IDEAL INVARIANT INJECTIONS

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Abstract. For an ideal $\mathcal{I}$ on ω, we introduce the notions of $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections from ω to ω. We study injections that are invariant with respect to selected classes of ideals. We show some applications to ideal convergence.

1. Introduction

Let $\omega := \{0, 1, \dots\}$, $\mathbb{Z}$ stands for the set of all integers, and id is the identity function on ω. By *an ideal* $\mathcal{I}$ *on* ω we mean an ideal of subsets of ω such that $\omega \notin \mathcal{I}$ and $\{n\} \in \mathcal{I}$ for all $n \in \omega$. If $\mathcal{I}$ is an ideal on ω then $\mathcal{I}^\star$ denotes its dual filter $\{\omega \setminus A \colon A \in \mathcal{I}\}$. Several examples of ideals on ω were considered in [8] (see also [16], [18] and [14]). The ideal of all finite subsets of ω is denoted by Fin.

Through the paper, we will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \mathbf{Inj}$. We say that f is $\mathcal{I}$-*invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is $\mathcal{I}$-*invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-*$\mathcal{I}$*-invariant*. Note that every $f \in \mathbf{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact 1. *Let* $\mathcal{I}$ *be an ideal on* ω *and let* $f \in \mathbf{Inj}$.

(i) f^{-1} *is* $\mathcal{I}$*-invariant if and only if* $f[A] \notin \mathcal{I}$ *for every* $A \notin \mathcal{I}$.
(ii) *If* $f[\omega] \in \mathcal{I}$ *then* f *is* $\mathcal{I}$*-invariant and it is not bi-*$\mathcal{I}$*-invariant.*

Proof. (i) "$\Rightarrow$" Let $A \notin \mathcal{I}$ and suppose that $f[A] \in \mathcal{I}$. Then $A = f^{-1}[f[A]] \in \mathcal{I}$, a contradiction.

"$\Leftarrow$" Assume that $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$. Suppose that f^{-1} is not $\mathcal{I}$-invariant. Hence $f^{-1}[B] \notin \mathcal{I}$ for some $B \in \mathcal{I}$. Then $B \supseteq f[f^{-1}[B] \notin \mathcal{I}$, a contradiction.

(ii) The first part is obvious, and the second part follows from $f^{-1}[f[\omega]] = \omega \notin \mathcal{I}$. $\square$

To show an example based on Fact 1(ii), recall the definition of the *classical density ideal* $\mathcal{I}_d$. For a set $A \subseteq \omega$, consider the following numbers

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap \{0,\dots,n-1\}|}{n}\ , \quad \overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{0,\dots,n-1\}|}{n}\ .$$

If $\underline{d}(A) = \overline{d}(A)$, we denote this common value by $d(A)$ and call the *asymptotic density* of A. Then define $\mathcal{I}_d := \{A \subseteq \omega \colon \overline{d}(A) = 0\}$.

Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$. Hence f is not bi-$\mathcal{I}_d$-invariant by Fact 1(ii).

The next example shows an ideal $\mathcal{I}$ on ω and a bijection f from ω onto ω which is $\mathcal{I}$-invariant but f^{-1} is not so. If $k, l \in \omega$ and $k > 0$, we denote $k\omega + l := \{kn + l \colon n \in \omega\}$.

Example 2. Let $f \colon \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n+1) = 4n+2$, $f(4n+3) := 2n+1$ for $n \in \omega$. Then f is a bijection. Consider the ideal $\mathcal{I}$ defined as follows

$$\mathcal{I} := \{A \cup B \colon A \in \text{Fin},\ B \subseteq 2\omega\}.$$

Clearly, f is $\mathcal{I}$-invariant. Note that $4\omega + 1 \notin \mathcal{I}$ but $f[4\omega + 1] \in \mathcal{I}$. Let $B := f[4\omega + 1]$. Then $B \in \mathcal{I}$ and $f^{-1}[B] \notin \mathcal{I}$. $\square$

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An ideal $\mathcal{I}$ on ω is called *tall* if every infinite subset of ω contains an infinite set belonging to $\mathcal{I}$ (see [8]). Note that the ideal in the Example 2 is not tall. The respective example with a tall ideal will be presented in Section 4.

For $f\colon \omega \to \omega$ let $\mathrm{Fix}(f) := \{n \in \omega\colon f(n) = n\}$. The following fact is obvious.

Fact 3. *Let $\mathcal{I}$ be an ideal on ω and $f \in \mathbf{Inj}$. If $\mathrm{Fix}(f) \in \mathcal{I}^\star$ then f is bi-$\mathcal{I}$-invariant.*

The purpose of our paper is to describe $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections for selected classes of ideals. In some cases, we also study topological features of the sets of such injections. It is easy to see that $\mathbf{Inj}$ is a G_δ subset of the Baire space ω^ω (cf. [24, p. 66]), so it is a Polish space, by the Alexandrov theorem. Sets of the form $\{f \in \mathbf{Inj}\colon f(k_i) = l_i \text{ for } i = 1, \dots, p\}$ constitute a base of the topology in $\mathbf{Inj}$. We are interested in the Baire category and levels of the Borel hierarchy for the sets of $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections in the space $\mathbf{Inj}$.

Proposition 4. *The set $\{f \in \mathbf{Inj} : \omega \setminus \mathrm{Fix}(f) \in \mathrm{Fin}\}$ is dense in $\mathbf{Inj}$. In particular, the set $\{f \in \mathbf{Inj} : f \text{ is bi-}\mathcal{I}\text{-invariant}\}$ is dense in $\mathbf{Inj}$ for every ideal $\mathcal{I}$ containing all singletons. Moreover, if $\mathcal{I}$ contains infinite sets and all singletons, the set $\{f \in \mathbf{Inj} : f \text{ is not } \mathcal{I}\text{-invariant}\}$ is dense in $\mathbf{Inj}$ as well.*

Proof. Let $V := \{f \in \mathbf{Inj}\colon f(k_i) = l_i \text{ for } i = 1, \dots, p\}$ be a basic set. To prove the first assertion, define $g : \omega \to \omega$ as follows. Pick $n \in \omega$ such that $k_i \leq n$ and $l_i \leq n$ for $i = 1, \dots, p$. Put $g(k_i) := l_i$ for $i = 1, \dots, p$ and extend g on $\{0, \dots, n\}$ to be a bijection of this set onto itself. Finally put $g(k) := k$ for $k > n$. Then $g \in V$ and $\omega \setminus \mathrm{Fix}(g) \in \mathrm{Fin}$. Next use Fact 3.

To prove the second assertion, fix distinct $k_1, \dots, k_p \in \omega$ and distinct $l_1, \dots, l_p \in \omega$. Set $B := \{k_1, \dots, k_p, l_1, \dots l_p\}$ and consider a bijection h from B onto itself that $h_1(k_i) = l_i$ for $i \in \{1, \dots, p\}$. Now take an infinite $A \in \mathcal{I}$ disjoint from B. Then $\omega \setminus (A \cup B)$ is also infinite. Now take any bijection $h_2 : \omega \setminus B \to \omega \setminus B$ such that $h_2[A] = \omega \setminus (A \cup B)$. Finally put g as the common extension of h_1 and h_2. Then $A \in \mathcal{I}$ but $g[A] = \omega \setminus (A \cup B) \notin \mathcal{I}$, so g is not $\mathcal{I}$-invariant. $\square$

Instead of ideals on ω, one can consider ideals on an arbitrary infinite countable set X. Then using a fixed bijection φ between X and ω, one can transform an ideal $\mathcal{I}$ on X onto the ideal $\mathcal{I}' := \{\varphi[A]\colon A \in \mathcal{I}\}$ on ω, without loss of any reasonable properties. Thus we can consider an $\mathcal{I}$-invariant (or bi-$\mathcal{I}$-invariant) injection f from X to X defined analogously as in the case of ω. Then $\varphi \circ f \circ \varphi^{-1}$ is an $\mathcal{I}'$-invariant (bi-$\mathcal{I}'$-invariant) injection from ω to ω.

