(y_n) = A(x_n)$ then $(y_n) \subset (x_n)$. By comparing sums of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ we get $y_n = x_n$. □

Remark 4.3. Recall that the conditon $x_n > 2x_{n+1} > 0$ implies that $x_n > r_n$ for each $n \in \mathbb{N}$. We call such a series quickly convergent. In [17] it is proved that an achievement set of a quickly convergent sequence is homeomorphic to the ternary Cantor set.

Example 4.4. All Cantor sets of the form $A(q^n)$ for $q < \frac{1}{2}$ are uniquely defined.

Theorem 4.2 can be used to obtain uniquely defined Cantor sets with positive Lebesgue measure.

Example 4.5. Let $q \in (0, \frac{1}{2})$ and $x_n = \frac{1}{2^n} + q^n$ for $n \in \mathbb{N}$. Then a sequence (x_n) satisfies a condition given in Theorem 4.2, so the set $A(x_n)$ is the achievement set of the only one sequence. Moreover the Lebesgue measure of the set $A(x_n)$ can be calculated by the formula given in [6], namely $\lambda(A(x_n)) = \lim_{n\to\infty} 2^n r_n = \lim_{n\to\infty} 2^n(\frac{1}{2^n} + \frac{q^{n+1}}{1-q}) = 1$. Hence we constructed a family of uniquely defined Cantors with positive Lebesgue measure.

The next example shows that the assumption $x_n > 2x_{n+1}$ for $n \in \mathbb{N}$ in Theorem 4.2 is optimal in some sense. One may think that if we assume weaker condition that a series is quickly convergent, in symbols $x_n > r_n$ for $n \in \mathbb{N}$, then the assertion of Theorem 4.2 is still true. However it is not, even when we additionally assume that $x_n \geq 2x_{n+1}$ for $n \in \mathbb{N}$.

Example 4.6. Let us consider the multigeometric sequence defined as $x_{2n-1} = \frac{2}{5^n}, x_{2n} = \frac{1}{5^n}$ for each $n \in \mathbb{N}$. Observe that $x_n > r_n$ and $x_n \geq 2x_{n+1}$ for each $n \in \mathbb{N}$, so the series $\sum_{n=1}^{\infty} x_n$ is quickly convergent, but the condition $x_n > 2x_{n+1}$ is satisfied only for even n's. Define $y_{3n-2} = y_{3n-1} = y_{3n} = \frac{1}{5^n}$. Then we have $A(x_n) = A(y_n)$.

Theorem 4.7. *Assume that $R = \operatorname{rng}(\mu)$ and $\{(a_n, b_n) : n \in \mathbb{N}\}$ is a sequence of gaps in R such that*

(1) *(a_1, b_1) is the longest gap in R and any other gap in R is shorter;*
(2) *$|b_{n+1} - a_{n+1}| < |b_n - a_n|$ for every $n \in \mathbb{N}$;*
(3) *(a_{n+1}, b_{n+1}) is the longest gap in $R \cap [0, a_n]$ and any other gap in $R \cap [0, a_n]$ is shorter.*

Then $\mu(\{n\}) = b_n$. Moreover, R is a Cantor set.

Proof. Since (a_1, b_1) is the only longest gap in R, then the middle point of (a_1, b_1) equals $\frac{1}{2}\mu(\mathbb{N})$. Thus $b_1 > \frac{1}{2}\mu(\mathbb{N})$. By Lemma 2.3 the number b_1 is equal to some $\mu(\{n\})$ and $a_1 = \mu(\mathbb{N} \setminus \{1, 2, \dots, n\})$. Since only one $\mu(\{n\})$ may be greater than $\frac{1}{2}\mu(N)$, then $b_1 = \mu(\{1\})$ and $a_1 = \mu(\mathbb{N} \setminus \{1\})$. Consider a measure μ_1 defined on $\mathbb{N} \setminus \{1\}$ given by $\mu_1(E) = \mu(E)$ for $E \subset \mathbb{N} \setminus \{1\}$. Then $\operatorname{rng}(\mu_1) = R \cap [0, a_1]$. Then (a_2, b_2) is the only longest gap in $\operatorname{rng}(\mu_1)$. Repeating the same argument we obtain that $b_2 = \mu_1(\{2\}) = \mu(\{2\})$. Proceeding inductively we obtain that $\mu(\{n\}) = b_n$.

The "moreover" part of the assertion follows from the inequality $b_n > \sum_{m>n} b_m$ for every n and from Kakeya's Theorem. $\square$

Note that the existence of a sequence $\{(a_n, b_n) : n \in \mathbb{N}\}$ of gaps in R fulfilling conditions (1)–(3) from the Theorem 4.7 is equivalent to the following statement: *between every two gaps of the same length there is a longer gap.*

Theorem 4.8. *Assume that R is a compact subset of the real line with $\min R = 0$, $a_0 = \max R > 0$ and $\{(a_n, b_n) : n \in \mathbb{N}\}$ is a sequence of gaps in R such that*

(1) *$|b_{n+1} - a_{n+1}| < |b_n - a_n|$ for every $n \in \mathbb{N}$;*
(2) *(a_{n+1}, b_{n+1}) is the longest gap in $R \cap [0, a_n]$ and any other gap in $R \cap [0, a_n]$ is shorter for every $n \geq 0$;*
(3) *$\frac{1}{2}a_n$ is a point of reflection of $R \cap [0, a_n]$ for every $n \geq 0$.*

Then $R = \operatorname{rng}(\mu)$ with $\mu(\{n\}) = b_n$. Moreover, R is a Cantor set.

