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LOCAL AND GLOBAL MONOTONICITY

Abstract

We give characterizations of sets $E \subset [0,1]$ for which the local monotonicity of each function $f : [0,1] \to \mathbb{R}$ from a given class $\mathcal{F}$, at all points $x \in E$, implies the global monotonicity of f on $[0,1]$. We consider as $\mathcal{F}$ – the families of continuous functions, differentiable functions, absolutely continuous functions, functions of class C^n ($n = 1, 2, ..., \infty$), real analytic functions and polynomials.

We shall consider real-valued functions defined on $[0,1]$. However, all our results remain true when $[0,1]$ is replaced by any interval J. This remark will be used without comments in some of our proofs. Let us recall the notions of local monotonicity for real functions defined on $[0,1]$. They can be found in Bruckner's monograph [1]. As we know, they were introduced by E. Borel.

A function $f : [0,1] \to \mathbb{R}$ is called left non-decreasing (LND) at a point $x \in (0,1]$ if

$$\exists_{\delta>0} \forall_{y \in (x-\delta, x)} f(y) \le f(x).$$

Analogously we mean a right non-decreasing (RND) function at $x \in [0,1)$. A function $f : [0,1] \to \mathbb{R}$ is called non-decreasing (ND) at $x \in (0,1)$ if it is LND and RND at x. The following theorem shows the connections between local and global monotonicity of real functions.

Theorem 1. *(J. Jachymski [3]) Let $f : [0,1] \to \mathbb{R}$. The following conditions are equivalent:*

(i) f is non-decreasing;

(ii) $\forall_{x \in [0,1]}$(f is ND at x);

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(iii) $\forall_{x\in(0,1]}(f$ *is LND at* $x)$ *and*
$\forall_{x\in[0,1)}(f$ *is right lower-semicontinuous at* $x)$*.*

(iv) $\forall_{x\in[0,1)}(f$ *is RND at* $x)$ *and*
$\forall_{x\in(0,1]}(f$ *is left upper-semicontinuous at* $x)$*.*

Proof. $(i)\Rightarrow(ii)$ obvious.

$(ii)\Rightarrow(iii)$ If f is RND at x, then $f(x)\le\liminf_{y\to x^+}f(y)$. This means that f is right lower-semicontinuous at x.

$(iii)\Rightarrow(i)$ Let $0\le c<d\le 1$, $M=\{x<d:\forall_{t\in[x,d]}f(t)\le f(d)\}$ and $m=\inf M$. We now show that $m\in M$. Let $\{s_n\}$ be a sequence in M such that $s_n\to m$. We have

$$f(m)\le\liminf_{x\to m^+}f(x)\le\liminf_{n\to\infty}f(s_n)\le f(d),$$

so $m\in M$. Suppose that $m>0$. Since f is LND at m, there exists $\delta>0$ such that $f(t)\le f(m)$ for all $t\in(m-\delta,m)$ which together with $m\in M$ contradicts the definition of m. Thus $m=0$ which shows that $f(c)\le f(d)$.

$(ii)\Rightarrow(iv)$ goes similarly as $(ii)\Rightarrow(iii)$.

$(iv)\Rightarrow(i)$ goes similarly as $(iii)\Rightarrow(i)$. □

By $C([0,1])$ we denote the set of all continuous functions on $[0,1]$.

Corollary 1. *Let* $f\in C([0,1])$*. The following conditions are equivalent:*

(i) $\forall_{x\in(0,1]}(f$ *is LND at* $x)$*;*

(ii) f *is non-decreasing.*

Easy examples witness that in Theorem 1 we cannot weaken conditions (ii) and (iii) for every function $f:[0,1]\to\mathbb{R}$ in such a way that we consider "$\forall_{x\in E}$" instead of "$\forall_{x\in[0,1]}$" for a proper subset E of $[0,1]$. The aim of this paper is to study how small can be a set $E\subset[0,1]$ to make implication "$(\forall_{x\in E}(f$ is ND at $x))\Rightarrow f$ is non-decreasing" true for every function f from a particular class. First we consider the class $C([0,1])$.

Lemma 1. *Let* $f:[0,1]\to\mathbb{R}$ *and* $x\in(0,1)$*. If* f *is continuous and ND at any point of the set* $[0,1]\setminus\{x\}$*, then* f *is non-decreasing.*

Proof. By Theorem 1, f is non-decreasing in intervals $[0,x)$ and $(x,1]$. Since f is continuous at x, we have

$$\forall_{t<x}\forall_{s>x}(f(t)\le f(x)\le f(s)).$$ □

Lemma 2. *Let* $f\in C([0,1])$*. The following conditions are equivalent:*

(i) f is non-decreasing;

(ii) $\forall_{x\in[0,1]}$ (f is LND at x or f is RND at x).

