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MEASURE-CATEGORY PROPERTIES OF BOREL PLANE SETS AND BOREL FUNCTIONS OF TWO VARIABLES

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Abstract. Let $\mathcal{I}\otimes\mathcal{J}$ stand for the Fubini-type product of σ-ideals $\mathcal{I},\mathcal{J}\subset\mathcal{P}(\mathbb{R})$. We consider mixed measure-category σ-ideals $\mathcal{M}\otimes\mathcal{N}$ and $\mathcal{N}\otimes\mathcal{M}$ (called the Mendez σ-ideals), and study some features of their structure. We show that $\mathcal{M}\otimes\mathcal{N}$ and $\mathcal{N}\otimes\mathcal{M}$ are not invariant with respect to nonzero rotations. Using Fremlin's results, we describe nice Borel bases of $\mathcal{M}\otimes\mathcal{N}$, $\mathcal{N}\otimes\mathcal{M}$, $\{\emptyset\}\otimes\mathcal{N}$ and $\{\emptyset\}\otimes\mathcal{M}$. The rest of the paper is devoted to uniform versions of the Nikodym theorem and the Lusin theorem for Borel functions of two variables.

1. Some σ-ideals on the plane

The monograph of Oxtoby [15] presents several similarities and differences between two families of small subsets of the real line that form σ-ideals: the family $\mathcal{N}$ of Lebesgue null sets (i.e. sets of Lebesgue measure zero) and the family $\mathcal{M}$ of meager sets (i.e. sets of the first Baire category). Oxtoby continues these studies for the σ-algebras associated with $\mathcal{N}$ and $\mathcal{M}$ (they consist of Lebesgue measurable sets and of sets with the Baire property, respectively) and real-valued functions measurable with respect to these σ-algebras. The analogous investigations can be conducted for subsets of $\mathbb{R}^2$ or $\mathbb{R}^k$ with $k>2$. Interesting properties arise when one examines small sections of small plane sets – this leads to the theorems of Fubini and of Kuratowski and Ulam.

A new idea appears when one considers plane sets whose almost all sections in one sense are small. Families (in fact σ-ideals) of such plane sets fulfilling the "Fubini-type mixed condition" were investigated by Mendez [13], [14]. These objects have some interesting properties and applications; see e.g. [6], [7], [8], [9], [2], [3], [4]. We show that detailed studies of the structure of the Mendez σ-ideals yield new properties of plane Borel sets and

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Borel functions of two variables. Some results of [8] and [1] are included, we derive from them several consequences. We also consider nice bases of the σ-ideals of plane sets whose all sections are meager (of measure zero), and we establish related consequences for Borel functions.

Let $\mathbb{N} = \{1, 2, \dots\}$. By $\triangle$ we denote the operation of symmetric difference of sets, and by λ (respectively, λ_k) – the one-dimensional (k-dimensional) Lebesgue measure. Throghout the paper X will stand for $\mathbb{R}$ or any non-degenerate subinterval of $\mathbb{R}$. The σ-algebra of all Borel subsets of X^2 will be written as $\mathcal{B}$. For a σ-ideal $\mathcal{J} \subset \mathcal{P}(X)$ and a set $B \subset X^2$, we denote $\Phi_{\mathcal{J}}(B) := \{ x \in X : B(x) \notin \mathcal{J} \}$ where $B(x) := \{ y \in X : (x, y) \in B \}$ is the section of B generated by $x \in X$. For a function $f : X^2 \to \mathbb{R}$ and $x \in X$, we put $f_x(y) := f(x, y)$, $y \in X$. If $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(X)$ are σ-ideals, we define

$$\mathcal{I} \otimes \mathcal{J} := \left\{ A \subset X^2 : (\exists B \in \mathcal{B}) \big(A \subset B \text{ and } \Phi_{\mathcal{J}}(B) \in \mathcal{I} \big) \right\}.$$

This is a σ-ideal of subsets of X^2. In particular, $\mathcal{N} \otimes \mathcal{N}$ and $\mathcal{M} \otimes \mathcal{M}$ produce exactly the σ-ideals of Lebesgue null sets and of meager sets in X^2, cf. [15, Chs. 14 and 15]. If we do not use a Borel cover B in the definition of $\mathcal{I} \otimes \mathcal{J}$, some patological sets appear in $\mathcal{N} \otimes \mathcal{N}$ and in $\mathcal{M} \otimes \mathcal{M}$ (cf. [15, Theorems 14.4 and 15.5], [17, Section 4]), and thus we would obtain essentially larger σ-ideals. The mixed product σ-ideals $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ will be called *the Mendez σ-ideals*. We also study the σ-ideals $\mathcal{N}^* := \{\emptyset\} \otimes \mathcal{N}$ and $\mathcal{M}^* := \{\emptyset\} \otimes \mathcal{M}$; they were considered for instance in [6].