Proof. Since (a_1, b_1) is the only longest gap in R and $a_0/2$ is the point of reflection of R, then $a_0/2$ is the middle point of (a_1, b_1). Similarly (a_2, b_2) is the only longest gap in $R \cap [0, a_1]$ and $a_0/2$ is the point of reflection of $R \cap [0, a_2]$. Thus $a_1/2$ is the middle point of (a_2, b_2). Since $a_0/2$ is the point of reflection of R, then $(b_1 + a_2, b_1 + b_2)$ is a gap in R. Note that $|a_0 - (b_1 + b_2)| = a_2$.

The same as in the previous two steps one can show that $a_2/2$ is the middle point of (a_3, b_3) and since $a_0/2$ is the point of reflection of R, then $(b_1 + b_2 + a_3, b_1 + b_2 + b_3)$ is a gap in R. Note that $|a_0 - (b_1 + b_2 + b_3)| = a_3$. Since $a_n \to 0$, then proceeding inductively we obtain that $\sum_{n=1}^{\infty} b_n = a_0 = \max R$.

Let $R' = A(b_n)$. Note that $b_n > a_n = \sum_{m>n} b_m$. Therefore R' is a Cantor set. By Lemma 2.2 the gaps in R' are of the form $(a_n + f_{j-1}^{(n)}, f_j^{(n)})$ where $F_n = \{0 = f_1^{(n)} < f_2^{(n)} < \dots < f_{m(n)}^{(n)}\}$. Since $b_1 > b_2 > \dots$, then there are no elements of $A(b_n)$ in $(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i)$ where $\varepsilon_i = 0, 1$, which shows that these intervals are gaps in R'. Clearly any gap of the length $|b_n - a_n|$ must be of the form $(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i)$ for some $\varepsilon_i = 0, 1$. Therefore the set of all gaps in R' is the following $\{(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i) : n \in \mathbb{N}, \varepsilon_i = 0, 1\}$.

Now we will prove inductively that every gap of R' is also a gap of R. Clearly R has exactly one gap (a_1, b_1) of the length $|b_1 - a_1|$. Suppose that we have already proved that R has 2^{n-1} gaps of the length $|b_n - a_n|$ of the form $(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i)$ for $\varepsilon_i = 0, 1$.

Since $a_n/2$ is the middle point of (a_{n+1}, b_{n+1}), then $a_n = b_{n+1} + a_{n+1}$. Since $a_{n-1}/2$ is the point of reflection of $[0, a_{n-1}] \cap R$, then $(a_{n+1}, b_n, b_{n+1} + b_n)$ is a gap in $[0, a_{n-1}] \cap R$. Now, since $a_{n-2}/2$ is the point of reflection of $[0, a_{n-2}] \cap R$, then $(a_{n+1} + b_{n-1}, b_{n+1} + b_{n-1})$ and $(a_{n+1} + b_n + b_{n-1}, b_{n+1} + b_n + b_{n-1})$ are gap in $[0, a_{n-2}] \cap R$. By a simple induction we obtain that each interval of the form $(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i)$ for $\varepsilon_i = 0, 1$ is a gap in R.

Note that R' is the closure of the endpoints of its gaps. These endpoints belong also to R. Since R is compact, then $R' \subset R$. This shows that R has no other gaps than those described above (each such gap would be a gap of R' as well). Since

$$R' = [0, a_0] \setminus \bigcup \{(a_n + \sum_{i=1}^{n-1} \varepsilon_i b_i, b_n + \sum_{i=1}^{n-1} \varepsilon_i b_i) : n \in \mathbb{N}, \varepsilon_i = 0, 1\},$$

then $R \subset R'$, and consequently $R = R' = A(b_n)$. □

Theorem 4.9. *Assume that $R = \operatorname{rng}(\mu)$ and there is $\varepsilon > 0$ such that between any two gaps of the same length smaller than ε there is a longer gap in R. Then R is a Cantor set.*

Proof. There exists a sequence of gaps $\{(a_n, b_n) : n \in \mathbb{N}\}$ of R such that

(1) (a_1, b_1) is the longest gap from the left of the length $b_1 - a_1 < \varepsilon$;
(2) $|b_{n+1} - a_{n+1}| < |b_n - a_n|$ for every $n \in \mathbb{N}$;
(3) (a_{n+1}, b_{n+1}) is the longest gap in $R \cap [0, a_n]$ and any other gap in $R \cap [0, a_n]$ is shorter for every $n \in \mathbb{N}$.

By Lemma 2.3 $b_1 = \mu(\{m\})$ and $a_1 = \mu(\{m+1, m+2, \dots\})$. Let $F_m = \{\mu(E) : E \subset \{1, 2, \dots, m\}\}$. Then $\operatorname{rng}(\mu) = F_m + \operatorname{rng}(\mu_1)$ where μ_1 is a measure on $\{m+1, m+2, \dots\}$ given by $\mu_1(E) = \mu(E)$ for $E \subset \{m+1, m+2, \dots\}$. By Theorem 4.7 $\operatorname{rng}(\mu_1)$ is a Cantor set. Thus $\operatorname{rng}(\mu)$ is a Cantor set as well as a union of finitely many shifts of a Cantor set $\operatorname{rng}(\mu_1)$. □

Immediately by Theorem 4.9 we obtain the necessary condition for a measure range to be a Cantorval.