PROOF. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Suppose that f is not non-decreasing. Then there exist numbers a_0 and b_0 such that $0 \le a_0 < b_0 \le 1$ and $f(a_0) > f(b_0)$. The Darboux property applied to f implies that there exist numbers c and d such that $a_0 < c < d < b_0$ and

$$f(c) = \frac{2f(a_0) + f(b_0)}{3},\ f(d) = \frac{f(a_0) + 2f(b_0)}{3}$$

and there exists $g \in (c, d)$ such that $f(g) = \frac{f(c)+f(d)}{2}$. If $g - c \le d - g$, then we take $a_1 = c$ and $b_1 = g$, otherwise $a_1 = g$ and $b_1 = d$. So $0 < b_1 - a_1 \le \frac{d-c}{2} < \frac{b_0-a_0}{2} \le \frac{1}{2}$.

Proceeding inductively we get strictly increasing sequences $\{a_n\}$, $\{f(b_n)\}$ and strictly decreasing sequences $\{b_n\}$, $\{f(a_n)\}$. Furthermore $0 < b_n - a_n < \frac{1}{2^n}$ for all n. Let $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Suppose for instance that f is LND at x. Then

$$\exists_{\delta>0}\forall_{y\in(x-\delta,x)}(f(y) \le f(x)).$$

But for large enough $n \in \mathbb{N}$ we have $f(a_n) > f(x)$ and $a_n \in (x - \delta, x)$. This is a contradiction. Hence f is non-decreasing. □

Lemma 3. *Let $f \in C([0, 1])$ and let $E \subset [0, 1]$ be countable and G_δ. If*

$$\forall_{x\in[0,1]\setminus E}(f \textit{ is LND at } x \textit{ or } f \textit{ is RND at } x),$$

then f is non-decreasing.

PROOF. Since E is countable and G_δ, there exists an ordinal $\alpha < \omega_1$ such that $(*)$ $E^\alpha = \emptyset$ (the αth Cantor–Bendixson derivative of E is empty).
This is an easy exercise. See e.g. [4, 2.5.14]. We shall show inductively that f is RND or LND at any point of the set $E \setminus E^\beta$ for $\beta \le \alpha$.

Let $x \in E \setminus E^1$; i.e., x is an isolated point of E. There exists $\varepsilon > 0$ such that $E \cap (x - \varepsilon, x + \varepsilon) = \{x\}$. So

$$\forall_{y\in(x-\varepsilon,x)}(f \text{ is LND at } x \text{ or } f \text{ is RND at } x).$$

Applying Lemma 2 we obtain that $f|_{(x-\varepsilon,x)}$ is non-decreasing. Similarly $f|_{(x,x+\varepsilon)}$ is non-decreasing. By Lemma 1, we get that $f|_{(x-\varepsilon,x+\varepsilon)}$ is non-decreasing. Hence, f is ND at x. From these observations we get

$$\forall_{x\in[0,1]\setminus E^1}(f \text{ is LND at } x \text{ or } f \text{ is RND at } x).$$

Let $\beta < \alpha$. Suppose that

$$\forall_{\gamma<\beta}\forall_{x\in[0,1]\setminus E^\gamma}(f \text{ is LND at } x \text{ or } f \text{ is RND at } x).$$

If β is a limit ordinal, then it follows that f is LND or RND at any point of the set $\bigcup_{\gamma<\beta}([0,1)\setminus E^\gamma) = [0,1]\setminus\bigcap_{\gamma<\beta}E^\gamma = [0,1]\setminus E^\beta$. If β is a successor ordinal, then there exists ξ such that $\beta = \xi+1$. With isolated points of E^ξ, we repeat the same reasoning as in the first step of induction.

By $(*)$ we have

$$\forall_{x\in[0,1]}(f \text{ is LND at } x \text{ or } f \text{ is RND at } x).$$

The assertion follows from Lemma 2. □

Theorem 2. *Let $E\subset[0,1]$. The following conditions are equivalent:*

(i) $\forall_{f\in C([0,1])}[(\forall_{x\in E}(f$ *is LND at* x *or* f *is RND at* $x)) \Rightarrow$ f *is non-decreasing*];

(ii) The set $[0,1]\setminus E$ *does not contain a homeomorph of the Cantor set.*

PROOF. $(ii)\Rightarrow(i)$ Fix $f\in C([0,1])$. Denote by $\mathbb{Q}^+$ the positive rationals. Let

$$A = \{x\in[0,1] : f \text{ is LND at } x \text{ or } f \text{ is RND at } x\}.$$