Let us quote two important properties of the Mendez σ-ideals. From the first one it follows that these σ-ideals are mutually incomparable (with respect to inclusion) and also incomparable with $\mathcal{M} \otimes \mathcal{M}$ and $\mathcal{N} \otimes \mathcal{N}$. Namely, Mendez [13] observed that if A, B are disjoint Borel sets such that $A \cup B = X$, $A \in \mathcal{M}$, $B \in \mathcal{N}$ (the celebrated decomposition of the interval X into two small sets, cf. [15, Theorem 1.6]) then $A \times X \in \mathcal{M} \otimes \mathcal{N}$, $B \times X \in \mathcal{N} \otimes \mathcal{M}$ and $C \in (\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$, $D \in (\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ where $C := (A \times B) \cup (B \times A)$, $D := (A \times A) \cup (B \times B)$. This yields the new decompositions $(A \times X) \cup (B \times X)$ and $C \cup D$ of X^2 into two small sets. The second property is due to Fremlin [7, 8G(a)] and it says that any disjoint family of Borel subsets of X^2 that are not in $\mathcal{N} \otimes \mathcal{M}$ (respectively, $\mathcal{M} \otimes \mathcal{N}$) is countable. This is the so called *countable chain condition* useful in several kinds of investigations.

We will show that the Mendez σ-ideals on $\mathbb{R}^2$ are not invariant under nonzero rotations. This answers a question of T. Natkaniec (oral communication). In fact we will obtain a more general result.

THEOREM 1. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals of subsets of $\mathbb{R}$ such that $\mathcal{I}$ is invariant with respect to affine mappings, $\mathcal{J}$ is invariant with respect to translations, and there is a Borel set in $\mathcal{I} \setminus \mathcal{J}$. Then for each $\alpha \in (0, \pi/2]$ there is a Borel* *set* $E \in \mathcal{I} \otimes \mathcal{J}$ *such that* $\varphi_\alpha(E) \notin \mathcal{I} \otimes \mathcal{J}$ *where* φ_α *stands for the rotation of* $\mathbb{R}^2$ *by angle* α *around the point* $(0, 0)$.

PROOF. Fix a Borel set $D \in \mathcal{I} \setminus \mathcal{J}$ and an angle $\alpha \in (0, \pi/2]$. Let $D^* := \mathrm{pr}_1\big(\varphi_{-\alpha}(\{0\} \times D)\big)$ where pr_1 is the projection onto the first axis. If $\alpha = \pi/2$ then $D^* = D$, and if $\alpha \in (0, \pi/2)$ then D^* is the image of D under the respective affine function from $\mathbb{R}$ to $\mathbb{R}$. Hence D^* is Borel and $D^* \in \mathcal{I}$. Put $E := D^* \times \mathbb{R}$. Then $E \in \mathcal{I} \otimes \mathcal{J}$. Consider $E^* := \varphi_\alpha(E)$. Observe that E^* is the union of all straight lines parallel to $y = x \tan(\pi/2 - \alpha)$ and intersecting $\{0\} \times D$. Hence each section $E^*(x)$, $x \in \mathbb{R}$, is of the form $D + t$ for some $t \in \mathbb{R}$. So, by the assumption, $E^*(x) \notin \mathcal{J}$. Hence $E^* \notin \mathcal{I} \otimes \mathcal{J}$. $\square$

In Section 2 we give a survey of regularity properties of σ-ideals $\mathcal{N} \otimes \mathcal{M}$, $\mathcal{M}^*$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N}^*$. We apply them (in Sections 3 and 4) to discuss uniform and almost-uniform versions of Nikodym-type and Lusin-type theorems for sections of a Borel function of two variables.

2. Regularity properties

In this section we collect some known results concerning the structure of $\mathcal{M}^*$ and $\mathcal{N}^*$, and of the Mendez σ-ideals. It is a simple observation that every Lebesgue null set is contained in a Lebesgue null set of type G_δ, and every meager set is contained in a meager set of type F_σ. For a σ-ideal $\mathcal{I}$, a subfamily $\mathcal{F} \subset \mathcal{I}$ is called a *base* of $\mathcal{I}$ if for every $A \in \mathcal{I}$ we can find $B \in \mathcal{F}$ such that $A \subset B$. By definition, every set A from $\mathcal{N} \otimes \mathcal{M}$ is contained in a Borel set B from this σ-ideal. But of which (possibly low) Borel level is B? This will be derived from Proposition 2. Thus we will describe a nice Borel base of $\mathcal{N} \otimes \mathcal{M}$. In general, by a *nice base* of a σ-ideal $\mathcal{I}$ on the plane we mean a base of $\mathcal{I}$ consisting of Borel sets of possibly low level or consisting of Borel sets whose sections are of possibly low level.

We need the following result due to Fremlin. By $\mathrm{cl}(\cdot)$ we denote the closure operation in X^2.

PROPOSITION 2 (cf. [8], 3F). *The* σ*-ideal* $\mathcal{N} \otimes \mathcal{M}$ *is generated by the family*

$$\{C \times X : C \in \mathcal{N}\} \cup \big\{D \subset X^2 : (\forall x \in X)(\mathrm{cl}\, D)(x) \text{ is nowhere dense}\big\}.$$

COROLLARY 3. *For each set* $A \in \mathcal{N} \otimes \mathcal{M}$ *there exist a set* $C \in \mathcal{N}$ *(of type* G_δ*) and an* F_σ *set* $F \subset X^2$ *with meager sections such that* $A \subset (C \times X) \cup F$.

Another question is whether a Borel subset of X^2 can differ (with respect to the symmetric difference) from a Borel set, of a possibly low Borel level, by a set from $\mathcal{N} \otimes \mathcal{M}$. (This question, in the case of the difference by a null set or by a meager set, has an easy answer, via the known characterization of the Lebesgue measurable sets and the definition of sets with the Baire property, cf. [15, Chs. 3 and 4].)