Corollary 4.10. *If $\operatorname{rng}(\mu)$ is a Cantorval, then there are infinitely many pairs of gaps in $\operatorname{rng}(\mu)$ of the same length which are not separated by a longer gap.*

5. Multigeometric sequences and attractors of the affine IFS's

Let us consider the sequences of the form

$$x_{km+1} = k_1 q^k, x_{km+2} = k_2 q^k, \dots, x_{km+m} = k_m q^k$$

for $k = 0, 1, 2, \dots$. Such a sequence we call multigeometric of the rank m and denote by $(k_1, k_2, \dots, k_m; q)$. As we have mentioned in the Introduction, almost all known examples of sequences whose achievement sets are Cantorvals belong to this class, see [1], [2], [3], [7] and [16]. Let us observe that the Guthrie-Nymann Cantorval $\mathbf{T}$ (described in Theorem 1.3) is an achievement set of the bigeometric sequence $(\frac{3}{4}, \frac{2}{4}; \frac{1}{4}) = (\frac{3}{4}, \frac{2}{4}, \frac{3}{16}, \frac{2}{16}, \dots)$. It is not difficult to see that the achievement set $A(k_1, \dots, k_m; q)$ is equal to the set $\{\sum_{n=1}^{\infty} \delta_n q^{n-1} : (\delta_n) \in \Sigma^{\mathbb{N}}\}$ where $\Sigma = \{\sum_{i=1}^{m} \varepsilon_i k_i : (\varepsilon_i) \in \{0,1\}^m\}$. Consequently, $A(k_1, \dots, k_m; q)$ is an attractor for the iterated function system, in short IFS, consisting of the affine functions of the form $f_\sigma(x) = qx + \delta$ where $\delta \in \Sigma$, and therefore it is the unique set $A = A(\Sigma, q)$ satisfying the equality $A = \Sigma + qA$. Not all attractors of affine IFS's are achievement sets of sequences (or ranges of purely atomic measures). Let us observe that if $A = A(\Sigma, q) = A(x_n)$ for some sequence (x_n) of positive terms, then $0 \in A$ and $\frac{1}{2}\sum_{n=1}^{\infty} x_n$ is a point of reflection of A. Hence $0 \in \Sigma$ and Σ is symmetric as well. It turns out that these two conditions for Σ are not sufficient. Recently the authors of [5] showed that the Cantorval N related to the construction of the ternary Cantor set is not an achievement set of any sequence but it is an attractor of some affine IFS.

Let us use the multigeometric sequences to show that there are Cantor sets as well as Cantorvals which can be defined by continuum many different sequences.

Example 5.1. Consider the Jones-Velleman sequence $(x_n) = (4, 3, 2; q)$, defined as follows $x_{3n-2} = 4q^{n-1}, x_{3n-1} = 3q^{n-1}, x_{3n} = 2q^{n-1}$ and its modification $(y_n) = (3, 2, 2, 2; q)$, defined as follows $y_{4n-3} = 3q^{n-1}, x_{4n-2} = 2q^{n-1}, x_{4n-1} = 2q^{n-1}, x_{4n} = 2q^{n-1}$, where $q \in (0, 1)$. For more details see [16], where the author considered among others the sequence (x_n) with $q = \frac{1}{5}$. Let us observe that the given modification does not change the achievement set and we have $A(x_n) = A(y_n)$ (compare the proof of Lemma 2.5). We define a family of sequences F as a family of all sequences (z_n) which are constructed as follows:

- in each step we define three or four succeeding elements of (z_n)
- in n-th step we define $z_{k_{n-1}+i} = x_{3n-3+i}$ for $i \in \{1, 2, 3\}$ or $z_{k_{n-1}+i} = y_{4n-4+i}$ for $i \in \{1, 2, 3, 4\}$ if we have decided to define three or four elements respectively, where k_{n-1} is the number of defined elements in the first $n-1$ steps, $k_0 = 0$

Then $A(z_n) = A(x_n)$ for each sequence (z_n) which belongs to F. Moreover, if we have two sequences $(s_n), (w_n) \in F$ then $w_n = s_n$ for each $n \in \mathbb{N}$ if and only if in each step of constructions of (s_n) and (w_n) we define the same numbers of elements. Hence the cardinality of F is continuum. We will call the sequences belonging to F as multigeometric-like. It is known that the achievement set $A(x_n)$ for some q can be an interval $(q \geq \frac{2}{11})$, a Cantor set with Lebesgue measure zero $(q < \frac{1}{8})$ or a Cantorval $(q \in [\frac{1}{6}, \frac{2}{11}))$. For more details see [16] and [7].

For the next theorem the fact, proved by Bielas, Plewik and Walczyńska in [8], that the Guthrie–Nymann–Cantorval's center of distances consists exactly of the terms of its generating sequence and zero will be crucial.

Theorem 5.2. *Let $X = A(x_n)$, where $x_{2n-1} = \frac{3}{4^n}, x_{2n} = \frac{2}{4^n}$. If $X = A(y_n)$ and $y_1 \geq y_2 \geq y_3 \geq \dots$, then $y_n = x_n$.*

Proof. First note that $\{y_n : n \in \mathbb{N}\} \subset \{x_n : n \in \mathbb{N}\}$. Take any $k \in \mathbb{N}$. By Lemma 2.4 we obtain that $y_k \in S(A(y_n)) = S(A(x_n))$. By the result of Bielas, Plewik and Walczyńska mentioned above, $S(A(x_n)) = \{x_n : n \in \mathbb{N}\} \cup \{0\}$. Thus $y_k \in \{x_n : n \in \mathbb{N}\}$.

Now we will prove that the set $\{y_n :\in \mathbb{N}\}$ of all terms of (y_n) contains every even term of the basic sequence (x_n). Let $m \in \mathbb{N}$. Observe that $x_{2m} > r_{2m}$. Indeed

$$r_{2m} = \sum_{n=2m+1}^{\infty} x_n = \sum_{n=m+1}^{\infty} \frac{3}{4^n} + \sum_{n=m+1}^{\infty} \frac{2}{4^n} = \frac{4}{3} \cdot \frac{5}{4^{m+1}} = \frac{5}{3 \cdot 4^m} < \frac{2}{4^m} = x_{2m}.$$

Therefore the interval (r_{2m}, x_{2m}) is a gap in X and by Lemma 2.3 we obtain that $x_m \in \{y_n : n \in \mathbb{N}\}$.