We say that ideals $\mathcal{I}$ and $\mathcal{J}$ on infinite countable sets X and Y, respectively, are *isomorphic* if there exists a bijection g between X and Y such that $E \in \mathcal{I}$ if and only if $g[E] \in \mathcal{J}$ for every $E \subseteq X$.

Recall two methods of building ideals when two of them are given. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Define

$$\mathcal{I} \oplus \mathcal{J} := \{A \subseteq \omega \times \{0,1\}\colon\ \mathrm{pr}_1[A \cap (\omega \times \{0\})] \in \mathcal{I} \text{ and } \mathrm{pr}_1[A \cap (\omega \times \{1\})] \in \mathcal{J}\}.$$

where pr_1 is the projection on the first factor. Then $\mathcal{I} \oplus \mathcal{J}$ is an ideal on $\omega \times \{0,1\}$. This also holds if one of $\mathcal{I}$, $\mathcal{J}$ equals the power set $\mathcal{P}(\omega)$. *The Fubini product* of $\mathcal{I}$ and $\mathcal{J}$ is given by

$$\mathcal{I} \times \mathcal{J} := \{A \subseteq \omega \times \omega\colon \{m \in \omega\colon A[m] \notin \mathcal{J}\} \in \mathcal{I}\}$$

where $A[m] := \{n \in \omega\colon (m,n) \in A\}$. Then $\mathcal{I} \times \mathcal{J}$ is an ideal on $\omega \times \omega$. This is also true if one of $\mathcal{I}, \mathcal{J}$ equals $\{\emptyset\}$. In particular, $\emptyset \times \mathrm{Fin}$ and $\mathrm{Fin} \times \emptyset$ are ideals on $\omega \times \omega$ (for simplicity, $\{\emptyset\}$ is written as $\emptyset$).

The paper is organized as follows. In Section 2, we focus on injections invariant with respect to countably generated ideals. In Section 3, we study injections invariant with respect to maximal ideals. In Section 4, we discuss injections invariant with respect to various ideals induced by submeasures on ω. We show that every increasing injection is invariant with respect to ideals from a large class, however it it is not so for Erdős-Ulam ideals. In Section 5, we characterize increasing injections that are bi-invariant with respect to the classical density ideal $\mathcal{I}_d$ and the summable ideal $\mathcal{I}_{(1/n)}$. In Section 6, we show some applications of ideal invariant injections to ideal convergence of sequences.

2. Invariance with respect to countably generated ideals

We say that an ideal $\mathcal{I}$ on ω is *countably generated*, if there is a countable family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that every set A from $\mathcal{I}$ is contained in a set $A \in \mathcal{A}$. We say that $\mathcal{A}$ *generates* $\mathcal{I}$. Then $\mathcal{I}$ is is the smallest ideal which contains $\mathcal{A}$.

There are three types of countably generated ideals: Fin, Fin $\oplus\mathcal{P}(\omega)$ and Fin $\times\emptyset$ (cf. [8, Proposition 1.2.8]). We know that every $f \in \mathbf{Inj}$ is bi-Fin-invariant. The case of Fin $\oplus\mathcal{P}(\omega)$ is discussed in the following example.

Example 5. By the definition of $\mathcal{I} := \mathrm{Fin} \oplus\mathcal{P}(\omega)$, a set is in $\mathcal{I}$ iff its intersection with $A := \omega \times \{0\}$ is finite and its intersection with $B := \omega \times \{1\}$ is arbitrary. For simplicity, we can treat $\{A, B\}$ as a partition of ω into infinite sets. Let $f \in \mathbf{Inj}$. Observe that

- f is $\mathcal{I}$-invariant iff $f[B] \cap A \in \mathrm{Fin}$;
- f is bi-$\mathcal{I}$-invariant iff $f[B] \cap A \in \mathrm{Fin}$ and $f[A] \cap B \in \mathrm{Fin}$.

Hence f is $\mathcal{I}$-invariant if and only if

$$(\exists k \in \omega)(\forall n \in B)\ (f(n) \in B \text{ or } f(n) \leq k),$$

and it is bi-$\mathcal{I}$-invariant if and only if

$$(\exists k \in \omega)(\forall n \in B)\ (f(n) \in B \text{ or } f(n) \leq k) \text{ and } (\exists k \in \omega)(\forall n \in A)\ (f(n) \in A \text{ or } f(n) \leq k).$$

This shows that the sets of all $\mathcal{I}$-invariant injections and all bi-$\mathcal{I}$-invariant injections are F_σ subsets of $\mathbf{Inj}$. By the Baire category theorem and Proposition 4 it follows that those sets are true F_σ sets in the Polish space $\mathbf{Inj}$ (i.e. an F_σ sets which is are not G_δ sets). □

Let us turn to the case of $\mathcal{I} := \mathrm{Fin} \times\emptyset$.

Theorem 6. *Let $\mathcal{I} := \mathrm{Fin} \times\emptyset$. Then the sets $\mathcal{I}$-$\mathbf{Inv}$, of all $\mathcal{I}$-invariant injections, and* **bi**-$\mathcal{I}$-$\mathbf{Inv}$*, of all bi-$\mathcal{I}$-invariant injections, are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega\times\omega}$. Moreover,* **bi**-$\mathcal{I}$-$\mathbf{Inv}$ *is a true $F_{\sigma\delta}$ set in $\mathbf{Inj}$ (i.e. an $F_{\sigma\delta}$ set which is not a $G_{\delta\sigma}$ set).*

Proof. Since $\mathcal{I}$ "lives" in $\omega \times \omega$, we consider $\mathbf{Inj}$ as the respective Polish subspace of $(\omega \times \omega)^{\omega\times\omega}$. Let $B_n := \{n\} \times \omega$ for $n \in \omega$. Clearly, the family $\{B_n : n \in \omega\}$ generates $\mathcal{I}$. Take the family of all finite unions of B_n's and arrange it into a sequence $(A_n)_{n\in\omega}$. It is easy to see that, for $f \in \mathbf{Inj}$, the statement "f is $\mathcal{I}$-invariant" means that

$$(\forall n \in \omega)(\exists m \in \omega)\ f[B_n] \subseteq A_m. \tag{1}$$

This is equivalent to

$$(\forall n \in \omega)(\exists m \in \omega)(\forall (k, l) \in B_n)\ f(k, l) \in A_m.$$

Note that, for fixed m and (k, l), the set $\{f \in (\omega \times \omega)^{\omega\times\omega} : f(k, l) \in A_m\}$ is clopen. Hence

$$\mathcal{I}\text{-}\mathbf{Inv} = \mathbf{Inj} \cap \bigcap_{n\in\omega} \bigcup_{m\in\omega} \bigcap_{(k,l)\in B_n} \{f \in (\omega \times \omega)^{\omega\times\omega} : f(k, l) \in A_m\}$$

is an $F_{\sigma\delta}$ subset of $\mathbf{Inj}$. Observe that the closed set $A_{nm} := \bigcap_{(k,l)\in B_n} \{f \in (\omega \times \omega)^{\omega\times\omega} : f(k, l) \in A_m\}$ has empty interior in the space $\mathbf{Inj}$. Indeed, A_{nm} does not contain any basic open set of the form $\{f \in \mathbf{Inj} : f(k_i, l_i) = (r_i, s_i) \text{ for } i = 1, \dots, p\}$ since the set B_n is infinite. Hence A_{nm} is nowhere dense, and consequently, $\mathcal{I}$-$\mathbf{Inv}$ is meager.