PROPOSITION 4 [1, Proposition 2.1]. *For every Borel set $B \subset X^2$ and a fixed base $(U_n)_{n\in\mathbb{N}}$ of open subsets of X, there is a sequence $(F_n)_{n\in\mathbb{N}}$ of F_σ subsets of X such that $B \triangle \bigcup_{n\in\mathbb{N}}(F_n \times U_n) \in \mathcal{N} \otimes \mathcal{M}$.*

PROOF. (Sketch.) We have $\Phi_{\mathcal{M}}(B) = \bigcup_{n\in\mathbb{N}} A_n$ where $A_n := \{x \in X : U_n \setminus B(x) \in \mathcal{M}\}$, $n \in \mathbb{N}$. The sets A_n, $n \in \mathbb{N}$, are Borel [11, 22.22]. For each $n \in \mathbb{N}$ choose an F_σ set $F_n \subset A_n$ such that $\lambda(A_n \setminus F_n) = 0$. Put $B_* := \bigcup_{n\in\mathbb{N}}(F_n \times U_n)$. We have $(B_* \setminus B)(x) \in \mathcal{M}$ for each $x \in X$. It remains to show that $B \setminus B_* \in \mathcal{N} \otimes \mathcal{M}$. To this end consider $A := \bigcup_{n\in\mathbb{N}}(A_n \setminus F_n)$ belonging to $\mathcal{N}$. Next observe that $(B \setminus B_*)(x) \in \mathcal{M}$ for each $x \in X \setminus A$. □

COROLLARY 5. *For every Borel set $B \subset X^2$ there are an F_σ set $D \subset X^2$ and $C \in \mathcal{N}$ such that $B \subset (C \times X) \cup D$ and $(D \setminus B)(x) \in \mathcal{M}$ for each $x \in X \setminus C$.*

PROOF. By Proposition 4 we find open sets $U_n \subset X$, $n \in \mathbb{N}$, and F_σ sets $F_n \subset X$, $n \in \mathbb{N}$, such that for $B_* := \bigcup_{n\in\mathbb{N}}(F_n \times U_n)$ we have $B \triangle B_* \in \mathcal{N} \otimes \mathcal{M}$. By Corollary 3 we can choose a set $C \in \mathcal{N}$ and an F_σ set $F \subset X^2$ with meager sections such that $B \triangle B_* \subset (C \times X) \cup F$. Put $D := B_* \cup F$. Then D is as desired (observe that $(D \setminus B)(x) \subset F(x)$ for each $x \in X \setminus C$). □

An equivalent formulation of Corollary 5 with an inner approximation of B can be obtained when we apply Corollary 5 to $X \setminus B$ and then consider complements. Namely, we then find a G_δ set $E \subset X^2$ and $C \in \mathcal{N}$ such that $\big((X \setminus C) \times X\big) \cap E \subset B$ and $(B \setminus E)(x) \in \mathcal{M}$ for each $x \in X \setminus C$.

By the definition of $\mathcal{M}^*$, each set from $\mathcal{M}^*$ can be covered by a Borel set $D \subset X^2$ such that each section $D(x)$, $x \in X$, is meager. However, $\mathcal{M}^*$ does not have a base that consists of Borel sets of a bounded level (see [6, p. 565]). On the other hand, Fremlin established a nice base of $\mathcal{M}^*$. His result was included to [5].

PROPOSITION 6 [5, Lemma 1.7]. (a) *For each set $A \in \mathcal{M}^*$ there is a Borel set $F \supset A$ with all sections meager of type F_σ.*

(b) *For every Borel set $B \subset X^2$ there is a Borel set $W \subset X^2$ such that for each $x \in X$, $W(x)$ is open and $B \triangle W \in \mathcal{M}^*$.*

We have been informed that Proposition 6(a) can be also derived from a general result of Hillard [10]. Next let us consider nice bases of $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N}^*$.

PROPOSITION 7 [1, Lemma 2.3]. *Let $B \subset X^2$ be a Borel set. The following property holds:*

$$(\star)\quad \begin{cases} \text{for every } \varepsilon > 0 \text{ there exist an open set } G \subset X^2 \text{ and a set } C \in \mathcal{M} \text{ such} \\ \text{that } B \subset (C \times X) \cup G \text{ and } \lambda\big((G \setminus B)(x)\big) \leqq \varepsilon \text{ for each } x \in X \setminus C. \end{cases}$$

PROOF. (Sketch.) Taking into account the hierarchies $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\alpha < \omega_1$, of Borel sets it suffices to show three facts: property $(\star)$ holds for

- every open set $B \subset X^2$;
- $B = \bigcup_{m\in\mathbb{N}} B_m$ provided it holds for Borel sets B_m, $m \in \mathbb{N}$;
- $B = \bigcap_{m\in\mathbb{N}} B_m$ provided it holds for Borel sets B_m, $m \in \mathbb{N}$, such that $B_{m+1} \subset B_m$ for all $m \in \mathbb{N}$. □

COROLLARY 8. *For every Borel set $B \subset X^2$ there exist a G_δ set $D \subset X^2$ and $C \in \mathcal{M}$ such that $B \subset (C \times X) \cup D$ and $(D \setminus B)(x) \in \mathcal{N}$ for each $x \in X \setminus C$.*

We can also describe a nice base of $\mathcal{M} \otimes \mathcal{N}$.