We have already proved that $\{x_{2n} : n \in \mathbb{N}\} \subset \{y_n : n \in \mathbb{N}\} \subset \{x_n : n \in \mathbb{N}\}$. Since the sequence (x_n) is one-to-one and $S(X) = \{x_n : n \in \mathbb{N}\} \cup \{0\}$, then by Lemma 2.5 none term of (y_n) can be repeated more than two times. However, if $\{y_n : n \in \mathbb{N}\} \neq \{x_n : n \in \mathbb{N}\}$, then some terms of (y_n) must be repeated. This easily follows from the equality $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n$.

Now we are ready to prove the assertion. Let us start from the first step of the inductive prove. Since (y_n) is non-increasing, then y_1 equals x_1 or x_2 (all even terms of (x_n) are among terms of (y_n)). Suppose that $y_1 \neq x_1 = \frac{3}{4}$. Since every term of (y_n) can be repeated at most two times, we have the following inequality

$$\sum_{n=1}^{\infty} y_n \leq 2 \cdot \sum_{n=2}^{\infty} x_n = \frac{11}{6}.$$

Moreover $\frac{5}{3} = \max X$ and $\frac{11}{6} - \frac{5}{3} = \frac{1}{6}$, which means that to obtain (y_n) from the sequence $(x_2, x_2, x_3, x_3, x_4, x_4, \ldots)$ we need to remove elements which sum equals precisely $\frac{1}{6}$. Since $\frac{1}{2}$ and $\frac{3}{16}$ are greater than $\frac{1}{6}$, then $y_1 = y_2 = \frac{1}{2}$, $y_3 = y_4 = \frac{3}{16}$. Note that $y_5 = x_4 = \frac{1}{8}$ because we have to use all even terms of (x_n). Observe that $y_6 \neq \frac{1}{8}$. Indeed, if $y_6 = \frac{1}{8}$ then $y_3 + y_5 + y_6 = \frac{7}{16} \in (\frac{5}{12}, \frac{1}{2})$ but $(r_2, x_2) = (\frac{5}{12}, \frac{1}{2})$ is a gap in X. Moreover $y_6 \neq x_5 = \frac{3}{64}$. Indeed, if $y_6 = \frac{3}{64}$ then $y_3 + y_4 + y_6 = \frac{27}{64} \in (\frac{5}{12}, \frac{1}{2})$. It means that we need to remove one element x_4 and two elements x_5 from the sequence $(x_2, x_2, x_3, x_3, x_4, x_4, \ldots)$. But

$$x_4 + 2x_5 = \frac{1}{8} + \frac{6}{64} = \frac{14}{64} = \frac{42}{192} > \frac{32}{192} = \frac{1}{6}$$

which yields a contradiction. Thus $y_1 = x_1$.

Now assume that $y_i = x_i$ for each $i \in \{1, \ldots, 2m-1\}$ for some $m \in \mathbb{N}$. We will show that $y_{2m} = x_{2m}$ and $y_{2m+1} = x_{2m+1}$. If $y_{2m} \neq x_{2m}$ then $y_{2m} = x_{2m-1}$ and $y_{2m+1} = x_{2m}$. Hence $\sum_{k=1}^{2m+1} y_k = \sum_{k=1}^{2m} x_k + x_{2m-1} > \sum_{k=1}^{2m} x_k + r_{2m} = \frac{5}{3}$, which brings a contradiction. Therefore $y_{2m} = x_{2m}$. Suppose that $y_{2m+1} \neq x_{2m+1}$. Observe that $(A(4^m x_n)_{n=2m+1}^{\infty}) = A(x_n)$. Moreover, if $A(x_n) = A(y_n)$ and $y_i = x_i$ for each $i \in \{1, \ldots, 2m\}$ then $A((x_n)_{n=2m+1}^{\infty}) = A((y_n)_{n=2m+1}^{\infty})$. Thus $A(x_n) = 4^m A((y_n)_{n=2m+1}^{\infty}) = A((4^m y_n)_{n=2m+1}^{\infty})$. By the first step of induction we obtain that $4^m y_{2m+1} = x_1 = \frac{3}{4}$. Thus $y_{2m+1} = \frac{x_1}{4^m} = x_{2m+1}$. This ends the inductive proof. $\square$

6. The Ferens fractals

Let us consider $A = A(\Sigma; q) = \{\sum_{n=1}^{\infty} x_n q^{n-1} : (x_n) \in \Sigma^{\mathbb{N}}\}$, where Σ is a finite set. We have $\Sigma + qA = A$ which means that A is the attractor of the affine IFS system $\{f_\sigma\}_{\sigma \in \Sigma}$, where $f_\sigma(x) = qx + \sigma$. We also call the set A a fractal - it is more general than the theory of multigeometric sequences, because Σ does not have to be the achievement set of any finite sequence. The important class of attractors are so called Ferens fractals for which $\Sigma = \{0, p, p+1, \ldots, p+r, 2p+r\}$ for some $p, r \in \mathbb{N}$, $p \geq 2$. It is known that for $q \geq \frac{p}{3p+r}$ the set $A(\Sigma; q)$ is an interval and for $q < \frac{p}{3p+r}$ the set $A(\Sigma; q)$ is not a union of closed intervals, in particular for $q < \frac{1}{|\Sigma|} = \frac{1}{r+3}$ it is a null Cantor set, see [1], [4] and [7].

