DESCRIPTIVE PROPERTIES RELATED TO POROSITY AND DENSITY FOR COMPACT SETS ON THE REAL LINE

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Abstract. Given $r \in [0,1]$ we study descriptive complexity of the set P_r (respectively D_r) of all compact sets K in the hyperspace $\mathcal{K}(\mathbb{R})$ with porosity r (density r) at 0. We also show that the set NBP of all nowhere bilaterally porous compact sets in $\mathcal{K}(\mathbb{R})$ is Π^1_1–complete, and we prove a similar fact for density.

1. introduction

Consider the hyperspace $\mathcal{K}(\mathbb{R})$ of all nonempty compact sets equipped with the Vietoris topology (i.e. the one generated by the sets of the form $\{K \in \mathcal{K}(\mathbb{R}) : K \subset U\}$ and $\{K \in \mathcal{K}(\mathbb{R}) : K \cap U \neq \emptyset\}$, for U open in $\mathbb{R}$; equivalently it is given by the Hausdorff metric). The aim of our paper is to investigate descriptive complexity of families in $\mathcal{K}(\mathbb{R})$ that consist of sets with prescribed porosity and density at a given point. Both notions of porosity and metric density describe in various manners a local size of sets. They play significant role in real analysis (see [2]). The origins of our studies come from the paper [5] by Zajiček and Zelený where it was shown that compact σ–porous sets form a Π^1_1–complete subset of $\mathcal{K}(\mathbb{R})$.

Let us recall the notion of porosity on the real line. (Many facts on the porosity can be found in the survey papers [6] and [7].) Let $E \subset \mathbb{R}$, $x \in \mathbb{R}$ and $R > 0$. Denote by $\lambda^+(x, R, E)$ the length of the largest open subinterval of $(x, x+R)$ which does not intersect E. The right-hand porosity of E at x is defined by the formula

$$p^+(E,x) = \limsup_{R \to 0^+} \frac{\lambda^+(x,R,E)}{R}.$$

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Analogously we define the left-hand porosity of E at x, and denote it by $p^-(E,x)$. If $\lambda^+(x,R,E)$ denotes the length of the largest open subinterval of $(x-R,x+R)$ which does not intersect E, then $p(E,x)=\limsup_{R\to 0^+}\frac{\lambda(x,R,E)}{2R}$ is the porosity of E at x. A set E is called porous (strongly porous) from the right at x if $p^+(E,x)>0$ ($p^+(E,x)=1$). A set E is called bilaterally porous (strongly porous) at x if it is porous (strongly porous) both from the right and from the left at x.

Let μ stand for Lebesgue measure on $\mathbb{R}$. For a measurable $E\subset\mathbb{R}$ and $x\in\mathbb{R}$, by $d^+(x,E)$ we denote the right-hand density of E at the point x, that is $d^+(x,E)=\lim_{h\to 0^+}\frac{\mu([x,x+h]\cap E)}{h}$, provided the limit exists, and by $d(x,E)=\lim_{h\to 0^+}\frac{\mu([x-h,x+h]\cap E)}{2h}$ we denote density of E at the point x, provided the limit exists. By symbol $\underline{d^+}(x,E)$ we denote the lower right-hand density of E at x, that is, the number $\liminf_{h\to 0^+}\frac{\mu([x,x+h]\cap E)}{h}$. Similarly we define $\overline{d^+}(x,E)$ – the upper right-hand density of E at the point x. If $d^+(x,E)=1$ then x is called a right-hand density point of E. Analogously we introduce the notions of left-hand density and left-hand density points.

We use standard set theoretic notation. See [4]. Let $\mathbb{N}=\{0,1,2,...\}$. Let X be a Polish space. For ordinals α, $1\leq\alpha\leq\omega_1$, we define the following pointclasses of Borel sets by transfinite induction: Σ^0_1 – open sets, Π^0_1 – closed sets; and for $1<\alpha<\omega_1$, $\Sigma^0_\alpha=\{\bigcup_{n\in\mathbb{N}}A_n : A_n\in\bigcup_{\beta<\alpha}\Pi^0_\beta\}$ and $\Pi^0_\alpha=\{\mathbb{R}\setminus A : A\in\Sigma^0_\alpha\}$. A subset A of X is called analytic if it is the projection of a Borel subset B of $X\times X$. A subset C of X is called coanalytic if $X\setminus C$ is analytic. The pointclasses of analytic and coanalytic sets are denoted by Σ^1_1 and Π^1_1, respectively. The symbol $\exists^\infty$ means "for infinitely many". By Tr we denote the Polish space of all trees on $\mathbb{N}$.