COROLLARY 9. *For each set $A \in \mathcal{M} \otimes \mathcal{N}$ there exist a set $C \in \mathcal{M}$ (of type F_σ) and a G_δ set $D \subset X^2$ with Lebesgue null sections such that $A \subset (C \times X) \cup D$.*

To obtain a nice base for $\mathcal{N}^*$ we can follow a similar scheme. In fact, this was done by Fremlin in his result included in [5].

PROPOSITION 10 [5, Lemma 1.7]. *Let $B \subset X^2$ be a Borel set. Then for every $\varepsilon > 0$ there is a Borel set $G \subset X^2$ such that $B \subset G$ and for each $x \in X$, $G(x)$ is open with $\lambda\big((G \setminus B)(x)\big) \leqq \varepsilon$.*

COROLLARY 11. *For each set $A \in \mathcal{N}^*$ there is a Borel set $H \supset A$ with all sections of type G_δ and of measure zero.*

Note that $\mathcal{N}^*$ does not have a base that consists of Borel sets of a bounded level (see [6, p. 565]).

3. Uniform versions of the Nikodym theorem

The Nikodym theorem (cf. [15, Theorem 8.1], [12, §32, II], [11, 8.38]) states that every real-valued function with the Baire property, defined on a Polish space, while restricted to some residual set, is continuous. Consider a Borel function $f : X^2 \to \mathbb{R}$. Then all its sections f_x are Borel measurable. By the Nikodym theorem, each section f_x restricted to a residual subset A_x of X is continuous and we may assume that A_x is of type G_δ. We ask whether all or almost all residual sets A_x can be chosen as the sections $B(x)$ of the same set $B \subset X^2$ of type G_δ. A similar question is asked when we derive from the Lusin theorem that the sections f_x, $x \in X$, restricted to sets of big measure are continuous (this question is discussed in the next section). The phrase "almost all" will be meant in the sense of the σ-ideals $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$.

PROPOSITION 12. *Let $f : X^2 \to \mathbb{R}$ have the Baire property. Then there exist an F_σ set $F \subset X^2$ and $C \in \mathcal{M}$ such that $F(x) \in \mathcal{M}$ for all $x \in X \setminus C$*

and $f\big|\big(((X\setminus C)\times X)\setminus F\big)$ *is continuous. In particular,* $f_x\big|\big(X\setminus F(x)\big)$ *is continuous for all* $x\in X\setminus C$.

Proof. By the Nikodym theorem pick an F_σ meager set $F\subset X^2$ such that the restriction $f|(X^2\setminus F)$ is continuous. Next, using the Kuratowski–Ulam theorem pick a set $C\in\mathcal{M}$ such that $F(x)\in\mathcal{M}$ for all $x\in X\setminus C$. Plainly $f\big|\big(((X\setminus C)\times X)\setminus F\big)$ is continuous. $\square$

It is natural to ask whether, for a Borel function $f: X^2\to\mathbb{R}$, the above statement remains true if "$C\in\mathcal{M}$" is replaced by "$C\in\mathcal{N}$". We do not know an answer if the continuity of $f\big|\big(((X\setminus C)\times X)\setminus F\big)$ is required. However, the answer is "yes" if we require only the continuity of all sections $f_x\big|\big(X\setminus F(x)\big)$, $x\in X\setminus C$. A real-valued function on a metric space is called *Borel measurable of class* 1 if all its preimages of open sets are of type F_σ in the domain, or equivalently, if it is a pointwise limit of a sequence of continuous functions.

Theorem 13. *Let* $f: X^2\to\mathbb{R}$ *be a Borel function. Then there is a set* $D\in\mathcal{N}\otimes\mathcal{M}$ *such that* $h:=f|(X^2\setminus D)$ *is Borel measurable of class* 1 *with* $\mathcal{N}$-*almost all sections* h_x *continuous. Moreover, there exist an* F_σ *set* $F\subset X^2$ *and* $C\in\mathcal{N}$ *such that* $F(x)\in\mathcal{M}$ *for all* $x\in X\setminus C$, *and* $g:=f\big|\big(((X\setminus C)\times X)\setminus F\big)$ *is Borel measurable of class* 1 *with* $g_x=f_x\big|\big(X\setminus F(x)\big)$ *continuous for all* $x\in X\setminus C$.

Proof. Let $(V_m)_{m\in\mathbb{N}}$ and $(U_m)_{m\in\mathbb{N}}$ be bases of open sets in $\mathbb{R}$ and in X, respectively. Then using Proposition 4 to the sets $B_m:=f^{-1}(V_m)$ for each $m\in\mathbb{N}$, we find sequences $(F_{mn})_{n\in\mathbb{N}}$, $m\in\mathbb{N}$, of F_σ subsets of X such that

$$D_m:=B_m\triangle\bigcup_{n\in\mathbb{N}}(F_{mn}\times U_n)\in\mathcal{N}\otimes\mathcal{M}.$$

Then $D:=\bigcup_{m\in\mathbb{N}}D_m\in\mathcal{N}\otimes\mathcal{M}$. By Corollary 5 we find an F_σ set $F\subset X^2$ and $C\in\mathcal{N}$ such that $D\subset(C\times X)\cup F$ and $F(x)\in\mathcal{M}$ for each $x\in X\setminus C$.