Theorem 6.1. *Let $p \in \mathbb{N}$, $p \geq 2$. The Ferens fractal $A = A(\Sigma; q)$ for $r = p$, that is $\Sigma = \{0, p, p+1, \ldots, p+r, 2p+r\} = \{0, p, p+1, \ldots, 2p, 3p\}$ and $q < \frac{1}{4}$ cannot be obtained as an achievement set for any sequence.*

Proof. Note that $(a, b) = (\frac{3pq}{1-q}, p)$ is the longest gap in A from the left. By Lemma 2.3 and the properties of the center of distances we get $p \in S(A)$. We consider the gaps (a, b) and $(2p + a, 2p + b)$. Firstly assume that

$q \in (0, \frac{p-1}{4p-1})$, which is equivalent to $a < p-1$. Fix $x = 2p-1 \in A$. Then $x+p = 3p-1 \in (2p+a, 2p+b)$ and $x-p = p-1 \in (a,b)$. Hence $p \notin S(A)$. Now assume that $q \in (\frac{p-1}{4p-1}, \frac{1}{4})$. Then $p-1 < a < p$. Since (a,b) is the longest gap in $A \cap [0,b)$ one can find $y \in (1+a-b, 1) \cap A$. Fix $x = 2p-1+y \in A$. Then $x+p = 3p-1+y$ and $2p+a = 3p+a-b < 3p-1+y < 3p = 2p+b$, so $x+p \in (2p+a, 2p+b)$. Analogously we prove $x-p \in (a,b)$. Hence $p \notin S(A)$.

If $q = \frac{p-1}{4p-1}$ then $(a,b) = (p-1,p)$ and we take any $z \in (0,1) \cap A$ and then define $x = 2p-1+z$. Thus $x-p \in (a,b)$ and $x+p \in (2p+a, 2p+b)$. The proof is finished. $\square$

On the other hand there exist $p \in \mathbb{N}$, $p \geq 2$ and $r \neq p$ such that the Ferens fractal $A = A(\Sigma; q)$ with $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\}$ is obtained as an achievement set for each $q \in (0,1)$. Let us consider the following examples.

Example 6.2. Let us consider the Ferens fractal $A = A(\Sigma; q)$ for $\Sigma = \{0,2,3,4,6\}$. It is known that for $q \geq \frac{1}{4}$ the set A is the interval. By Theorem 6.1 the set A for $q < \frac{1}{4}$ cannot be obtained as an achievement set for any sequence.

Example 6.3. Let $\Sigma = \{0,2,3,5\}$. Here we have $r = 1 < 2 = p$. Then $A = A(x_n)$ for the multigeometric sequence $x_{2n+1} = 3q^n$, $x_{2n+2} = 2q^n$ for $n = 0,1,2,\dots$. In particular for $q = \frac{1}{4}$ we get rescaled by 4 Guthrie and Nymann's Cantorval. It is also the Ferens fractal for $p = 2, r = 1, q = \frac{1}{4}$. Note that for each $p \in \mathbb{N}$ and $r = 1$ we obtain a Ferens fractal, which can be obtained by the multigeometric sequence.

Example 6.4. Let $\Sigma = \{0,2,3,4,5,7\}$. Here we have $r = 3 > 2 = p$. Then $A = A(x_n)$ for a multigeometric sequence $x_{3n+1} = 3q^n$, $x_{3n+2} = 2q^n$, $x_{3n+3} = 2q^n$ for $n = 0,1,2,\dots$.

So, there are Ferens fractals which are also achievement sets. The next theorem gives the example of large class of such fractals and shows that for each natural $p \geq 2$ we can find r such that the Ferens fractal $A = A(\Sigma; q)$ is also an achievement set. We will base our calculation on a simple observation that if Σ is the achievement set of a finite sequence $\{a_1, \dots, a_k\}$ then $A(\Sigma; q)$ can be obtained by the multigeometric sequence (x_n) defined as follows $x_{kn+j} = a_j q^n$ for $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \dots, k\}$.

Lemma 6.5. *Let $p \in \mathbb{N}$, $p \geq 2$, $r = \frac{3p^2-3p}{2}$. Then $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\} = \{0, p, p+1, \dots, \frac{3p^2-p}{2}, \frac{3p^2+p}{2}\}$ is the set of subsums for some finite sequence.*

Proof. Define $a_1 = p$, $a_j = (p+j-2)$ for $j \in \{2, \dots, p+1\}$. Then $\Sigma = A((a_n)_{n=1}^{p+1})$. $\square$

As a result we immediately obtain:

Theorem 6.6. *Let $p \in \mathbb{N}$, $p \geq 2$, $q \in (0,1)$. The Ferens fractal $A = A(\Sigma; q)$ for $r = \frac{3p^2-3p}{2}$ (so $\Sigma = \{0, p, p+1, \dots, \frac{3p^2-p}{2}, \frac{3p^2+p}{2}\}$) is an achievement set for some multigeometric sequence.*

Proof. Define $x_{(p+1)n+1} = pq^n$, $x_{(p+1)n+j} = (p+j-2)q^n$ for $n \in \mathbb{N} \cup \{0\}$, $j \in \{2, \dots, p+1\}$. Then $A = A(x_n)$. $\square$

Lemma 6.7. *Let $p \in \mathbb{N}$, $p \geq 2$, $r \geq \frac{3p^2-p}{2}$. Then $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\}$ is the set of subsums for some finite sequence.*

Proof. Let us first consider $r = \frac{3p^2-p}{2}$. Define $a_j = (p+j-1)$ for $j \in \{1, \dots, p+1\}$. Then $\Sigma = A((a_n)_{n=1}^{p+1})$. Let now consider $r > \frac{3p^2-p}{2}$, $r = \frac{3p^2-p}{2} + k$, where $k = mp + r$ for $m \in \mathbb{N} \cup \{0\}$, $r \in \{0, 1, \dots, p-1\}$. Define $a_j = p$ for $j \in \{1, \dots, 2+m\}$, $a_j = (p+j-m-2)$ for $j \in \{3+m, \dots, 2+m+k\}$, $a_j = (p+j-m-3)$ for $j \in \{3+m+k, \dots, 2+m+p\}$. Then $\Sigma = A((a_n)_{n=1}^{2+m+p})$. $\square$

Corollary 6.8. *Let $p \in \mathbb{N}$, $p \geq 2$, $q \in (0,1)$. The Ferens fractal $A = A(\Sigma; q)$ for $r \geq \frac{3p^2-p}{2}$ is an achievement set for some multigeometric sequence.*