2. Compact sets with prescribed porosity and density

Lemma 1. *Let $0<\varepsilon<1$ and $R>0$. Then $\{K\in\mathcal{K}(\mathbb{R}) : \lambda^+(0,R,K)>\varepsilon R\}$ and $\{K\in\mathcal{K}(\mathbb{R}) : \lambda^+(0,R,K)<\varepsilon R\}$ are open sets.*

Proof. Let $K_0 \in \mathcal{K}(\mathbb{R})$ be such that $\lambda^+(0,R,K_0) > \varepsilon R$. Then there is a closed interval $I \subset (0,R)$ of length εR such that $I \cap K_0 = \emptyset$. The set $\{K \in \mathcal{K}(\mathbb{R}) : K \subset \mathbb{R} \setminus I\}$ is an open neighbourhood of K_0 contained in $\{K \in \mathcal{K}(\mathbb{R}) : \lambda^+(0,R,K) > \varepsilon R\}$.

For every finite sequence $0 \le x_0 < x_1 < ... < x_n$ such that $x_i - x_{i-1} < \varepsilon R$ for $i = 1,2,...,n$ there is $\delta > 0$ such that $x_i - x_{i-1} + 2\delta < \varepsilon R$ and $x_i - x_{i-1} > 2\delta$ for $i = 1,2,...,n$. We have

$$\{K \in \mathcal{K}(\mathbb{R}) : \lambda^+(0,R,K) < \varepsilon R\} = \bigcup \{K \in \mathcal{K}(\mathbb{R}) : K \cap (x_i - \delta, x_i + \delta) \neq \emptyset, \ \ i = 0,1,...,n\}.$$

where the union is taken over all described above finite sequences $x_0,...,x_n$. □

Theorem 2. *Let $r \in [0,1]$ and $P_r^+ = \{K \in \mathcal{K}(\mathbb{R}) : p^+(K,0) = r\}$. Then*

(a) *P_r^+ is Π^0_3–complete;*

(b) *P_1^+ is Π^0_2–complete.*

Proof. (a) Denote by $\mathbb{Q}_+$ the set of all positive rationals. Observe that

$$P_r^+ = \{K \in \mathcal{K}(\mathbb{R}) : \forall \varepsilon > 0\ \exists R_0 > 0\ \forall R \in (0,R_0) \quad \frac{\lambda^+(0,R,K)}{R} \le (1+\varepsilon)r\} \cap$$
$$\{K \in \mathcal{K}(\mathbb{R}) : \forall \varepsilon > 0\ \forall R_0 > 0\ \exists R \in (0,R_0) \quad \frac{\lambda^+(0,R,K)}{R} \ge (1-\varepsilon)r\} =$$
$$\bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcup_{R_0 \in \mathbb{Q}_+} \bigcap_{R \in (0,R_0)\cap \mathbb{Q}_+} \{K \in \mathcal{K}(\mathbb{R}) : \frac{\lambda^+(0,R,K)}{R} \le (1+\varepsilon)r\} \cap$$
$$\bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcap_{R_0 \in \mathbb{Q}_+} \bigcup_{R \in (0,R_0)\cap \mathbb{Q}_+} \{K \in \mathcal{K}(\mathbb{R}) : \frac{\lambda^+(0,R,K)}{R} \ge (1-\varepsilon)r\}.$$

Hence by Lemma 1 we have that P_r^+ is Π^0_3.