To prove that h is Borel measurable of class 1 we will show that the preimage $h^{-1}(V_m)$ is of type F_σ in $X^2\setminus D$ for each $m\in\mathbb{N}$. This implies that $g^{-1}(V_m)$ is of type F_σ in $\mathrm{dom}(g)\subset X^2\setminus D$, and so g would be Borel measurable of class 1. Thus fix $m\in\mathbb{N}$. Then

$$f^{-1}(V_m)=B_m=\bigcup_{n\in\mathbb{N}}(F_{mn}\times U_n)\triangle D_m.$$

Consequently, $h^{-1}(V_m)=f^{-1}(V_m)\setminus D=\bigcup_{n\in\mathbb{N}}(F_{mn}\times U_n)\setminus D$ is of type F_σ in $X^2\setminus D$.

Now, fix $x \in X \setminus C$. We will prove the continuity of $h_x = f_x\big|\big(X \setminus D(x)\big)$ (since $D(x) \subset F(x)$, this implies the continuity of $g_x = f_x\big|\big(X \setminus F(x)\big)$). For a fixed $m \in \mathbb{N}$ we have

$$h_x^{-1}(V_m) = f_x^{-1}(V_m) \setminus D(x) = B_m(x) \setminus D(x)$$

$$= \Big(D_m(x) \triangle \bigcup_{n\in\mathbb{N}} (F_{mn} \times U_n)(x)\Big) \setminus D(x) = \bigcup \{U_n : x \in F_{mn},\ n \in \mathbb{N}\} \setminus D(x)$$

and this set is open in $X \setminus D(x)$. $\square$

We can connect Proposition 12 with Theorem 13 to obtain the respective result for sections f_x of a Borel function $f : X^2 \to \mathbb{R}$ where an exceptional set C is in $\mathcal{M} \cap \mathcal{N}$ but F has a bit more complicated Borel structure.

Corollary 14. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then there exist a set $C \in \mathcal{M} \cap \mathcal{N}$ and a Borel set $F \subset (X \setminus C) \times X$ (of type $F_{\sigma\delta}$ and $G_{\delta\sigma}$) such that for all $x \in X \setminus C$, the section $F(x)$ is meager of type F_σ and $f_x\big|\big(X \setminus F(x)\big)$ is continuous.*

Proof. By Proposition 12, pick an F_σ set $F_1 \subset X^2$ and a set $C_1 \in \mathcal{M}$ such that $F_1(x) \in \mathcal{M}$ and $f_x\big|\big(X \setminus F_1(x)\big)$ is continuous for each $x \in X \setminus C_1$. Next, by Theorem 13, pick an F_σ set $F_2 \subset X^2$ and a set $C_2 \in \mathcal{N}$ such that $F_2(x) \in \mathcal{M}$ and $f_x\big|\big(X \setminus F_2(x)\big)$ is continuous for each $x \in X \setminus C_2$. We may enlarge C_1 and C_2 to be of types F_σ and G_δ, respectively. Put $C := C_1 \cap C_2$ and

$$F := \big(F_1 \cap \big((X \setminus C_1) \times X\big)\big) \cup \big(F_2 \cap \big((C_1 \setminus C_2) \times X\big)\big).$$

Clearly $C \in \mathcal{M} \cap \mathcal{N}$ and $F \subset (X \setminus C) \times X$. It is not hard to check that F is of type $F_{\sigma\delta}$ and $G_{\delta\sigma}$. Fix $x \in X \setminus C$. If $x \in X \setminus C_1$ then $F(x) = F_1(x)$, and if $x \in C_1 \setminus C_2$ then $F(x) = F_2(x)$. So, the assertion follows. $\square$

We do not know whether we can use F of type F_σ in this corollary. Another question is how much we can decrease an exceptional set C. More precisely, we do not know how much thinner a σ-ideal $\mathcal{I}$ can be used with $C \in \mathcal{I}$. If we want F to be a Borel set of bounded level with meager sections for all $x \notin C$, the following theorem shows that the trivial σ-ideal $\mathcal{I} = \{\emptyset\}$ is bad.

Theorem 15. *Let $2 \leqq \alpha < \omega_1$. There is a Borel function $f : X^2 \to \mathbb{R}$ such that for every set F in the Borel class $\mathbf{\Sigma}^0_\alpha(X^2)$ with $F(x) \in \mathcal{M}$ for all $x \in X$ we can find $x_0 \in X$ such that $f_{x_0}\big|\big(X \setminus F(x_0)\big)$ is not continuous.*

Proof. We follow some ideas from [6, Lemma 2.3]. Let $U \in \mathbf{\Sigma}^0_\alpha(X^3)$ be a universal set for $\mathbf{\Sigma}^0_\alpha(X^2)$, cf. [11, 22.3]. Put $A := \big\{(x, y) \in X^2 : (x, x, y)$