Lemma 6.9. *Let $p \in \mathbb{N}$, $p \geq 3$, $r \in (1, \frac{3p^2-3p}{2})$. Then $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\}$ is not the set of subsums for any finite sequence.*

Proof. Let $r \geq p$. Assume that $\Sigma = A(x_n)$ for some finite sequence (x_n). Since p is the smallest non-zero element, we know that the smallest sum of two or more elements equals $2p$. We know that $p+r \geq 2p$. Therefore we get $\{p, p+1, \dots, 2p-1\} \subset (x_n)$. Since $2p \in \Sigma$ we have to add the another term x_n equal to $2p$ or one more term x_n equal to p. Thus its sum is an element of Σ, but $p + (p+1) + \ldots + 2p \geq p + p + (p+1) + \ldots 2p - 1 = \frac{3p^2+p}{2} = 2p + \frac{3p^2-3p}{2} > 2p + r = \max \Sigma$. We get contraditions for both cases.
Let $r \in (1, \dots, p)$. Since $p, p+1, p+2 \in \Sigma$ and $p+2 < 2p$ we get $p, p+1, p+2 \in (x_n)$. Therefore we have $3p+3 \in \Sigma$, but $\max \Sigma = 2p + r < 3p < 3p+3$, which gives us a contradition. $\square$

Lemma 6.10. *Let $p \in \mathbb{N}$, $p \geq 2$, $r \in (\frac{3p^2-3p}{2}, \frac{3p^2-p}{2})$. Then $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\}$ is not the set of subsums for any finite sequence.*

Proof. Note that $2p + r > \frac{3p^2+p}{2} > p + (p+1) + \ldots + (2p-1)$. Hence $2p \in \Sigma$. We can obtain it by adding $2p$ or one more p to our terms. If we add $2p$ then $p + (p+1) + \ldots + (2p-1) + 2p = \frac{3p^2+3p}{2} > 2p + r$, which yields a contradition. So let us consider $(x_n) = \{p, p, p+1, \dots, 2p-1\}$. We have $\sum x_n = \frac{3p^2+p}{2} \in (p+r, 2p+r)$. Since $\sum x_n < 2p + r$ we have to add next element to the sequence (x_n), but we cannot add an element which is smaller than p. Therefore $\sum x_n + p > 2p + r$, which yields a contradition. $\square$

Corollary 6.11. *Let $p, r \in \mathbb{N}$, q be a positive real number and $\Sigma = \{0, p, p+1, \dots, p+r, 2p+r\}$.*

1. *If $p \geq 3$, $q \in (0, \frac{p}{3p+r})$ and $r \in (1, \frac{3p^2-3p}{2}) \cup (\frac{3p^2-3p}{2}, \frac{3p^2-p}{2})$, then the Ferens fractal $A = A(\Sigma; q)$ is not an achievement set for the multigeometric sequence $(k_1, \dots, k_m; q)$ with $\{\sum_{i=1}^m \varepsilon_i k_i : (\varepsilon_i) \in \{0,1\}^m\} = \Sigma$.*
2. *If $p = 2$, $q \in (0, \frac{p}{3p+r})$ and $r \in (\frac{3p^2-3p}{2}, \frac{3p^2-p}{2})$, then the Ferens fractal $A = A(\Sigma; q)$ is not an achievement set for the multigeometric sequence $(k_1, \dots, k_m; q)$ with $\{\sum_{i=1}^m \varepsilon_i k_i : (\varepsilon_i) \in \{0,1\}^m\} = \Sigma$.*

7. Guthrie–Nymann–Jones Cantorvals (GNJ Cantorvals)

In this section we will deal with Ferens fractals of the type $A(r) = A(\Sigma, q)$ for $\Sigma = \{0, 2, 3, \dots, r+2, r+4\}$ and $q = \frac{1}{|\Sigma|} = \frac{1}{r+3}$. It is known that sets $A(r)$ for $r = 1, 2, \dots$ are Cantorvals. It follows from Kenyon Theorem, (see [21] and [22]) which states that if $\{n \bmod r : n \in \Sigma\} = \mathbb{Z}_r$, then $A(\Sigma, 1/r)$ has nonempty interior (it can be also deduced from proofs presented in [7]).

Note that

1. for $r = 1$ the set $A(r)$ is the rescaled Guthrie–Nymann Cantorval which, by Theorem 5.2 has the unique representation as an achievement set.
2. For $r = 2m-1$ the set $A(r)$ equals to $A(x_n)$ where $(x_n) = (3, \underbrace{2, \dots, 2}_{m}; \frac{1}{r+3})$. If $r \geq 5$, the Cantorval $A(r)$ has continuum many representations as an achievement set of multigeometric-like series with the same set Σ – see Example 5.1.
3. By Theorem 6.1 the set $A(r)$ is not an achievement set (or a range of any measure) for $r = 2$.
4. For $r = 4$ we know that $A(r)$ is not an achievement set for any multigeometric series generating the same set Σ.

(5) For $r = 2m \geq 6$ the set $A(r)$ equals to $A(x_n)$ where $(x_n) = (3, 3, \underbrace{2, \ldots, 2}_{m-2}; \frac{1}{r+3})$. Using the method from Example 5.1 for $r \geq 10$ (or $m - 2 \geq 3$) we observe that the Cantorval $A(r)$ has continuum many representations as an achievement set of multigeometric-like series with the same set Σ.