Let $t \in (0, 1-r)$, $b_n = t^n$ and $a_n^k = (b_{n+1}\frac{k}{1-r} + b_n)/(k+1)$ for $n,k \in \mathbb{N}$. We have $b_{n+1} < a_n^k < b_n$ for all $n,k \in \mathbb{N}$. Define $F_r : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(\mathbb{R})$ by

$$F_r(\alpha) = \{0\} \cup \bigcup_{n \in \mathbb{N}} [a_n^{\alpha(n)}, b_n],\ \alpha \in \mathbb{N}^{\mathbb{N}}.$$

Let $\varepsilon > 0$. Fix $n \in \mathbb{N}$ such that $b_n < \varepsilon$ and let $\alpha, \alpha' \in \mathbb{N}^{\mathbb{N}}$ be such that $\alpha(k) = \alpha'(k)$ for $k \le n$. Then $d_H(F_r(\alpha), F_r(\alpha')) < \varepsilon$ where d_H stands for the Hausdorff metric induced by the natural metric on $\mathbb{R}$. This shows the continuity of F_r.

Recall that $C_3 = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \lim_{n\to\infty} \alpha(n) = \infty\}$ is Π^0_3–complete (see [4, 23.A]). Our proof will be finished if we show that $F_r(\alpha) \in P^+_r \iff \alpha \in C_3$.

Let $\alpha \in C_3$. Then

$$p^+(F_r(\alpha), 0) = \limsup_{n\to\infty} \frac{a_n^{\alpha(n)} - b_{n+1}}{a_n^{\alpha(n)}} = \limsup_{n\to\infty} \left(1 - \frac{b_{n+1}}{a_n^{\alpha(n)}}\right) =$$

$$\limsup_{n\to\infty} \left(1 - \frac{b_{n+1}(\alpha(n)+1)}{b_{n+1}\frac{\alpha(n)}{1-r} + b_n}\right) = 1 - (1-r) = r.$$

Hence $F_r(\alpha) \in P^+_r$.

Let $\alpha \notin C_3$. Then there are a strictly increasing sequence $\{n_k\}$ of natural numbers and a number $N \in \mathbb{N}$ such that $\alpha(n_k) = N$ for all $k \in \mathbb{N}$. Hence

$$p^+(F_r(\alpha), 0) = \limsup_{n\to\infty} \frac{a_n^{\alpha(n)} - b_{n+1}}{a_n^{\alpha(n)}} \geq \limsup_{k\to\infty} \frac{a_{n_k}^{\alpha(n_k)} - b_{n_k+1}}{a_{n_k}^{\alpha(n_k)}} =$$

$$\limsup_{k\to\infty} \left(1 - \frac{b_{n_k+1}N + b_{n_k+1}}{b_{n_k+1}\frac{N}{1-r} + b_{n_k}}\right) = \limsup_{k\to\infty} \left(1 - \frac{b_{n_k+1}N + b_{n_k+1}}{b_{n_k+1}\frac{N}{1-r} + t^{-1}b_{n_k+1}}\right) =$$

$$1 - \frac{N+1}{\frac{N}{1-r} + t^{-1}} = 1 - \frac{(N+1)(1-r)}{N + t^{-1}(1-r)}.$$

Note that

$$1 - \frac{(N+1)(1-r)}{N + t^{-1}(1-r)} - r = \frac{r^2 t^{-1} + r(1 - 2t^{-1}) + t^{-1} - 1}{N + t^{-1}(1-r)}.$$

Let $f(r) = r^2 t^{-1} + r(1-2t^{-1}) + t^{-1} - 1$. It is easy to check that $f(r) = 0$ if and only if either $r = 1 - t$ or $r = 1$. Then $f(r) > 0$ for $r \in [0, 1-t)$. Hence $p^+(F_r(\alpha), 0) > r$ and $F_r(\alpha) \notin P^+_r$.

(b) We have

$$P^+_1 = \{K \in \mathcal{K}(\mathbb{R}) : \limsup_{R\to 0^+} \frac{\lambda^+(0,R,K)}{R} = 1\} =$$

$$\{K \in \mathcal{K}(\mathbb{R}) : \forall \varepsilon > 0\ \forall R_0 > 0\ \exists R \in (0, R_0) \quad \frac{\lambda^+(0,R,K)}{R} > (1-\varepsilon)\} =$$

$$\bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcap_{R_0\in\mathbb{Q}_+} \bigcup_{R\in(0,R_0)\cap\mathbb{Q}_+} \{K \in \mathcal{K}(\mathbb{R}) : \frac{\lambda^+(0,R,K)}{R} > (1-\varepsilon)\}.$$

By Lemma 1 we infer that P^+_1 is Π^0_2.