Then

$$A = \{x\in[0,1] : \exists_{\delta_x>0}\forall_{y\in(x-\delta_x,x)}f(y)\le f(x)\}$$

$$\cup\{x\in[0,1] : \exists_{\delta_x>0}\forall_{y\in(x,x+\delta_x)}f(y)\ge f(x)\}$$

$$= \{x\in[0,1] : \exists_{\delta\in\mathbb{Q}^+}\forall_{y\in(x-\delta_x,x)}f(y)\le f(x)\}$$

$$\cup\{x\in[0,1] : \exists_{\delta\in\mathbb{Q}^+}\forall_{y\in(x,x+\delta_x)}f(y)\ge f(x)\},$$

so A is F_σ. Since $[0,1]\setminus A\subset[0,1]\setminus E$, then $[0,1]\setminus A$ does not contain a homeomorph of the Cantor set. Hence $[0,1]\setminus A$ is countable. By Lemma 3 we conclude that f is non-decreasing.

$(i)\Rightarrow(ii)$ Assume that $[0,1]\setminus E$ contains a homeomorph of the Cantor set, say K. Let $f:[0,1]\to\mathbb{R}$ be a Cantor-type function associated with the set K such that $f(0)=1$ and $f(1)=0$. Since f is constant on each component of $[0,1]\setminus K$, it is ND at any point of $[0,1]\setminus K$, but it is not non-decreasing. □

Lemma 4. *Let $f \in C([0,1])$ and let $E \subset (0,1)$ be dense. If*

$$\forall_{x\in E}(f \text{ is ND at } x),$$

then there exists a set $U \supset E$ open in $(0,1)$ such that for each component (a,b) of U we have $f(a) \le f(b)$.

PROOF. Put $U = \bigcup_{x\in E}(x-\delta_x, x+\delta_x)$ where $\delta_x > 0$ is a number obtained from the definition of ND at the point x. Let (a,b) be a component of U. Consider two sequences $\{a_n\}$, $\{b_n\}$ such that $a < a_n < b_n < b$, for all $n \in \mathbb{N}$ and $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Fix $n \in \mathbb{N}$. The family $\{(x-\delta_x, x+\delta_x) : x \in E \cap (a,b)\}$ is an open covering of $[a_n, b_n]$. There exists a finite subcovering $\{(x_k^n - \delta_k^n, x_k^n + \delta_k^n) : k = 0, 1, ..., k_n\}$ such that $x_0^n < x_1^n < ... < x_{k_n}^n$ and $\delta_k^n = \delta_{x_k^n}$. If necessary we remove each interval contained in any other interval in the subcovering. For $k = 0, 1, ..., k_n$ fix $y_k^n \in (x_{k-1}^n, x_{k-1}^n + \delta_{k-1}^n) \cap (x_k^n - \delta_k^n, x_k^n)$ such that $x_{k-1}^n < y_k^n < x_k^n$. Then $f(x_{k-1}^n) \le f(y_k^n) \le f(x_k^n)$ for $k = 1, ..., k_n$. Hence $f(x_0^n - \delta_0^n) \le f(x_{k_n}^n + \delta_{k_n}^n)$. Since $\lim_{n\to\infty}(x_0^n - \delta_0^n) = a$, $\lim_{n\to\infty}(x_{k_n}^n + \delta_{k_n}^n) = b$, and f is continuous we have $f(a) \le f(b)$. $\square$

Let μ stand for Lebesgue measure on $\mathbb{R}$.

Lemma 5. *Assume that $F \in C([0,1])$ and F satisfies condition (N) of Luzin. Let $E \subset [0,1]$ be such that $\mu([0,1] \setminus E) = 0$. If*

$$\forall_{x\in E}(F \text{ is ND at } x),$$

then F is non-decreasing.

PROOF. Suppose that F is not non-decreasing. There exist numbers a and b such that $0 \le a < b \le 1$ and $F(a) > F(b)$. By Lemma 4 applied to $E \cap (a,b)$ and $[a,b]$ in place of E and $[0,1]$, there exists a sequence of pairwise disjoint intervals $(a_n, b_n) \subset [a,b]$, $n \in \mathbb{N}$, such that $F(a_n) \le F(b_n)$ and $\mu([a,b] \setminus \bigcup_n (a_n, b_n)) = 0$. We shall prove that $[F(b), F(a)] \subset F([a,b] \setminus \bigcup_{i=1}^n (a_i, b_i))$ for all n.

The Darboux property of F shows that for every y between $F(a)$ and $F(a_1)$ there is $x \in (a, a_1)$ such that $F(x) = y$, and for every y between $F(b)$ and $F(b_1)$ there is $x \in (b_1, b)$ such that $F(x) = y$. But $F(a_1) \le F(b_1)$ so $[F(b), F(a)] \subset F([a,b] \setminus (a_1, b_1))$. Since (a_i, b_i) are pairwise disjoint intervals, either $(a_2, b_2) \subset (a, a_1)$ or $(a_2, b_2) \subset (b_1, b)$. Assume that $(a_2, b_2) \subset (a, a_1)$. Similarly as above we show that for every y between $F(a)$ and $F(a_1)$ there is $x \in (a, a_1) \setminus (a_2, b_2)$ such that $F(x) = y$. So $[F(b), F(a)] \subset F([a,b] \setminus \bigcup_{i=1}^2 (a_i, b_i))$.