Using methods from [8] one can get some information on geometry and the center of distances for Cantorvals $A(r)$:

(1) $A(r) \subset [0, \frac{(r+4)(r+3)}{r+2}]$.
(2) The interval $[\frac{2(r+3)}{r+2}, r+3]$ is the longest component of $A(r)$.
(3) $[0, \frac{r+4}{r+2}] \cap A(r) = \frac{1}{r+3} A(r)$.
(4) $(\frac{r+4}{r+2}, 2)$ is the longest gap from the left and it has the same length as the longest component of $[0, \frac{4+r}{(2+r)(3+r)}] \cap A(r)$.
(5) $\big(r+3+(A(r) \cap [0, \frac{2(r+3)}{r+2}])\big) \cup (A(r) \cap [r+3, \frac{(r+4)(r+3)}{r+2}]) = [r+3, \frac{(r+4)(r+3)}{r+2}]$, it follows from the fact that the gaps of the first summand in the above union are exactly in the same places as the copmonents of the second one and vice versa.

In Example 7.1 we present the idea of proving (1)–(5) based on an appropriate picture.

Note that if $t \in (\frac{2(r+3)}{r+2}, \frac{r+3}{2} - \frac{r+3}{r+2})$, then $t \in S(A(r))$. Recall that $\frac{2(r+3)}{r+2}$ is a left endpoint of the longest component of $A(r)$ and $\frac{r+3}{2} - \frac{r+3}{r+2}$ is a half of its length. Similarly we have for every longest component of $A(r)$ from the left. Therefore if $\frac{2}{2+r} < \frac{1}{2} - \frac{1}{2+r}$, that is if $r > 4$, then $S(A(r))$ contains a sequence of intervals.

This observation suggests that for $r > 4$ one can look for a multigeometric series (x'_n) with $\Sigma' \neq \Sigma$ and $A(x'_n) = A(x_n)$.

Example 7.1. At Figure 1 we present a GNJ Cantorval $A := A(6)$, i.e. $\Sigma = \{0, 2, 3, \ldots, 8, 10\}$ and $q = \frac{1}{9}$; there are also nine its copies $\tau + \frac{1}{9}A$, $\tau \in \Sigma$. The first and the last copies, $\frac{1}{9}A$ and $10 + \frac{1}{9}A$, are equal to the left $A \cap [0, 1\frac{2}{8}]$ and the right $A \cap [10, 11\frac{2}{8}]$ parts of the original Cantorval A, respectively. Other copies cover the rest of A; note that $2 + \frac{1}{9}A$ and $3 + \frac{1}{9}A$ cover the interval $[3, 3\frac{2}{8}]$, since the components interiors of $(2 + \frac{1}{9}A) \cap [3, 3\frac{2}{8}]$ are precisely gaps of $(3 + \frac{1}{9}A) \cap [3, 3\frac{2}{8}]$, and vice versa.

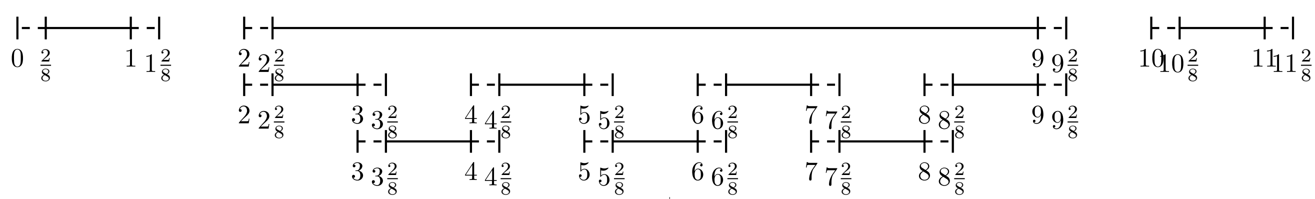

Figure 1

On the other hand $A(6)$ satisfies also the equality $A = \Sigma' + \frac{1}{9}A$ for $\Sigma' = \{0, 2, 2\frac{2}{8}, 2\frac{4}{8}, 3\frac{2}{8}, 4\frac{2}{8}, 4\frac{4}{8}, 4\frac{6}{8}, 5\frac{2}{8}, 5\frac{4}{8}, 5\frac{6}{8}, 6\frac{6}{8}, 7\frac{4}{8}, 7\frac{6}{8}, 8, 10\}$. Let us observe that Σ' is an achievement set for the finite sequence $\{3\frac{2}{8}, 2\frac{4}{8}, 2\frac{2}{8}, 2\}$ and hence $A(6) = A(3\frac{2}{8}, 2\frac{4}{8}, 2\frac{2}{8}, 2; \frac{1}{9})$ as well as $A(6) = A(3, 3, 2, 2; \frac{1}{9})$. For the clarity and readers' convenience we present the next picture - Figure 2.