For $f \in \mathbf{Inj}$, the statement "f^{-1} is $\mathcal{I}$-invariant" means that

$$(\forall n \in \omega)(\exists m \in \omega)\ f^{-1}[B_n] \subseteq A_m. \tag{2}$$

This is equivalent to

$$(\forall n \in \omega)(\exists m \in \omega)(\forall (k, l) \notin A_m)\ f(k, l) \notin B_n.$$

Note that, for fixed n and (k, l), the set $\{f \in (\omega \times \omega)^{\omega\times\omega} : f(k, l) \notin B_n\}$ is clopen. Hence

$$\mathbf{bi}\text{-}\mathcal{I}\text{-}\mathbf{Inv} = \mathcal{I}\text{-}\mathbf{Inv} \cap \bigcap_{n\in\omega} \bigcup_{m\in\omega} \bigcap_{(k,l)\notin A_m} \{f \in (\omega \times \omega)^{\omega\times\omega} : f(k, l) \notin B_n\}$$

is an $F_{\sigma\delta}$ subset of $\mathbf{Inj}$.

To prove the final assertion, define $F\colon \omega^\omega \to \mathbf{Inj}$ by $F(x) := f_x$ for $x \in \omega^\omega$ where $f_x(i,n) := (x(i), 2^i(2n-1)-1)$ for $(i,n) \in \omega \times \omega$. Evidently, $F(x) \in \mathbf{Inj}$ and the mapping F is continuous. It is known that the set $E := \{x \in \omega^\omega : x(n) \to \infty\}$ is a Π^0_3-complete set (see [15, Definition 22.9 and Exercise 23.2]). We will show that

$$E = F^{-1}[\mathbf{bi}\text{-}\mathcal{I}\text{-}\mathbf{Inv}] \tag{3}$$

which implies that $\mathbf{bi}$-$\mathcal{I}$-$\mathbf{Inv}$ is also Π^0_3-complete. Consequently, it is a true $F_{\sigma\delta}$ set in $\mathbf{Inj}$, as desired.

Let $f \in \mathbf{Inj}$. Note that condition (2) can be written as

$$(\forall n \in \omega)(\exists m \in \omega)\ B_n \subseteq f[A_m]. \tag{4}$$

If we recall the definitions of B_n and A_m and consider (1) and (4), we see that f is bi-$\mathcal{I}$-invariant if and only if

$$(\forall n \in \omega)(\exists m \in \omega)\ f[\{n\} \times \omega] \subseteq \{0, \dots, m\} \times \omega$$

and

$$(\forall n \in \omega)(\exists m \in \omega)\ (\{n\} \times \omega) \cap \bigcup_{j>m} f[\{j\} \times \omega] = \emptyset.$$

Thus for every $x \in \omega^\omega$, f_x is bi-$\mathcal{I}$-invariant if and only if

$$(\forall n \in \omega)(\exists m \in \omega)\ x(n) \le m \text{ and } (\forall n \in \omega)(\exists m \in \omega)(\forall j > m)\ x(j) > n.$$

The first part of this conjunction is always true, so $F(x) \in \mathbf{bi}$-$\mathcal{I}$-$\mathbf{Inv}$ if and only if $x(j) \to \infty$ which yields (3). □

3. Invariance with respect to maximal ideals

One can ask whether the implication in Fact 3 can be reversed for some ideal. We will show that the answer is positive for maximal ideals.

A known characterization of maximal ideals states that an ideal $\mathcal{I}$ on ω is maximal if and only if, for every $A \subseteq \omega$, either $A \in \mathcal{I}$ or $\omega \setminus A \in \mathcal{I}$. It follows that if $\mathcal{I}$ is a maximal ideal then, for any disjoint sets $A, B \subseteq \omega$, at least one of them is in $\mathcal{I}$.

We are ready to present the main result of this section.

Theorem 7. *Let $\mathcal{I}$ be a maximal ideal on ω and let $f \in \mathbf{Inj}$ be such that $\mathrm{Fix}(f) \notin \mathcal{I}^\star$. Then f is $\mathcal{I}$-invariant if and only if $f[\omega] \in \mathcal{I}$.*

Proof. The "if" part follows from Fact 1(ii), so we will prove the "only if" part. Fix a maximal ideal $\mathcal{I}$ and an injection f such that $\mathrm{Fix}(f) \notin \mathcal{I}^\star$. Then $\mathrm{Fix}(f) \in \mathcal{I}$ by the maximality of $\mathcal{I}$. Note that the orbit $O_f(n) := \{f^k(n)\colon k \in \mathbb{Z}\}$ of any $n \in \omega \setminus \mathrm{Fix}(f)$ has at least two elements. Here $f^0 := \mathrm{id}_\omega$, $f^{k+1} := f \circ f^k$ and $f^{-k} := (f^{-1})^k$ for $k \in \omega$. Define:

- $A_1 := \{n \in \omega\colon |O_f(n)| < \infty \text{ is even}\}$
- $A_2 := \{n \in \omega\colon 2 \le |O_f(n)| < \infty \text{ is odd}\}$
- $A_3 := \{n \in \omega\colon |O_f(n)| = \omega \text{ and } O_f(n) \text{ has no initial point}\}$
- $A_4 := \{n \in \omega\colon |O_f(n)| = \omega \text{ and } O_f(n) \text{ has an initial point}\}$.

Note that $A_i = \bigcup\{O_f(n)\colon n \in A_i\}$ for $i = 1,2,3,4$. First, we will prove that $A_1 \cup A_3 \in \mathcal{I}$. Let X_1 be a selector of the family $\{O_f(n)\colon n \in A_1 \cup A_3\}$. Define $B_1 := \bigcup\{f^{2k}(n)\colon n \in X_1,\ k \in \mathbb{Z}\}$ and $C_1 := \bigcup\{f^{2k+1}(n)\colon n \in X_1,\ k \in \mathbb{Z}\}$. Then $B_1 \cap C_1 = \emptyset$, so $B_1 \in \mathcal{I}$ or $C_1 \in \mathcal{I}$ by the maximality of $\mathcal{I}$. But $f[B_1] = C_1$ and $f[C_1] = B_1$, so the both sets B_1 and C_1 are in $\mathcal{I}$ since f is $\mathcal{I}$-invariant. Thus we have shown that $A_1 \cup A_3 = B_1 \cup C_1 \in \mathcal{I}$.

Next we will prove that $A_2 \in \mathcal{I}$. Let X_2 be a selector of the family $\{O_f(n)\colon n \in A_2\}$. Define $B_2 := \bigcup\{f^{2k}(n)\colon n \in X_2,\ k \in \{0,1,\dots,(|O_f(n)|-1)/2\}\}$ and $C_2 := \bigcup\{f^{2k+1}(n)\colon n \in X_2,\ k \in \{0,1,\dots,(|O_f(n)|-3)/2\}\}$. Then $B_2 \cap C_2 = \emptyset$, so $B_2 \in \mathcal{I}$ or $C_2 \in \mathcal{I}$ by the maximality of $\mathcal{I}$. But $A_2 = B_2 \cup f[B_2] \cup f^2[B_2] = C_2 \cup f[C_2] \cup f^2[C_2]$, so in the both cases we obtain $A_2 \in \mathcal{I}$.