Recall that $N_2 = \{\alpha \in \{0,1\}^{\mathbb{N}} : \exists^\infty n \ (\alpha(n) = 0)\}$ is Π^0_2–complete (see [4, 23.A]). Define $G : \{0,1\}^{\mathbb{N}} \to \mathcal{K}(\mathbb{R})$ by

$$G(\alpha) = \{0\} \cup \bigcup_{\alpha(n)=1} \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right], \ \ \alpha \in \{0,1\}^{\mathbb{N}}.$$

To show continuity of G we employ exactly the same argument as for F_r. Hence our proof will be complete if we show that $G(\alpha) \in P^+_1 \iff \alpha \in N_2$.

Let $\alpha \in N_2$ and let $\{n_k\}$ be a strictly increasing sequence of natural numbers such that $\alpha(n_k) = 0$ for $k \in \mathbb{N}$. Then

$$p^+(G(\alpha), 0) \geq \limsup_{k\to\infty} \frac{\frac{1}{n_k!} - \frac{1}{(n_k+1)!}}{\frac{1}{n_k!}} = \limsup_{k\to\infty} \frac{n_k}{n_k+1} = 1.$$

Hence $G(\alpha) \in P^+_1$.

Let $\alpha \notin N_2$. So, there is $N \in \mathbb{N}$ such that $\alpha(n) = 1$ for $n \geq N$. Then $G(\alpha) \supset [0, \frac{1}{N!}]$ and $p^+(G(\alpha), 0) = 0$. Hence $G(\alpha) \notin P^+_1$.□

Corollary 3. *The set P^+_0 (SP^+_0) of all compact sets which are porous (strongly porous) from the right at* 0 *is Σ^0_3–complete (Π^0_2–complete).*

Now, we will prove the analogue of Theorem 2 when the operator of porosity is replaced by the operator of density.

Lemma 4. *Let $0 < \varepsilon < 1$, $h > 0$. Then $\{K \in \mathcal{K}(\mathbb{R}) : \mu([0,h] \cap K) < \varepsilon h\}$ is open.*

Proof. Observe that

$$\{K \in \mathcal{K}(\mathbb{R}) : \mu([0,h] \cap K) < \varepsilon h\} =$$

$$\{K \in \mathcal{K}(\mathbb{R}) : \exists U - \text{an open set}, \ \mu(U \cap [0,h]) < \varepsilon h \text{ and } K \subset U\}.$$

This is an open set as the union of basic open sets in $\mathcal{K}(\mathbb{R})$. □

Proposition 5. *Let $r \in [0,1]$ and let $D^+_r = \{K \in \mathcal{K}(\mathbb{R}) : d^+(0,K) = r\}$. Then D^+_r is Π^0_4.*

Proof. Let $r \in [0,1]$. For a measurable set $E \subset \mathbb{R}$ and $t \in [0,1]$ we have

$$\lim_{h\to 0^+} \frac{\mu(E\cap[0,h])}{h} = t \iff \lim_{n\to\infty} \frac{\mu(E\cap[0,1/n])}{1/n} = t.$$

Using this we obtain

$$D_r^+ = \{K \in \mathcal{K}(\mathbb{R}) : \forall \varepsilon > 0\ \exists N\ \forall n \geq N \quad r-\varepsilon \leq n\mu(K\cap[0,1/n]) < r+\varepsilon\} =$$

$$\bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \{K \in \mathcal{K}(\mathbb{R}) : r-\varepsilon \leq n\mu(K\cap[0,1/n]) < r+\varepsilon\}.$$

By Lemma 4 the set $\{K \in \mathcal{K}(\mathbb{R}) : r-\varepsilon \leq n\mu(K\cap[0,1/n]) < r+\varepsilon\}$ is Π_2^0. Hence D_r^+ is Π_4^0. □