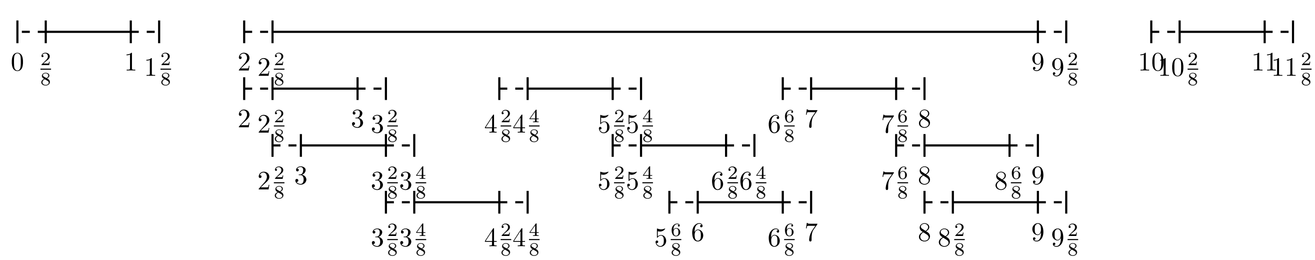

Figure 2

Example 7.2. In [8] the authors found a center of distances of the boundary $\partial\frac{A(1)}{4}$ of the Guthrie–Nymann Cantorval $\frac{1}{4}A(1)$. The set $\partial\frac{A(1)}{4}$ is a Cantor set arisen from $\frac{A(1)}{4}$ by removing all interiors of its nontrivial components. It turns out that $S(\partial\frac{A(1)}{4}) = \{1, \frac{1}{4}, \frac{1}{4^2}, \dots\}$. Therefore if $\partial\frac{A(1)}{4} = A(y_n)$ for some sequence (y_n), then $\{y_n : n \in \mathbb{N}\} \subset \{1, \frac{1}{4}, \frac{1}{4^2}, \dots\}$. The authors, according to this observation, claimed that $\partial\frac{A(1)}{4}$ is not an achievement set for any sequence, since $1 + \frac{1}{4} + \frac{1}{4^2} + \dots < \frac{5}{3} = \max\frac{A(1)}{4}$. However, they did not observe that terms of (y_n) may repeat. By Lemma 2.5 none of the terms may repeat more than twice, since the doubling of such term would be in $S(\partial\frac{A(1)}{4})$. But $1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^2} + \dots = \frac{5}{3}$. It turns out that for any positive integer r

$$\partial\frac{A(r)}{r+3} = A(1, \frac{1}{r+3}, \frac{1}{r+3}, \frac{1}{(r+3)^2}, \frac{1}{(r+3)^2}, \dots).$$

Indeed, by geometric properties of $A(r)$ it follows that

$$[0, \frac{2}{2+r}] \cap \partial\frac{A(r)}{r+3} = C_{\frac{1}{r+3}} + C_{\frac{1}{r+3}}.$$

Thus

$$\partial\frac{A(r)}{r+3} = (C_{\frac{1}{r+3}} + C_{\frac{1}{r+3}}) \cup (1 + C_{\frac{1}{r+3}} + C_{\frac{1}{r+3}}).$$

References

[1] T. Banakh, A. Bartoszewicz, M. Filipczak, E. Szymonik, Topological and measure properties of some self-similar sets, Topol. Methods Nonlinear Anal., 46 (2015), 1013–1028.

[2] T. Banakh, A. Bartoszewicz, S. Głąb, E. Szymonik, Algebraic and topological properties of some sets in ℓ_1, Colloq. Math. 129 (2012), 75–85.

[3] T. Banakh, A. Bartoszewicz, S. Głąb, E. Szymonik, Erratum to "Algebraic and topological properties of some sets in ℓ_1" (Colloq. Math. 129 (2012), 75–85) Colloq. Math. 135 (2014), no. 2, 295–298.

[4] M. Banakiewicz, F. Prus-Wiśniowski, M-Cantorvals of Ferens type, Math. Slovaca 67 (2017) no.4, 907-918.

[5] A. Bartoszewicz, M. Filipczak, S. Głąb, J. Swaczyna, F. Prus-Wiśniowski, On generating regular Cantorvals connected with geometric Cantor sets, arXiv:1706.03523v1.

[6] A. Bartoszewicz, M. Filipczak, F. Prus-Wiśniowski, Topological and algebraic aspects of subsums of series. Traditional and present-day topics in real analysis, 345–366, Faculty of Mathematics and Computer Science. University of Łódź, Łódź, 2013.

[7] A. Bartoszewicz, M. Filipczak, E. Szymonik, Multigeometric sequences and Cantorvals, Cent. Eur. J. Math. 12(7) (2014), 1000–1007.

[8] W. Bielas, S. Plewik, M. Walczyńska, On the center of distances, M. European Journal of Mathematics (2018) 4: 687.

[9] Z. Daróczy, A. Járai, I. Katái, Intervallfullende Folgen und volladditive Funktionen, Acta Sci. Math., 50 (1986), 337-350.

[10] Z. Daróczy, I. Katái, Interval filling and additive functions, Acta Sci. Math., 52 (1988), 337-347.

[11] Z. Daróczy, I. Katái, T. Szabó, On completely additive functions related to interval-filling sequences, Arch. Math., 54 (1990), 173-179.

[12] C. Ferens, On the range of purely atomic probability measures, Studia Math., 77(3) (1984), 261–263.

[13] J.A. Guthrie, J.E. Nymann, The topological structure of the set of subsums of an infinite series, Colloq. Math. 55:2 (1988), 323–327.

[14] H. Steinhaus, Nowa własność mnogości Cantora, Wektor (1917) 1–3.

[15] H. Steinhaus, A new property of the Cantor set, in Hugo Steinhaus. Selected Papers, PWN Warszawa (1985), 205–207.

[16] R. Jones, Achievement sets of sequences, Am. Math. Mon. 118:6 (2011), 508–521.

[17] S. Kakeya, On the partial sums of an infinite series, Tôhoku Sic. Rep. 3 (1914), 159–164.

[18] S. Leth, A uniqueness condition for sequences, Proc. Amer. Math. Soc. 93 (1985), 287-290.

[19] J. Malitz, A strengthening of Leth's uniqueness condition for sequences, Proc. Amer. Math. Soc. 98 (1986), 641-642.

[20] P. Mendes, F. Oliveira, On the topological structure of the arithmetic sum of two Cantor sets, Nonlinearity. 7 (1994), 329–343.

[21] Z. Nitecki, The subsum set of a null sequence, arXiv:1106.3779v1.

[22] Z. Nitecki, Cantorvals and subsum sets of null sequences, Amer. Math. Monthly, 122 (2015), no. 9, 862–870.

[23] J.E. Nymann, A uniqueness condition for finite measures, Proc. Amer. Math. Soc. 108 (1990), 913-919.

[24] J.E. Nymann, R.A. Sáenz, On the paper of Guthrie and Nymann on subsums of infinite series, Colloq. Math. 83 (2000), 1–4.

[25] A. Rényi, Probability theory, North-Holland, Amsterdam (1970).

[26] A. D. Wainshtein, B. Z. Shapiro, Structure of a set of $\overline{\alpha}$-representable numbers, Izv. Vyssh. Uchebn. Zaved. Mat., 5 (1980), 8–11 (in Russian).

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