Now we focus on A_4. We define X_4 as the set of the initial points of all orbits used in A_4. We set $B_4 := \bigcup\{f^{2k}(n)\colon n \in X_4,\ k \in \omega\}$ and $C_4 := \bigcup\{f^{2k+1}(n)\colon n \in X_4,\ k \in \omega\}$. We have $B_4 \cap C_4 = \emptyset$, so $B_4 \in \mathcal{I}$ or $C_4 \in \mathcal{I}$ by the maximality of $\mathcal{I}$. If $B_4 \in \mathcal{I}$ then also $C_4 = f[B_4] \in \mathcal{I}$ and

$A_4 = B_4 \cup C_4 \in \mathcal{I}$. If $B_4 \notin \mathcal{I}$ then $C_4 \in \mathcal{I}$ and $C_4 \cup f[C_4] = A_4 \setminus X_4 \in \mathcal{I}$. Finally, observe that $\mathrm{Fix}(f) \cup A_1 \cup A_2 \cup A_3 \cup (A_4 \setminus X_4) = f[\omega]$. Hence $f[\omega] \in \mathcal{I}$. □

By Fact 3 and Theorem 7 we obtain

Corollary 8. *Let $\mathcal{I}$ be a maximal ideal on ω and $f \in \mathbf{Inj}$. Then f is $\mathcal{I}$-invariant if and only if either $\mathrm{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.*

Now, we infer that, for maximal ideals, in the assertion of Fact 3 we can replace the implication by the equivalence.

Corollary 9. *Let $\mathcal{I}$ be a maximal ideal on ω and $f \in \mathbf{Inj}$. Then condition $\mathrm{Fix}(f) \in \mathcal{I}^\star$ is equivalent to the bi-$\mathcal{I}$-invariace of f.*

Proof. Suppose $f \in \mathbf{Inj}$ is bi-$\mathcal{I}$-invariant and $\mathrm{Fix}(f) \notin \mathcal{I}^\star$. Then $f[\omega] \in \mathcal{I}$ by Theorem 7. This together with Fact 1(ii) yield a contradiction. □

It is natural to ask whether the equivalence stated in Corollary 9 characterizes maximal ideals on ω. The following example gives a negative answer.

Example 10. Let $\mathcal{I}$ and $\mathcal{J}$ be non-isomorphic maximal ideals on ω. The ideal $\mathcal{A} := \mathcal{I} \oplus \mathcal{J}$ will give an answer to our question. For our purpose, it will be convenient to assume that $\mathcal{I}$ and $\mathcal{J}$ are maximal ideals that "live" on infinite sets A and B, respectively, where $\{A, B\}$ is a partition of ω. We will show that, for any bi-$\mathcal{A}$-invariant injection f, one has $\mathrm{Fix}(f) \in \mathcal{A}^\star$.

Fix a bi-$\mathcal{A}$-invariant $f \in \mathbf{Inj}$ and define the following sets (which form a partition of ω):

- $H_1 := \{n \in \omega \colon |O_f(n)| \geq 3$ and $O_f(n) \cap A \neq \emptyset$ and $O_f(n) \cap B \neq \emptyset\}$
- $H_2 := \{n \in \omega \colon |O_f(n)| \geq 2$ and $O_f(n) \subseteq A\}$
- $H_3 := \{n \in \omega \colon |O_f(n)| \geq 2$ and $O_f(n) \subseteq B\}$
- $H_4 := \{n \in \omega \colon |O_f(n)| = 2$ and $O_f(n) \cap A \neq \emptyset$ and $O_f(n) \cap B \neq \emptyset\}$
- $H_5 := \mathrm{Fix}(f) \cap A$
- $H_6 := \mathrm{Fix}(f) \cap B$.

Note that $H_2 \cup H_3 \in \mathcal{A}$ by an argument analogous to that used in the proof of Theorem 7. Now we focus on H_1. Let $X \subseteq A$ be a selector of the family $\{O_f(n) \colon n \in H_1\}$. Fix $O_f(n)$ for $n \in X$. Using recursion, we will define a partition of $O_f(n)$ into sets V_n^1, V_n^2, W_n^1, W_n^2. At first set $n \in V_n^1$. Assume that for some $k \in \omega$ we have already assigned $n, f(n), f^2(n) \dots, f^k(n)$ to sets V_n^1, V_n^2, W_n^1, W_n^2. If $f^{k+1}(n)$ has not been assigned yet, put

$$u(k, A) := \max\{i \leq k \colon f^i(n) \in A\}; \quad u(k, B) := \max\{i \leq k \colon f^i(n) \in B\}$$

and proceed as follows:

- if $f^{k+1}(n) \in A$ and $f^{u(k,A)}(n) \in V_n^1$, set $f^{k+1}(n) \in W_n^1$;
- if $f^{k+1}(n) \in A$ and $f^{u(k,A)}(n) \in W_n^1$, set $f^{k+1}(n) \in V_n^1$;
- if $f^{k+1}(n) \in B$ and $f^{u(k,B)}(n) \in V_n^2$, set $f^{k+1}(n) \in W_n^2$;
- if $f^{k+1}(n) \in B$ and $f^{u(k,B)}(n) \in W_n^2$, set $f^{k+1}(n) \in V_n^2$.

Now we have to deal with $f^k(n)$ for $k \in \mathbb{Z}$, $k < 0$. We also use recursion in this case. Of course we do it only in the case of infinite orbits, since for finite orbits all points are already asssigned. Put

$$l(k, A) := \min\{i \geq k \colon f^i(n) \in A\}; \quad l(k, B) := \min\{i \geq k \colon f^i(n) \in B\}$$

and proceed as follows:

- if $f^{k-1}(n) \in A$ and $f^{l(k,A)}(n) \in V_n^1$, set $f^{k-1}(n) \in W_n^1$;
- if $f^{k-1}(n) \in A$ and $f^{l(k,A)}(n) \in W_n^1$, set $f^{k-1}(n) \in V_n^1$;
- if $f^{k-1}(n) \in B$ and $f^{l(k,B)}(n) \in V_n^2$, set $f^{k-1}(n) \in W_n^2$;
- if $f^{k-1}(n) \in B$ and $f^{l(k,B)}(n) \in W_n^2$, set $f^{k-1}(n) \in V_n^2$.

Clearly, the sets V_n^1, V_n^2, W_n^1, W_n^2 defined as above, form a partition of $O_f(n)$. Moreover, V_n^1 and W_n^1 form a partition of $A \cap O_f(n)$ while V_n^2 and W_n^2 form a partition of $B \cap O_f(n)$. Moreover, for any $i, j \in \{1, 2\}$ we have $O_f(n) = \bigcup_{l \in \{-2,-1,0\}} f^l[V_n^i \cup W_n^j]$. Indeed, fix $i, j \in \{1, 2\}$ and take an arbitrary

$x \in O_f(n)$. As $O_f(n) \subseteq H_1$, observe that $x, f(x), f^2(x)$ are distinct and at least one of them belongs to $V_n^i \cup W_n^j$ due to the construction of those sets.

Now define $V^i := \bigcup_{n\in X} V_n^i$ and $W^i := \bigcup_{n\in X} W_n^i$ for $i \in \{1,2\}$. Then V^1 and W^1 form a partition of $A\cap H_1$ while V_n^2 and W_n^2 form a partition of $B\cap H_1$. Hence at least one of sets V^1, W^1 belongs to $\mathcal{I}$ and at least one of sets V^2, W^2 belongs to $\mathcal{J}$. This observation, together with $H_1 = \bigcup_{l\in\{-2,-1,0\}} f^l[V^i \cup W^j]$ for any $i,j \in \{1,2\}$, yields $H_1 \in \mathcal{A}$ by the bi-$\mathcal{A}$-invariance of f.

So far, we have shown that $H_1 \cup H_2 \cup H_3 \in \mathcal{A}$. Now we will deal with H_i for $i = 4,5,6$. Let $C := H_4 \cap A$ and $D := H_4 \cap B$. Then H_5 and C are disjoint subsets of A, so at least one of them belongs to $\mathcal{I}$. Analoguously, at least one of sets H_6 and D belongs to $\mathcal{J}$. Consider two cases:

(i) $C \in \mathcal{I}$ or $D \in \mathcal{J}$. Then it must be $C \in \mathcal{I}$ and $D \in \mathcal{J}$ thanks to the bi-$\mathcal{A}$-invariance of f. So $\mathrm{Fix}(f) = H_5 \cup H_6 = \omega \setminus (H_1 \cup H_2 \cup H_3 \cup C \cup D) \in \mathcal{A}^\star$.