Proceeding inductively we have $[F(b),F(a)] \subset F([a,b] \setminus \bigcup_{i=1}^{n}(a_i,b_i))$ for all n. Since $A_n = [0,1] \setminus \bigcup_{i=1}^{n}(a_i,b_i)$, $n \in \mathbb{N}$, is decreasing sequence of compact sets, we have $\bigcap_n F(A_n) = F(\bigcap_n A_n)$. Consequently $[F(b),F(a)] \subset F([a,b] \setminus \bigcup_n (a_n,b_n))$. But $\mu([a,b] \setminus \bigcup_n (a_n,b_n)) = 0$, so F does not satisfy the condition (N) of Luzin, a contradiction. □

Theorem 3. *Assume that $\mathcal{F}$ is one of the following classes of functions on $[0,1]$: continuous functions satisfying condition (N), absolutely continuous functions, differentiable functions, Lipschitz functions. Let $E \subset [0,1]$. The following conditions are equivalent:*

(i) $\forall_{F \in \mathcal{F}}[(\forall_{x \in E}(F \textit{ is ND at } x)) \Rightarrow F \textit{ is non-decreasing}]$*;*

(ii) The set $[0,1] \setminus E$ does not contain any set of positive measure.

Proof. $(ii) \Rightarrow (i)$ Observe that the set $A = \{x \in [0,1] : F \text{ is ND at } x\}$ is F_σ. So A is measurable and $\mu([0,1] \setminus A) = 0$. Now (i) follows from Lemma 5; note that functions from $\mathcal{F}$ satisfy (N) (see e.g. [2, Thm. 6.12, Lemma 6.14]).

$(i) \Rightarrow (ii)$ Suppose that $[0,1] \setminus E$ contains a set of positive measure. So $[0,1] \setminus E$ contains a closed set D of positive measure. Then $E \subset [0,1] \setminus D$ and $[0,1] \setminus D = \bigcup_n U_n$, where U_n are pairwise disjoint intervals which are open sets in $[0,1]$. We shall construct a differentiable function F satisfying Lipschitz condition, which is ND at any point of $[0,1] \setminus D$ but is not non-decreasing. This will yield a contradiction.

Let $H \subset D$ be a set of type F_σ of points of density of D, with $\mu(H) = \mu(D)$. By [1, Thm. 6.5, p. 22], we can find an approximately continuous function $g : [0,1] \to \mathbb{R}$ which takes values from $(0,1]$ on H and which is 0 outside of H. Put $f = g - 1$. Pick a real α such that $\int_0^1 f < \alpha < 1$. For $x \in [0,1]$ put

$$F(x) = \int_0^x (f(y) - \alpha)\, dy.$$

Then F is differentiable [2, Thm. 14.8] and F satisfies Lipschitz condition. Additionally, $F(0) = 0$, $F(1) \leq \mu([0,1] \setminus D) - \alpha < 0$ and $F|_{U_n}$ is non-decreasing for every n. □

Theorem 4. *Assume that $\mathcal{F}$ is one of the following classes of real functions on $[0,1]$: functions of class $C^{(n)}$ $(n = 1,2,...,\infty)$, analytic functions, polynomials. Let $E \subset [0,1]$. The following conditions are equivalent:*

(i) $\forall_{f \in \mathcal{F}}[(\forall_{x \in E}(F \textit{ is ND at } x)) \Rightarrow F \textit{ is non-decreasing}]$*;*

(ii) the set E is dense in $[0,1]$.

PROOF. $(ii) \Rightarrow (i)$ It suffices to show that $f'(x) \geq 0$ for all $x \in (0,1)$.

Suppose that there exists a point $x \in (0,1)$ such that $f'(x) < 0$. There is an open interval $U = (x-\delta, x+\delta)$ such that $f'(y) < 0$ for all $y \in U$. Hence f is decreasing on U and f cannot be ND at any point of U. This violates (ii), since $U \cap E = \emptyset$ is a contradiction.

$(i) \Rightarrow (ii)$ Suppose that E is not dense. There exists an interval $[a,b] \subset [0,1]$ such that $E \cap [a,b] = \emptyset$. Consider a function $f : [0,1] \to \mathbb{R}$ defined as an antiderivative of $(x-a)(x-b)$. Then f is ND at all points of E but is not non-decreasing. This is a contradiction. □

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