Theorem 6. *D_1^+ is Π_3^0–complete.*

Proof. Observe that

$$D_1^+ = \{K \in \mathcal{K}(\mathbb{R}) : \forall \varepsilon > 0\ \exists h' > 0\ \forall h \in (0,h') \quad \mu([0,h]\cap K) \geq (1-\varepsilon)h\} =$$

$$\bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcup_{h'\in\mathbb{Q}_+} \bigcap_{h\in\mathbb{Q}_+\cap(0,h)} \{K \in \mathcal{K}(\mathbb{R}) : \mu([0,h]\cap K) \geq (1-\varepsilon)h\}.$$

By Lemma 4 the set D_1^+ is Π_3^0.

Put $b_n = \frac{1}{2^n}$ and $a_n^k = \frac{kb_{n+1}+b_n}{k+1}$. Define $F : \mathbb{N}^\mathbb{N} \to \mathcal{K}(\mathbb{R})$ by

$$F(\alpha) = \{0\} \cup \bigcup_{n\in\mathbb{N}} [a_n^{\alpha(n)}, b_n],\ \alpha \in \mathbb{N}^\mathbb{N}.$$

The argument for the continuity of F is the same as in the proof of Theorem 2 for F_r. Hence our proof will be finished if we show that $F(\alpha) \in D_1^+ \iff \alpha \in C_3$.

Fix $\alpha \in C_3$. Let $0 < \varepsilon < 1$ and pick $N \in \mathbb{N}$ be such that $\frac{N}{N+2} > 1-\varepsilon$. Then there is $k \in \mathbb{N}$ such that for all $n \geq k$ we have $\alpha(n) \geq N$. Notice that

$$\underline{d^+}(0, F(\alpha)) \geq \liminf_{k\to\infty} \left(\frac{1}{a_k^{\alpha(k)}} \sum_{n=k+1}^{\infty} \left(b_n - a_n^{\alpha(n)}\right) \right) =$$

$$\liminf_{k\to\infty} \left(\frac{\alpha(k)+1}{\alpha(k)b_{k+1}+b_k} \sum_{n=k+1}^{\infty} \left(b_n - \frac{\alpha(n)b_{n+1}+b_n}{\alpha(n)+1} \right) \right) =$$

$$\liminf_{k\to\infty}\left(\frac{\alpha(k)+1}{\alpha(k)2^{-k-1}+2^{-k}}\sum_{n=k+1}^{\infty}\frac{\alpha(n)(b_n-b_{n+1})}{\alpha(n)+1}\right)\geq$$

$$\liminf_{k\to\infty}\left(\frac{N+1}{N2^{-k-1}+2^{-k}}\cdot\frac{N}{N+1}\sum_{n=k+1}^{\infty}\frac{1}{2^{n+1}}\right)=\frac{N}{N+2}>1-\varepsilon.$$

Hence $F(\alpha)\in D_1^+$.

Let $\alpha\notin C_3$. There are a strictly increasing sequence $\{n_k\}$ of natural numbers and a number $N\in\mathbb{N}$ such that $\alpha(n_k)=N$ for all $k\in\mathbb{N}$. Then

$$\overline{d^+}(0,F(\alpha))\leq\limsup_{k\to\infty}\left(\frac{1}{b_{n_k}}\sum_{n=n_k}^{\infty}(b_n-a_n^{\alpha(n)})\right)=\limsup_{k\to\infty}\left(2^{n_k}\sum_{n=n_k}^{\infty}\frac{\alpha(n)}{\alpha(n)+1}\cdot\frac{1}{2^{n+1}}\right)\leq$$

$$\limsup_{k\to\infty}\left(2^{n_k}\left(\frac{N}{N+1}\cdot\frac{1}{2^{n_k+1}}+\sum_{n=n_k+1}^{\infty}\frac{1}{2^{n+1}}\right)\right)=\left(\frac{N}{N+1}+1\right)\frac{1}{2}<1.$$

Hence $F(\alpha)\notin D_1^+$. $\square$

Remark. All results of this section remain true if we consider $p^-(K,0)$ or $p(K,0)$ instead of $p^+(K,0)$, and $d^-(0,K)$ or $d(0,K)$ instead of $d^+(0,K)$.