$\notin U\}$. Then $A \in \mathbf{\Pi}^0_\alpha(X^2)$. For each $F \in \mathbf{\Sigma}^0_\alpha(X^2)$ we can find $x \in X$ such that $F = U(x)$, hence

$$F(x) = \big(U(x)\big)(x) = U(x,x) = X \setminus A(x).$$

Put $B := \{x \in X : A(x) \notin \mathcal{M}\}$. Then B is a Borel set ([11, 22.22]). By [11, 18.6] applied with the σ-ideal $\mathcal{M}$ we can find a Borel function $g : B \to X$ with $g(x) \in A(x)$ for all $x \in B$. Let $f : X^2 \to \mathbb{R}$ stand for the characteristic function of the graph of g. Let $F \in \mathbf{\Sigma}^0_\alpha(X^2)$ and $F(x) \in \mathcal{M}$ for all $x \in X$. Pick $x_0 \in X$ such that $F(x_0) = X \setminus A(x_0)$. Then $x_0 \in B$. Suppose that $h := f_{x_0}\big|\big(X \setminus F(x_0)\big)$ is continuous. Then $h\big(g(x_0)\big) = 1$ and $h(y) = 0$ for the remaining $y \in X \setminus F(x_0)$. Hence $g(x_0)$ should be an isolated point of $X \setminus F(x_0) = A(x_0)$ which is impossible since $A(x_0)$ is residual in X. □

On the other hand, we have the following positive result

THEOREM 16. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then there is a Borel set $F \subset X^2$ such that $F(x) \in \mathcal{M}$, $F(x)$ is of type F_σ, and $f_x\big|\big(X \setminus F(x)\big)$ is continuous for all $x \in X$.*

PROOF. Let $(V_m)_{m\in\mathbb{N}}$ be a base of open sets in $\mathbb{R}$. Put $B_m := f^{-1}(V_m)$, $m \in \mathbb{N}$. By Proposition 6(b) it follows that for every Borel set B_m we can find a Borel set $W_m \subset X^2$ such that for each $x \in X$, $W_m(x)$ is open and $A_m \in \mathcal{M}^*$ where $A_m := B_m \triangle W_m$. Put $A := \bigcup_{m\in\mathbb{N}} A_m$. It is easy to show that $f_x\big|\big(X \setminus A(x)\big)$ is continuous for all $x \in X$ (cf. [15, Theorem 8.1]). By Proposition 6(a) pick a Borel set $F \supset A$ with all sections meager of type F_σ. Then $f_x\big|\big(X \setminus F(x)\big)$ is continuous for all $x \in X$. □

4. Uniform versions of the Lusin theorem

Let us start with the following observation.

PROPOSITION 17. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then for every $\varepsilon > 0$ there exist open sets $G \subset X^2$ and $C \subset X$ such that $f\Big|\Big(\big((X \setminus C) \times X\big) \setminus G\Big)$ is continuous, $\lambda(C) \leqq \varepsilon$ and $\lambda\big(G(x)\big) \leqq \varepsilon$ for all $x \in X \setminus C$.*

PROOF. By the Lusin theorem (cf. [15, Theorem 8.2], [11, 17.12]), for $\varepsilon > 0$ pick an open set $G \subset X^2$ such that $\lambda_2(G) \leqq \varepsilon^2$ and $f|(X^2 \setminus G)$ is continuous. Let $C := \big\{x \in X : \lambda\big(G(x)\big) > \varepsilon\big\}$. Then C is open (cf. [11, 22.25]). Plainly $f\Big|\Big(\big((X \setminus C) \times X\big) \setminus G\Big)$ is continuous and $\lambda\big(G(x)\big) \leqq \varepsilon$ for all $x \in X \setminus C$. To show that $\lambda(C) \leqq \varepsilon$ we use the Fubini theorem and obtain

$$\varepsilon^2 \geqq \lambda_2(G) = \int_X \lambda\big(G(x)\big)\,dx$$

$$= \int_C \lambda\big(G(x)\big)\,dx + \int_{X\setminus C} \lambda\big(G(x)\big)\,dx \geqq \varepsilon\lambda(C). \quad \square$$

We propose a modified version of Proposition 17 where an exceptional set C is meager. We extract a basic idea (method) contained in the proof of Lusin's theorem (cf. [15, Theorem 8.2]); this is formulated in the following (rather obvious) lemma.

LEMMA 18. *Let $(V_m)_{m\in\mathbb{N}}$ be a base of open sets in $\mathbb{R}$. For a measurable function $f : E \to \mathbb{R}$ defined on a measurable set $E \subset \mathbb{R}^k$ let $B_m := f^{-1}(V_m)$, $m \in \mathbb{N}$. If for every $m \in \mathbb{N}$, G_m and $G^*_m \subset \mathbb{R}^k$ are open sets such that $B_m \subset G_m$ and $E \setminus B_m \subset G^*_m$, then for the open set $G := \bigcup_{m\in\mathbb{N}}(G_m \cap G^*_m)$ the restriction $f|(E \setminus G)$ is continuous. Moreover, if $\varepsilon > 0$ and G_m, G^*_m are chosen so that $\lambda_k(G_m \setminus B_m) \leqq \varepsilon/2^{m+1}$ and $\lambda_k(G^*_m \cap B_m) \leqq \varepsilon/2^{m+1}$ for all $m \in \mathbb{N}$, then $\lambda_k(G) \leqq \varepsilon$.*

THEOREM 19. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then for every $\varepsilon > 0$ there is an open set $G \subset X^2$ and $C \in \mathcal{M}$ such that $f\Big|\Big(\big((X \setminus C) \times X\big) \setminus G\Big)$ is continuous and $\lambda\big(G(x)\big) \leqq \varepsilon$ for all $x \in X \setminus C$.*