(ii) $C \notin \mathcal{I}$ and $D \notin \mathcal{J}$. Then $H_5 \in \mathcal{I}$ and $H_6 \in \mathcal{J}$. Pick an infinite set $G \subseteq C$ such that $G \in \mathcal{I}$. Define $A_1 := (H_1 \cap A) \cup H_2 \cup G \cup H_5$ and $B := (H_1 \cap B) \cup H_3 \cup f[G] \cup H_6$. Then A_1, B_1 are infinite and $A_1 \in \mathcal{I}$, $B \in \mathcal{J}$. Let $g\colon A \to B$ be such that $g|_{A_1}$ is any bijection between A_1 and B_1, and $g|_{A\setminus A_1} := f|_{A\setminus A_1}$. Then g is a bijection between A and B witnessing that $\mathcal{I}$ and $\mathcal{J}$ are isomorphic which contradicts our assumption.

Hence indeed $\mathrm{Fix}(f) \in \mathcal{A}^\star$. □

Question 1. What are (reasonable) characterizations of two classes that consist of:

- ideals $\mathcal{I}$ such that every bi-$\mathcal{I}$-invariant injection f satisfies condition $\mathrm{Fix}(f) \in \mathcal{I}^\star$.
- ideals $\mathcal{I}$ such that every $\mathcal{I}$-invariant injection f satisfies either $f[\omega] \in \mathcal{I}$ or $\mathrm{Fix}(f) \in \mathcal{I}^\star$?

4. Invariance with respect to ideals induced by submeasures

An important class of ideals on ω consists of those of them which are induced by submeasures (see [8]). A *submeasure* on ω is a function $\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ such that:

- $\varphi(\emptyset) = 0$;
- if $A \subseteq B$ then $\varphi(A) \le \varphi(B)$,
- $\varphi(A \cup B) \le \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \omega$.

A submeasure φ is called *lower semicontinuous* (lsc, in short) if

$$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap \{0,\dots,n-1\}) \text{ for all } A \subseteq \omega.$$

For an lsc submeasure φ, let

$$\mathrm{Fin}(\varphi) := \{A \subseteq \omega\colon \varphi(A) < \infty\}, \quad \mathrm{Exh}(\varphi) := \left\{A \subseteq \omega\colon \lim_{n\to\infty} \varphi(A \cap \{n, n+1, \dots\}) = 0\right\}.$$

Identifying subsets of ω with their characteristic functions, one can equip the power set $\mathcal{P}(\omega)$ with the topology of the Cantor space 2^ω. Hence ideals on ω can by Borel (of a certain class), analytic, coanalytic, etc.

An ideal $\mathcal{I}$ on ω is called a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets in $\mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \subseteq^\star A$ for all $n \in \omega$ (where $A_n \subseteq^\star A$ means that $A_n \setminus A \in \mathrm{Fin}$).

It follows that for every lsc submeasure φ on ω, $\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, and $\mathrm{Fin}(\varphi)$ is an F_σ ideal which includes $\mathrm{Exh}(\varphi)$ [8, Lemma 1.2.2]. Some important examples can be found in [8, Example 1.2.3]. Note that $\mathcal{I}_d$ is of the form $\mathrm{Exh}(\varphi)$ where $\varphi(A) := \sup_{n\in\omega}(|A \cap \{0,\dots,n-1\}|/n)$ is the respective lsc submeasure. Let us mention about *summable ideals* of the form $\mathcal{I}_{(f(n))} := \{A \subseteq \omega\colon \sum_{n\in A} f(n) < \infty\}$ where $f\colon \omega \to [0,\infty)$ is such that $\sum_{n\in\omega} f(n) = \infty$. Note that $\mathcal{I}_{(f(n))} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ where $\varphi(A) := \sum_{n\in A} f(n)$ is a lsc submeasure on ω. Consequently, $\mathcal{I}_{(f(n))}$ is an F_σ ideal which is a P-ideal. The theorem of Solecki [23] states that each analytic P-ideal on ω is of the form $\mathrm{Exh}(\varphi)$ for some lsc submeasure φ on ω.

We propose the following useful criterion for the bi-invariance of injections with respect to ideals of the form $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.

Proposition 11. *Let φ be a lsc submeasure on ω. Let $f\colon \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on f such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then f is invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then f is bi-invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.*

Proof. Let $A \in \mathrm{Fin}(\varphi)$. Since $\varphi(f[A]) \leq \varphi(A)/C_f < \infty$, then $f[A] \in \mathrm{Fin}(\varphi)$.

Now, consider $\mathrm{Exh}(\varphi)$. Let $A \in \mathrm{Exh}(\varphi)$, $A \neq \emptyset$. We want to show that $\varphi(f[A] \cap \{n, n+1, \dots\}) \to 0$. Fix sufficiently large $n \in \omega$ and $m > n$. Then $f[A] \cap \{n, n+1, \dots, m\} = \{f(n_1), \dots, f(n_k)\}$ for some $n_1, \dots n_k \in \omega$. Put $n' := \min\{n_j : 1 \leq j \leq k\}$ and $m' := \max\{n_j : 1 \leq j \leq k\}$. Since f is increasing, then $f[A \cap \{n', \dots, m'\}] = f[A] \cap \{n, \dots, m\}$ and consequently $C_f \varphi(f[A] \cap \{n, \dots, m\}) \leq \varphi(A \cap \{n', \dots, m'\}) \leq \varphi(A \cap \{n', n'+1, \dots\})$. Now, letting $m \to \infty$, we have $C_f \varphi(f[A] \cap \{n, n+1, \dots\}) \leq \varphi(A \cap \{n', n'+1, \dots\})$. If $n \to \infty$ then $n' \to \infty$. Hence $A \in \mathrm{Exh}(\varphi)$ implies $\varphi(f[A] \cap \{n, n+1, \dots\}) \to 0$ as desired. The proof of the second part of the assertion goes similarly. $\square$

In the above proposition, by the lower semicontinuity of φ, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any A with $|A| \leq n$ for some fixed n. The following example shows that it is not true.

Example 12. Fix $n \in \omega$, $n \geq 1$ and define a submeasure φ on ω as follows. Put $x_{2j+1} := 1/(j+1)$ and $x_{2j} := 0$ for $j \in \omega$. For $A \subseteq \omega$ let $\mu(A) := \sum_{j \in A} x_j$. For $i \in \omega$ let $\mu_{2i}(A) := 1/(i+1)$ if $2i \in A$ and $\mu_{2i}(A) := 0$, otherwise. Define $\varphi(A) := \mu(A) + \sup_{i \in \omega} \mu_{2i}(A)$. Note that $\varphi(A) = \mu(A) + 1/(i+1)$ where $2i =: \min(A \cap 2\omega)$ and we use the following convention: $\min \emptyset := \infty$ and $1/\infty := 0$. Note that φ is an lsc submeasure on ω.

We will prove the following properties:

(i) For any increasing injection $f\colon \omega \to \omega$ with $f(m) > m$ for each $m \in \omega$, the inequality $\varphi(A) \geq \varphi(f[A])/n$ holds for any $A \subseteq \omega$ with $|A| \leq n$.

(ii) For any increasing injection $f\colon \omega \to \omega$ there is $N \in \omega$ such that the inequality $\varphi(A) \geq \varphi(f[A])/n$ holds for any $A \subseteq \omega \setminus \{0, \dots, N\}$ with $|A| \leq n$.