3. NOWHERE BILATERALLY POROUS SETS

Let us return to the notion of porosity. Consider a subspace of Tr defined by

$$\widetilde{Tr}=\{T\in Tr:\forall s\in\mathbb{N}^{<\mathbb{N}}\ \ \forall n\in\mathbb{N}\quad(s\hat{\ }n\in T\Rightarrow\forall m\in\mathbb{N}\quad s\hat{\ }m\in T)\}=$$

$$\bigcap_{s\in\mathbb{N}^{<\mathbb{N}}}\bigcap_{n\in\mathbb{N}}(\{T\in Tr:s\hat{\ }n\notin T\}\cup\bigcap_{m\in\mathbb{N}}\{T\in Tr:s\hat{\ }m\in T\}).$$

Note that $\widetilde{Tr}$ is a Polish space as a closed subset of Tr. By $\widetilde{WF}$ we denote $WF\cap\widetilde{Tr}$, where WF is the set of all well-founded trees on $\mathbb{N}$. We have

$$T\text{ is well-founded}\iff T\cup\{s\hat{\ }m\in\mathbb{N}^{<\mathbb{N}}:(\exists n\in\mathbb{N}\ s\hat{\ }n\in T)\text{ and }m\in\mathbb{N}\}\in\widetilde{WF}.$$

It is well know that WF is Π_1^1–complete. The following map

$$T\mapsto T\cup\{s\hat{\ }m\in\mathbb{N}^{<\mathbb{N}}:(\exists n\in\mathbb{N}\ s\hat{\ }n\in T)\text{ and }m\in\mathbb{N}\}$$

is Borel as a pointwise limit of continuous maps $R_k : Tr \to \widetilde{Tr}$ defined by

$$R_k(T) = T \cup \{s\hat{\ }m \in \mathbb{N}^{<\mathbb{N}} : (\exists n \in \mathbb{N}\ s\hat{\ }n \in T) \text{ and } m < k\}.$$

Since the notions of Π^1_1–completeness and Borel Π^1_1–completeness coincide (see [3]), $\widetilde{WF}$ is a Π^1_1–complete subset of $\widetilde{Tr}$.

Now, we are ready to prove the following:

Theorem 7. *Let $NBP \subset \mathcal{K}(\mathbb{R})$ be of the form*

$$NBP = \{K \in \mathcal{K}(\mathbb{R}) : \forall x \in \mathbb{R}\ [x \in K \Rightarrow (p^-(K,x) = 0 \textit{ or } p^+(K,x) = 0)]\}.$$

Then NBP is Π^1_1–complete.

Proof. First we will show that NBP is coanalytic. Plainly NBP is co-projection of the set

$$\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : x \in K \Rightarrow (p^-(K,x) = 0 \text{ or } p^+(K,x) = 0)\}.$$

We need only to show that

$$A = \{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : p^+(K,x) = 0\}$$

is Borel. We have

$$A = \{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \limsup_{R \to 0^+} \frac{\lambda^+(x,R,K)}{R} = 0\} =$$

$$\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \forall \varepsilon > 0\ \exists R_0 > 0\ \forall R \in (0,R_0)\ \ \lambda^+(x,R,K) < \varepsilon R\} =$$

$$\bigcap_{\varepsilon \in \mathbb{Q}_+} \bigcup_{R_0 \in \mathbb{Q}_+} \bigcap_{R \in (0,R_0) \cap \mathbb{Q}} \{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \lambda^+(x,R,K) < \varepsilon R\}.$$

To finish the proof of Borelness, it is enough to check that

$$\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \lambda^+(x,R,K) < \varepsilon R\}$$

is open. Let (K,x) be such that $\lambda^+(x,R,K) < \varepsilon R$, for fixed positive rational numbers R and ε. Put $\delta = \varepsilon R - \lambda^+(x,R,K)$. Using the compactness of K, pick a family $\{U_1,...,U_k\}$ of open intervals, with diameters less than $\delta/3$, such that

$$K \subset \bigcup_{i=1}^{k} U_i \text{ and } K \cap U_i \neq \emptyset \text{ for } i = 1,...,k.$$