PROOF. Fix a countable base $(V_m)_{m\in\mathbb{N}}$ of open sets in $\mathbb{R}$. Put $B_m := f^{-1}(V_m)$, $m \in \mathbb{N}$, and apply Proposition 7 to every set B_m with $\varepsilon/2^{m+1}$. We find an open set $G_m \subset X^2$ and a set $C_m \in \mathcal{M}$ such that $B_m \subset (C_m \times X) \cup G_m$ and $\lambda\big((G_m \setminus B_m)(x)\big) \leqq \varepsilon/2^{m+1}$ for each $x \in X \setminus C_m$. When we repeat it for $X^2 \setminus B_m$, we find an open set $G^*_m \subset X^2$ and a set $C^*_m \in \mathcal{M}$ such that $X^2 \setminus B_m \subset (C^*_m \times X) \cup G^*_m$ and $\lambda\big((G^*_m \cap B_m)(x)\big) \leqq \varepsilon/2^{m+1}$ for each $x \in X \setminus C^*_m$. Put $G := \bigcup_{m\in\mathbb{N}}(G_m \cap G^*_m)$ and $C := \bigcup_{m\in\mathbb{N}}(C_m \cup C^*_m)$. Then G is open and $C \in \mathcal{M}$. Let $X_0 := (X \setminus C) \times X$. Applying the first assertion of Lemma 18 to the function $f|X_0$ we have that $f|(X_0 \setminus G)$ is continuous. Then fix $x \in X \setminus C$. Applying Lemma 18 to $f_x : X \to \mathbb{R}$ with B_m, G_m, G^*_m, G replaced by $B_m(x)$, $G_m(x)$, $G^*_m(x)$, $G(x)$, we have $\lambda\big(G(x)\big) \leqq \varepsilon$. $\square$

COROLLARY 20. *For every Borel function $f : X^2 \to \mathbb{R}$ there is a set $D \in \mathcal{M} \otimes \mathcal{N}$ such that $f|(X^2 \setminus D)$ is Borel measurable of class* 1.

PROOF. We apply Theorem 19 with $\varepsilon := 1/k$, $k \in \mathbb{N}$, and obtain $C_k \in \mathcal{M}$ and an open set $G_k \subset X^2$ such that $g_k := f\Big|\Big(\big((X \setminus C_k) \times X\big) \setminus G_k\Big)$ is continuous and $\lambda\big(G_k(x)\big) \leqq 1/k$ for all $x \in X \setminus C_k$. Define $C := \bigcup_{k\in\mathbb{N}} C_k$, $G := \bigcap_{k\in\mathbb{N}} G_k$ and $D := (C \times X) \cup G$. Then $D \in \mathcal{M} \otimes \mathcal{N}$ and

$$X^2 \setminus D = \big((X \setminus C) \times X\big) \setminus G = \bigcup_{k\in\mathbb{N}} \big((X \setminus C) \times X\big) \setminus G_k.$$

Since $\big((X \setminus C) \times X\big) \setminus G_k$ is closed in $X^2 \setminus D$, we can (by the Tietze theorem) extend g_k to a continuous function $f_k : X^2 \setminus D \to \mathbb{R}$. Then $f_k \to f|(X^2 \setminus D)$ and so $f|(X^2 \setminus D)$ is Borel measurable of class 1. □

We can connect Proposition 17 with Theorem 19 to obtain the following analogue of Corollary 14 in which a set G is of type G_δ with open sections.

Corollary 21. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then for every $\varepsilon > 0$ there exist a set $C \in \mathcal{M}$ of type F_σ with $\lambda(C) \leqq \varepsilon$ and a G_δ set $G \subset (X \setminus C) \times X$ such that for all $x \in X \setminus C$, the section $G(x)$ is open with $\lambda\big(G(x)\big) \leqq \varepsilon$ and $f\Big|\Big(\big((X \setminus C) \times X\big) \setminus G\Big)$ is continuous.*

Proof. By Proposition 17, pick open sets $C_1 \subset X$ and $G_1 \subset X^2$ such that $\lambda(C_1) \leqq \varepsilon$, $f\Big|\Big(\big((X \setminus C_1) \times X\big) \setminus G_1\Big)$ is continuous and $\lambda\big(G_1(x)\big) \leqq \varepsilon$ for all $x \in X \setminus C_1$. Next, by Theorem 13, pick an open set $G_2 \subset X^2$ and a set $C_2 \in \mathcal{M}$ of type F_σ such that $f\Big|\Big(\big((X \setminus C_2) \times X\big) \setminus G_2\Big)$ is continuous and $\lambda\big(G_2(x)\big) \leqq \varepsilon$ for all $x \in X \setminus C_2$. Put $C := C_1 \cap C_2$ and

$$G := \Big(G_1 \cap \big((X \setminus C_1) \times X\big)\Big) \cup \Big(G_2 \cap \big((C_1 \setminus C_2) \times X\big)\Big).$$

Clearly $C \in \mathcal{M}$, $\lambda(C) \leqq \varepsilon$, and $G \subset (X \setminus C) \times X$ is a G_δ set. Fix $x \in X \setminus C$. If $x \in X \setminus C_1$ then $G(x) = G_1(x)$, and if $x \in C_1 \setminus C_2$ then $G(x) = G_2(x)$. So, the assertion follows. □

Having Corollary 21, it is interesting to know how much one can decrease an exceptional set C when G is required merely Borel with open sections of measure $\leqq \varepsilon$. The answer is analogous to the respective fact obtained in the category case. First we mimic the method applied in the proof of Theorem 15 to show that an exceptional set C, for the respectively chosen f and every G of bounded Borel level with all sections of measure $\leqq \varepsilon$, should be nonempty.