(iii) The injection $g(n) := n+1$, $n \in \omega$, is not $\mathrm{Exh}(\varphi)$-invariant.

To show (i) fix an increasing injection $f\colon \omega \to \omega$ with $f(m) > m$ for each $m \in \omega$. Fix $A \subseteq \omega$ with $|A| = n$. Let $2i_1 := \min(A \cap 2\omega)$ and $2i_0 := \min(f[A] \cap 2\omega)$. Put $A_1 := A \cap 2\omega$, $A_2 := A \setminus 2\omega$, $B_1 := \{m \in A : f(m) \in 2\omega\}$ and $B_2 := \{m \in A : f(m) \notin 2\omega\}$. Then

$$\mu(f[A]) = \sum_{m \in B_2} x_{f(m)} = \sum_{m \in A_1 \cap B_2} x_{f(m)} + \sum_{m \in A_2 \cap B_2} x_{f(m)} \leq \sum_{m \in A_1 \cap B_2} x_{f(m)} + \sum_{m \in A_2 \cap B_2} x_m \tag{5}$$

(if $A_i \cap B_j = \emptyset$, the respective sum is 0). First assume that the set $A_1 \cap B_2$ is nonempty and its elements are ordered as $2j_1 < 2j_2 < \dots < 2j_k$ for some $k \leq n$. Then

$$\sum_{m \in A_1 \cap B_2} x_{f(m)} \leq \sum_{l=1}^{k} x_{2j_l+1} = \sum_{l=1}^{k} \frac{1}{j_l+1} \leq \frac{k}{j_1+1}. \tag{6}$$

Note that $j_1 \geq i_1$ and this is also true if $A_1 \cap B_2 = \emptyset$ since then, by our convention, we may assume $2j_1 := \infty$ and $k := 0$. If $B_1 = \emptyset$ then by (5) and (6) we have

$$\varphi(f[A]) = \mu(f[A]) \leq \sum_{m \in A_2 \cap B_2} x_m + \frac{k}{j_1+1} \leq n \left(\sum_{m \in A_2} x_m + \frac{1}{i_1+1} \right) = n\varphi(A).$$

Assume now that $B_1 \neq \emptyset$. Pick $m' \in A$ with $f(m') = 2i_0$. By the assumption on f we have $m' < 2i_0$. Consider two cases.

1^0 Let $m' \in A_1$. Then $2i_1 < 2i_0$. From $2i_1 \in A_1 \setminus B_2$ it follows that $k < n$. We have

$$\varphi(f[A]) = \mu(f[A]) + \frac{1}{i_0+1} \leq \sum_{m \in A_2 \cap B_2} x_m + \frac{k}{j_1+1} + \frac{1}{i_0+1} \leq \sum_{m \in A_2} x_m + \frac{k+1}{i_1+1} \leq \sum_{m \in A_2} x_m + \frac{n}{i_1+1} \leq n\varphi(A).$$

2^0 Let $m' \in A_2$. Since $m' \in A_2 \cap B_1$, we have

$$\varphi(f[A]) = \mu(f[A]) + \frac{1}{i_0+1} \leq \sum_{m \in A_2 \cap B_2} x_m + \frac{k}{j_1+1} + x_{m'} \leq \sum_{m \in A_2} x_m + \frac{k}{i_1+1} \leq n\varphi(A).$$

To see (ii) note that an increasing injection f is either the identity or there exists $N \in \omega$ such that $f(m) > m$ for all $m \geq N$. If $f = \mathrm{id}$ then (ii) is obvious. If there is $N \in \omega$ such that $f(m) > m$ for all $m \geq N$ then from (i) it follows that $\varphi(A) \geq \varphi(f[A])/n$ for any $A \subseteq \omega \setminus \{0, \dots, N\}$ with $|A| \leq n$.

To see (iii) note that $2\omega \in \mathrm{Exh}(\varphi)$ while $2\omega + 1 \notin \mathrm{Exh}(\varphi)$. □

If $f\colon \omega \to \omega$ is an increasing injection then $|A \cap \{0, \dots, n-1\}| \geq |f[A] \cap \{0, \dots, n-1\}|$ for all $A \subseteq \omega$. Therefore $\varphi(A) = \sup_{n\in\omega}(|A \cap \{0, \dots, n-1\}|/n) \geq \sup_{n\in\omega}(|f[A] \cap \{0, \dots, n-1\}|/n) = \varphi(f[A])$. Thus by Proposition 11, every increasing injection is $\mathcal{I}_d$-invariant. Note also that if $g\colon \omega \to [0, \infty)$ is decreasing and $\sum_{n\in\omega} g(n) = \infty$ then every increasing injection is $\mathcal{I}_{(g(n))}$-invariant. In the next section, we will characterize bi-$\mathcal{I}$-invariant increasing injections for $\mathcal{I} := \mathcal{I}_d$ and for $\mathcal{I}$ equal to the summable ideal $\mathcal{I}_{(1/n)}$.

A general notion of density for subsets of ω was considered in [2]. Namely, denote by G the set of all functions $g\colon \omega \to [0, \infty)$ satisfying conditions $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$. Then we define the *upper density of weight* $g \in G$ by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap \{0, \dots, n-1\}|}{g(n)} \quad \text{for } A \subseteq \omega.$$

Then consider the following ideal

$$\mathcal{Z}_g := \{A \subseteq \omega \colon \overline{d}_g(A) = 0\}.$$

In particular, $\mathcal{I}_d = \mathcal{Z}_g$ for $g(n) := n$. Note also that $\mathcal{Z}_g$ is of the form $\mathrm{Exh}(\varphi)$ where $\varphi(A) = \sup_{n\in\omega}(|A \cap \{0, \dots, n-1\}|/g(n))$ for $A \subseteq \omega$. Hence using the same argument as for $\mathcal{I}_d$, we infer that every increasing injection is $\mathcal{Z}_g$-invariant.

Note that all ideals $\mathcal{Z}_g$, $g \in G$, are tall (see [2]). We will use this fact to show the promised improvement of Example 2.

Example 13. Fix α_0, α_1 with $0 < \alpha_0 < \alpha_1 \leq 1$ and consider $\mathcal{I}_i := \mathcal{Z}_{g_i}$ where $g_i(n) := n^{\alpha_i}$ for $i = 0, 1$ and $n \in \omega$. It is known that $\mathcal{I}_0 \subsetneq \mathcal{I}_1$ (see [2, Corollary 2.5]). Define $\mathcal{I} := \mathcal{I}_0 \oplus \mathcal{I}_1$. Note that $\mathcal{I}$ is a tall ideal on $\omega \times \{0, 1\}$. Consider a bijection $f\colon \omega \times \{0, 1\} \to \omega \times \{0, 1\}$ given by

$$f(2n+1, 0) := (2n+1, 1),\ f(2n, 0) := (n, 0),\ f(n, 1) := (2n, 1) \text{ for } n \in \omega.$$

It can be easily seen that f is $\mathcal{I}$-invariant. Pick $A \in \mathcal{I}_1 \setminus \mathcal{I}_0$. Then $A \times \{0\} \notin \mathcal{I}$ but $B := f[A \times \{0\}] \in \mathcal{I}$. So $f^{-1}[B] \notin \mathcal{I}$. Hence f^{-1} is not $\mathcal{I}$-invariant. □

There is a family of ideals on ω, larger than $\{\mathcal{Z}_g \colon g \in G\}$, which consists of the so-called density ideals (in the sense of Farah). To describe them, recall some definitions (see [8]).