Let

$$\mathcal{V} = \{L \in \mathcal{K}(\mathbb{R}) : L \cap U_i \neq \emptyset \text{ for } i = 1,..,k\}.$$

This is an open neighbourhood of K. Let $(L,y) \in \mathcal{V} \times (x-\delta/3, x+\delta/3)$. Then

$$\lambda^+(y,R,L) \leq \lambda^+(x,R,K) + \frac{2}{3}\delta < \varepsilon R.$$

We have shown that $\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \lambda^+(x,R,K) < \varepsilon R\}$ is open. Hence NBP is Π^1_1. For $a,b \in \mathbb{R}$ such that $a < b$, let

$$\phi_{[a,b]}(x) = \phi_{(a,b)}(x) = a + (b-a)x, \text{ for } x \in \mathbb{R}.$$

This is an affine function which maps $[0,1]$ onto $[a,b]$. Let $K_\emptyset = \emptyset$ and $L_\emptyset = [0,1]$. For $n,m \in \mathbb{N}$ let

$$K_n = \phi_{[\frac{1}{3},\frac{2}{3}]}\left(\left[\frac{1}{2n+2}, \frac{1}{2n+1}\right]\right),\ L_n = \phi_{[\frac{1}{3},\frac{2}{3}]}\left(\left(\frac{1}{2n+3}, \frac{1}{2n+2}\right)\right).$$

For $s \in \mathbb{N}^{<\mathbb{N}}$ and $m \in \mathbb{N}$ we define inductively

$$K_{s\hat{\ }m} = \phi_{L_s}(K_m),\ L_{s\hat{\ }m} = \phi_{L_s}(L_m).$$

Let $T \in \widetilde{Tr}$. Then the mapping

$$T \mapsto cl(\bigcup_{s\in T} K_s) \in \mathcal{K}(\mathbb{R})$$

is continuous. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ let x_α be the unique point of $\bigcap_{n\in\mathbb{N}} L_{\alpha|n}$. Then x_α is a limit of any sequence (x_n) such that $x_n \in K_{\alpha|n}$ for all n. For $s \in \mathbb{N}^{<\mathbb{N}}$ let $y_s = \inf L_s + \frac{1}{3}(\sup L_s - \inf L_s)$. Then y_s is a limit of any sequence (y_n) such that $y_n \in K_{s\hat{\ }n}$ for all n.

To prove the assertion, it suffices to show that $T \in \widetilde{WF} \iff cl(\bigcup_{s\in T} K_s) \in NBP$. This is exactly the reduction of the Π_1^1–complete set $\widetilde{WF}$ to NBP by a continuous function.

Suppose that $T \in \widetilde{WF}$ and let $x \in cl(\bigcup_{s\in T} K_s)$. If $x \in K_s$ for some $s \in T$, then x can not be both left-hand and right-hand porosity point of $cl(\bigcup_{s\in T} K_s)$. If $x \notin K_s$ for all $s \in T$, then $x = y_s$ for some $s \in T$. Indeed, suppose that $x \neq y_s$ for all $s \in T$. There exists a sequence $y_{s_n} \to x$ with $y_{s_n} \in K_{s_n}$ and $s_n \in T$. From $x \neq y_{\langle\rangle}$ (where $\langle\rangle$ stands for the empty sequence) it follows that $\{s_n(0)\}_{n\in\mathbb{N}}$ is bounded. Hence there is k_0 such that $\{n \in \mathbb{N} : s_n(0) = k_0\}$ is infinite. Proceeding inductively one can find a sequence $\alpha = (k_0, k_1, k_2, ...)$ such that $\alpha|n \in T$ for all $n \in \mathbb{N}$, which yields a contradiction.

Let $R > 0$ and let $n \in \mathbb{N}$ be the first number such that $L_{s\hat{\ }n} \subset (x, x + R)$. Then $\lambda^+(x, R, cl(\bigcup_{s\in T} K_s)) \leq \frac{1}{n+1}R$ and $p^+(cl(\bigcup_{s\in T} K_s), x) = 0$. Hence $cl(\bigcup_{s\in T} K_s) \in NBP$.