Theorem 22. *Let $1 \leqq \alpha < \omega_1$, $X := [0,1]$ and $\varepsilon := 1/2$. There is a Borel function $f : X^2 \to \mathbb{R}$ such that for every set $G \in \mathbf{\Sigma}^0_\alpha(X^2)$ with $\lambda\big(G(x)\big) \leqq \varepsilon$ for all $x \in X$ there exists $x_0 \in X$ such that $f_{x_0}\big|\big(X \setminus G(x_0)\big)$ is not continuous.*

Proof. Define a set $A \subset X^2$ as in the in the proof of Theorem 15. Put $B := \big\{x \in X : \lambda\big(A(x)\big) \geqq \varepsilon\big\}$. Then B is a Borel set ([11, 22.25]). We will define a Borel subset $A^* \subset A$ of positive measure with $A^*(x)$ dense in itself, for each $x \in B$ (in fact, each portion of $A^*(x)$, $x \in B$, will be of positive measure). This construction is due to Recław [16]. Fix a base $(V_n)_{n\in\mathbb{N}}$ of open sets in X. Then every set

$$W_n := \Big\{x \in X : \lambda\big(\big(A \cap (X \times V_n)\big)(x)\big) = 0\Big\}, \quad n \in \mathbb{N},$$

is Borel ([11, 22.25]). We claim that the Borel set $A^* := A \setminus \bigcup_{n\in\mathbb{N}}(W_n \times V_n)$ is as desired. Indeed, fix $x \in B$ and put $M_x := \big\{n \in \mathbb{N} : \lambda\big(A(x) \cap V_n\big) = 0\big\}$. By the definition of A^* we have

$$A^*(x) = A(x) \setminus \bigcup_{n\in M_x} \big(V_n \cap A(x)\big) \quad \text{and} \quad \lambda\Bigg(\bigcup_{n\in M_x} V_n \cap A(x)\Bigg) = 0.$$

Hence $\lambda\big(A^*(x)\big) = \lambda\big(A(x)\big) > 0$. Now, consider any set V_m such that $V_m \cap A^*(x) \neq \emptyset$. The case $\lambda\big(V_m \cap A(x)\big) = 0$ is impossible since then $m \in M_x$. Hence $\lambda\big(V_m \cap A(x)\big) > 0$ and so $\lambda\big(V_m \cap A^*(x)\big) > 0$.

By [11, 18.6] applied with the σ-ideal $\mathcal{N}$ we can find a Borel function $g : B \to X$ with $g(x) \in A^*(x)$ for all $x \in B$. Let $f : X^2 \to \mathbb{R}$ stand for the characteristic function of the graph of g. Let $G \in \mathbf{\Sigma}^0_\alpha(X^2)$ be such that $\lambda\big(G(x)\big) \leqq \varepsilon$ for all $x \in X$. Pick $x_0 \in X$ such that $G(x_0) = X \setminus A(x_0)$. Then $\lambda\big(A(x_0)\big) = 1 - \lambda\big(G(x_0)\big) \geqq \varepsilon$. Hence $x_0 \in B$. Since $A^*(x_0)$ is dense in itself, $f_{x_0}|A^*(x_0)$ is discontinuous at $g(x_0)$ and $f_{x_0}|A(x_0) = f_{x_0}\big|\big(X \setminus G(x_0)\big)$ remains discontinuous at $g(x_0)$. $\square$

In particular, from Theorem 22 (for $\alpha = 1$) it follows that an exceptional set C in Proposition 17 can be nonempty. We also have the following positive result.

THEOREM 23. *Let $f : X^2 \to \mathbb{R}$ be a Borel function. Then for every $\varepsilon > 0$ there is a Borel set $G \subset X^2$ such that $G(x)$ is open, $\lambda\big(G(x)\big) \leqq \varepsilon$ and $f_x\big|\big(X \setminus G(x)\big)$ is continuous for all $x \in X$.*

PROOF. Fix a countable base $(V_m)_{m\in\mathbb{N}}$ of open sets in $\mathbb{R}$. For every $m \in \mathbb{N}$ put $B_m := f^{-1}(V_m)$, and by Proposition 10 pick Borel sets $G_m, G^*_m \subset X^2$ such that $B_m \subset G_m$, $X \setminus B_m \subset G^*_m$ and $G_m(x)$, $G^*_m(x)$ are open with $\lambda\big((G_m \setminus B_m)(x)\big) \leqq \varepsilon/2^{m+1}$, $\lambda\big((G^*_m \cap B_m)(x)\big) \leqq \varepsilon/2^{m+1}$ for all $x \in X$. Since $f_x^{-1}(V_m) = B(x)$, we may apply Lemma 18 to f_x for every fixed $x \in X$. Thus for $G := \bigcup_{m\in\mathbb{N}}(G_m \cap G^*_m)$ we obtain the assertion. $\square$

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