For a measure μ defined on subsets of ω, the *support* of μ is the set $\{n \in \omega \colon \mu(\{n\}) > 0\}$. Consider a sequence $(\mu_i)_{i\in\omega}$ of measures with pairwise disjoint supports being finite subsets of ω. Put $\varphi := \sup_{i\in\omega} \mu_i$. Then φ is an lsc submeasure on ω. If $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a sequence $(\mu_i)_{i\in\omega}$ as above, we say that $\mathcal{I}$ is a *density ideal*, more exactly, this is the *density ideal generated by* $(\mu_i)_{i\in\omega}$.

An ideal $\mathcal{I}$ is called an *Erdős-Ulam ideal* (an EU ideal, in short) if, for some function $f\colon \omega \to [0, \infty)$ such that $\sum_{n\in\omega} f(n) = \infty$, we have

$$\mathcal{I} = \left\{ A \subseteq \omega \colon \lim_{n\to\infty} \frac{\sum_{i\leq n\, i\in A} f(i)}{\sum_{i\leq n} f(i)} = 0 \right\}.$$

It is known that each EU ideal is a density ideal but the converse need not be true (see [8]). It was proved in [2] that each $\mathcal{Z}_g$, $g \in G$, is a density ideal but there is no inclusion between $\{\mathcal{Z}_g \colon g \in G\}$ and the set of all EU ideals.

For a non-increasing $f\colon \omega \to [0, \infty)$ such that $\sum_{n\in\omega} f(n) = \infty$, let $\mathcal{I}$ be the corresponding EU ideal. Note that every increasing injection $f\colon \omega \to \omega$ is $\mathcal{I}$-invariant (the argument is similar to that for $\mathcal{I}_d$). However, this property does not hold for all EU ideals which will be shown in Proposition 15. We need the following useful characterization.

Theorem 14. [8, 1.13.3] *Let $\mathcal{I}$ be the density ideal generated by a sequence of measures $(\mu_n)_{n\in\omega}$. Then $\mathcal{I}$ is an EU ideal if and only if the following conditions hold:*

(D1) $\sup_{n\in\omega}\mu_n(\omega)<\infty$,
(D2) $\lim_{n\to\infty}\sup_{i\in\omega}\mu_n(\{i\})=0$,
(D3) $\limsup_{n\to\infty}\mu_n(\omega)>0$.

If $\mathcal{I}$ is an EU ideal then we may additionally assume that all μ_n's are probability measures.

Proposition 15. *There exist an Erdős-Ulam ideal $\mathcal{I}$ and an increasing injection $f\colon\omega\to\omega$ such that neither f is $\mathcal{I}$-invariant nor f^{-1} is $\mathcal{I}$-invariant.*

Proof. For $n\geq 3$ let $\mu_n(\{k\})$ be equal to $1/n$ if $k\in[2^n,2^n+n)$, and equal to 0, otherwise. Let $\mathcal{I}$ be the density ideal generated by $(\mu_n)_{n\geq 3}$. Note that conditions (D1)–(D3) hold, hence $\mathcal{I}$ is an EU-ideal. Set

$$A_n:=\omega\cap[2^n,2^n+n),\quad B_n:=\omega\cap[2^n+n,2^{n+1}),\text{ and } A:=\bigcup_{n\geq 4}A_n,\quad B:=\bigcup_{n\geq 3}B_n.$$

Obviously, $A\notin\mathcal{I}$ and $B\in\mathcal{I}$. Now, we define an increasing injection f as follows: $f|_{\{0,\dots,10\}}:=\mathrm{id}$ and for $n\geq 3$ let $f|_{B_n\cup A_{n+1}}:=\mathrm{id}+|B_n|$. Then we have $A_{n+1}\subseteq f[B_n]\subseteq A_{n+1}\cup B_{n+1}$ for $n\geq 3$, and $f[A_n]\subseteq B_n$ for $n\geq 4$. Now, one can see that $A\subseteq f[B]$, so f is not $\mathcal{I}$-invariant, and $f[A]\subseteq B$, so $A\subseteq f^{-1}[B]$ and thus f^{-1} is not $\mathcal{I}$-invariant. □

5. Bi-invariance with respect to the ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$

We are going to show that bi-$\mathcal{I}$-invariant increasing injections f for the ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$ are the same. It turns out that they can be characterized by condition $\underline{d}(f[\omega])>0$ or equivalently, by the linear growth of f.

Theorem 16. *Let $f\colon\omega\to\omega$ be an increasing injection. Then f is bi-$\mathcal{I}_d$-invariant if and only if $\underline{d}(f[\omega])>0$.*

Proof. "$\Leftarrow$" Assume that $\underline{d}(f[\omega])=0$. We will prove that f^{-1} is not $\mathcal{I}_d$-invariant. If $\overline{d}(f[\omega])=0$, then $f[\omega]\in\mathcal{I}_d$ and $f^{-1}[f[\omega]]=\omega\notin\mathcal{I}_d$, so we are done.

Now, assume that $\underline{d}(f[\omega])=0$ and $\overline{d}(f[\omega])=a>0$. We will find $A\in\mathcal{I}_d$ such that $f^{-1}[A]\notin\mathcal{I}_d$. Let $0<n_1<n_2<\dots$ be a sequence of integers such that for every $k\in\omega$ we have

$$\frac{|f[\omega]\cap\{0,\dots,n_{2k-1}-1\}|}{n_{2k-1}-1}<\frac{1}{2^k}\,,\tag{7}$$

$$\frac{|f[\omega]\cap\{0,\dots,n_{2k}\}|}{n_{2k}}>\frac{a}{2}\,.\tag{8}$$

Let $l_k\in\omega$ be the smallest number such that

$$\frac{|f[\omega]\cap\{n_{2k-1},\dots,l_k+1\}|}{l_k+1}>\frac{a}{2k}.\tag{9}$$

Put $A:=f[\omega]\cap\bigcup_{k=1}^{\infty}\{n_{2k-1},\dots,l_k\}$. By (7) and (9) we have

$$\frac{|A\cap\{0,\dots,l_k\}|}{l_k}=\frac{|A\cap\{0,\dots,n_{2k-1}-1\}|}{l_k}+\frac{|A\cap\{n_{2k-1},\dots,l_k\}|}{l_k}$$
$$=\frac{|A\cap\{0,\dots,n_{2k-1}-1\}|}{n_{2k-1}-1}\cdot\frac{n_{2k-1}-1}{l_k}+\frac{|A\cap\{n_{2k-1},\dots,l_k\}|}{l_k}\leq\frac{1}{2^k}+\frac{a}{2k}.$$

Thus $\overline{d}(A)=0$ which means that $A\in\mathcal{I}_d$.

Since $l_k+1\in f[\omega]$, we have

$$|f^{-1}[\{n_{2k-1},\dots,l_k+1\}]|=|f^{-1}[\{n_{2k-1},\dots,l_k\}]|+1$$

and consequently, by the definition of l_k,

$$\frac{l_ka+a-2k}{2k}=\frac{(l_k+1)a}{2k}-1<|f^{-1}[\{n_{2k-1},\dots,l_k\}]|\leq\frac{l_ka}{2k}.\tag{10}$$

Using (9) we have

$$\frac{a}{2k} < \frac{|f[\omega] \cap \{n_{2k-1}, \ldots, l_k+1\}|}{l_k+1} \leq \frac{l_k+2-n_{2k-1}}{l_k+1}$$

which implies that

$$n_{2k-1} - 1 < \left(1 - \frac{a}{2k}\right)(l_k+1). \tag{11}$$

Now, denote by m_k the largest element of $f^{-1}[\{n_{2k-1}, \ldots, l_k\}]$. Then by (7), (10) and (11) we obtain

$$\frac{|f^{-1}[\{n_{2k-1}, \ldots, l_k\}]|}{m_k} \geq \frac{(l_k a + a - 2k)/(2k)}{(n_{2k-1}-1)/2^k + (l_k a)/(2k)}$$

$$= \frac{a/(2k) + a/(2kl_k) - 1/l_k}{(n_{2k-1}-1)/(l_k 2^k) + a/(2k)} \geq \frac{a/(2k) + a/(2kl_k) - 1/l_k}{\frac{1-a/(2k)}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2k}} \to 1 \quad \text{if } k \to \infty.$$