For $a, b \in \mathbb{R}$ such that $a < b$, the intervals $(a, \frac{2a+b}{3})$, $(\frac{2a+b}{3}, \frac{a+2b}{3})$, $(\frac{a+2b}{3}, b)$ will be called the left, the central and the right subintervals of (a, b), respectively. Suppose that $T \notin \widetilde{WF}$. Then the body $[T]$ of T is nonempty. Let $\alpha \in [T]$. Then $\alpha|n \in T$ for all $n \in \mathbb{N}$, and $x_\alpha \in cl(\bigcup_{s\in T} K_s)$. For every $n \in \mathbb{N}$ the point x_α is in the central subinterval of $L_{\alpha|n}$ with the length $\frac{1}{3}|L_{\alpha|n}|$. Since both the left and the right subintervals are disjoint with $cl(\bigcup_{s\in T} K_s)$, we have $\lambda^\pm(x, \frac{2}{3}|L_{\alpha|n}|, cl(\bigcup_{s\in T} K_s)) > \frac{1}{2}$ and thus $p^\pm(cl(\bigcup_{s\in T} K_s), x) \geq \frac{1}{2}$. Hence $cl(\bigcup_{s\in T} K_s) \notin NBP$. $\Box$

Remark. One can slightly modify this proof to show that the set

$$\{K \in \mathcal{K}(\mathbb{R}) : K \text{ is not bilaterally strongly porous at } x, \text{ for all } x \in K\}$$

is Π_1^1–complete.

Given three sets A, B and C in the same Polish space, we say that C separates A and B if $A \subset C$ and $B \cap C = \emptyset$. A pair of disjoint coanalytic sets which cannot be separated by any Borel set is called Borel–inseparable (see [1]).

Corollary 8. *Let NBP' be the family of all compact sets K in $\mathcal{K}(\mathbb{R})$ that are bilaterally porous at exactly one point $x \in K$. Then NBP and NBP' is a Borel–inseparable pair of coanalytic sets.*

Proof. Let UB be the set of all trees on $\mathbb{N}$ with a unique infinite branch. It is known that WF and UB is a Borel–inseparable pair of coanalytic sets (see [4, Exercise 35.2]). Note that

$$T \in \widetilde{WF} \iff cl(\bigcup_{s \in T} K_s) \in NBP \text{ and } T \in \widetilde{UB} \iff cl(\bigcup_{s \in T} K_s) \in NBP',$$

where $\widetilde{UB} = UB \cap \widetilde{Tr}$. Hence NBP and NBP' are Borel–inseparable. □

The set NBP introduced in Theorem 7 consists of nowhere bilaterally porous compact sets. These sets have rather "large local size". Another variant of such sets constructed by operator of density is described by the family

$$OSD = \{K \in \mathcal{K}(\mathbb{R}) : \forall x \in K \ (d^-(x,K) = 1 \text{ or } d^+(x,K) = 1)\}.$$

Thus $K \in OSD$ iff every point of K is its at least one-sided density point.

Let (a_n) and (b_n) be sequences of real numbers such that $a_n < b_n < a_{n-1}$, $a_n \to 0$ and $d^+(0, \bigcup_{n+1}^{\infty}[a_n,b_n]) = 1$. If in the proof of Theorem 8 we put

$$K_n = \phi_{[\frac{1}{3},\frac{2}{3}]}([a_n,b_n]) \text{ and } L_n = \phi_{[\frac{1}{3},\frac{2}{3}]}((b_{n+1},a_n)),$$

then we can obtain the following counterpart of Theorem 7 for the operator of density:

Theorem 9. *The set OSD is Π^1_1–complete.*

At the first look, it is not obvious that sets NBP and OSD are different. To see this, observe that $p^+(\{1/n : n \in \mathbb{N}, \ n > 0\}, 0) = 0$. For every $n \geq 1$, let $[a_n,b_n]$ be an interval such that $(a_n+b_n)/2 = \frac{1}{n}$ and $d^+(0, \bigcup_{n=1}^{\infty}[a_n,b_n]) = 0$. Then $\bigcup_{n=1}^{\infty}[a_n,b_n] \cup \{0\} \in NBP \setminus OSD$.

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