Indeed, for $k \to \infty$,

$$\frac{a/(2k)}{\frac{1-a/(2k)}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2k}} \to 1, \quad \frac{a/(2kl_k)}{\frac{1-a/(2k)}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2k}} = \frac{a/(2l_k)}{\frac{k-a/2}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2}} \to 0,$$

and using $l_k > n_{2k-1} > 2^k$, we get

$$\frac{1/l_k}{\frac{1-a/(2k)}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2k}} = \frac{k/l_k}{\frac{k-a/2}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2}} \leq \frac{k/2^k}{\frac{k-a/2}{2^k} \cdot \frac{l_k+1}{l_k} + \frac{a}{2}} \to 0.$$

Thus $\overline{d}(f^{-1}[A]) = 1$ and therefore $f^{-1}[A] \notin \mathcal{I}_d$.

"$\Rightarrow$" Assume that $a := \underline{d}(f[\omega]) > 0$. We will prove that, for any $A \subseteq \omega$ with $\overline{d}(A) > 0$, we have $\overline{d}(f[A]) > 0$. So, fix any $A \subseteq \omega$ with $b := \overline{d}(A) > 0$. Pick $n_0 \in \omega$ such that $|f[\omega] \cap \{0, \ldots, n-1\}|/n > a/2$ for all $n > n_0$. Then pick $m_0 \in \omega$ such that $f(m_0) > n_0$ and $|A \cap \{0, \ldots, m-1\}|/m > b/2$ for all $m > m_0$. Then for all $k > f(m_0)$ we have

$$\frac{|f[A] \cap \{0, \ldots, k-1\}|}{k} = \frac{|f[A] \cap \{0, \ldots, k-1\}|}{|f[\omega] \cap \{0, \ldots, k-1\}|} \cdot \frac{|f[\omega] \cap \{0, \ldots, k-1\}|}{k} > \frac{b}{2} \cdot \frac{a}{2} > 0.$$

Hence $\overline{d}(f[A]) > 0$. Thus we have proved that if $A \notin \mathcal{I}_d$ then $f[A] \notin \mathcal{I}_d$. So, f^{-1} is $\mathcal{I}_d$-invariant by Fact 1(ii). □

In fact we have proved more than it is stated in the assertion of Theorem 16. Namely, we have shown that $\underline{d}(f[\omega]) > 0$ for **every** bi-$\mathcal{I}_d$-invariant injection. However, the fact that an injection f has the property $\underline{d}(f[\omega]) > 0$ does not imply that f is bi-$\mathcal{I}_d$-invariant. To see this, take an infinite set $A \in \mathcal{I}_d$, a bijection $h\colon A \to \omega \setminus A$ and put $f := h \cup h^{-1}$. Then neither f nor f^{-1} is $\mathcal{I}_d$-invariant but $f[\omega] = \omega$.

We will need a technical lemma. First note that, using twice l'Hôpital's rule, we have

$$\lim_{x \to 0} \frac{x(\exp\frac{x}{2} - 1)}{\exp x^2 - 1} = \frac{1}{2}.$$

Then replace x by $1/k$ for integers $k \to \infty$. Hence we can choose $k_0 \in \omega$ such that

$$\frac{\exp\frac{1}{2k} - 1}{k\left(\exp\frac{1}{k^2} - 1\right)} \leq \frac{3}{4} \quad \text{for every } k \geq k_0. \tag{12}$$

Lemma 17. *Let $k \geq k_0$. There exists $n_0 \in \omega$ such that for each $N \geq n_0$ one can find $r \in \omega$ satisfying the conditions*

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{N+r} \geq \frac{1}{2k}$$

and

$$\frac{1}{kN} + \frac{1}{kN+1} + \cdots + \frac{1}{kN+r} \leq \frac{1}{k^2}.$$

Proof. Fix $N \geq 2$. Note that

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{N+r} \geq \int_0^{r+1} \frac{dx}{N+x} = \log \frac{N+r+1}{N}.$$

The inequality

$$\log \frac{N+r+1}{N} \geq \frac{1}{2k}$$

is equivalent to

$$r \geq N\left(\exp \frac{1}{2k} - 1\right) - 1. \tag{13}$$

On the other hand,

$$\frac{1}{kN} + \frac{1}{kN+1} + \cdots + \frac{1}{kN+r} \leq \int_0^{r+1} \frac{dx}{kN-1+x} = \log \frac{kN+r}{kN-1}.$$

The inequality

$$\log \frac{kN+r}{kN-1} \leq \frac{1}{k^2}$$

is equivalent to

$$r \leq kN\left(\exp \frac{1}{k^2} - 1\right) - \exp \frac{1}{k^2}. \tag{14}$$

A common solution $r \in \omega$ of inequalities (13) and (14) will exist provided that

$$kN\left(\exp \frac{1}{k^2} - 1\right) - \exp \frac{1}{k^2} - \left(N\left(\exp \frac{1}{2k} - 1\right) - 1\right) \geq 1$$

which is equivalent to

$$\frac{kN\left(\exp \frac{1}{k^2} - 1\right) - \exp \frac{1}{k^2}}{N\left(\exp \frac{1}{2k} - 1\right)} \geq 1. \tag{15}$$

Pick $n_0 \geq 2$ such that

$$\frac{\exp \frac{1}{k^2}}{N\left(\exp \frac{1}{2k} - 1\right)} \leq \frac{1}{6} \text{ for all } N \geq n_0.$$

Then (15) is true for all $N \geq n_0$ since by (12) we have

$$\frac{kN\left(\exp \frac{1}{k^2} - 1\right) - \exp \frac{1}{k^2}}{N\left(\exp \frac{1}{2k} - 1\right)} = \frac{k\left(\exp \frac{1}{k^2} - 1\right)}{\exp \frac{1}{2k} - 1} - \frac{\exp \frac{1}{k^2}}{N\left(\exp \frac{1}{2k} - 1\right)} \geq \frac{4}{3} - \frac{1}{6} > 1.$$

□

Theorem 18. *Let $f\colon \omega \to \omega$ be an increasing injection. Then f is bi-$\mathcal{I}_{(1/n)}$-invariant if and only if there is a constant $C > 0$ such that $f(n) \leq Cn$ for every $n \geq 1$.*

Proof. Assume first that $f(n) \leq Cn$ for every $n \geq 1$. Let $A \subseteq \omega$ be such that $\sum_{n \in A} 1/f(n) < \infty$. Then

$$\sum_{n \in A} \frac{1}{f(n)} \geq \frac{1}{C} \sum_{n \in A} \frac{1}{n}$$

and consequently, $A \in \mathcal{I}_{(1/n)}$.

Now, let $f\colon \omega \to \omega$ be an increasing injection such that for every $C > 0$ there is $n \in \omega$ with $f(n) > Cn$. Note that for any $C > 0$ there are infinitely many numbers $n \in \omega$ with $f(n) > Cn$. Hence for any $C > 0$ we can find an arbitrarily large n such that $f(n) > Cn$. Now, we will define inductively sequences $N_0, N_1, \dots$ and $r_0, r_1, \dots$ of integers in the following way. By Lemma 17 there are $N_0 \in \omega$ and $r_0 \in \omega$ such that

$$\sum_{i=0}^{r_0} \frac{1}{N_0 + i} \geq \frac{1}{2k_0}, \quad \sum_{i=0}^{r_0} \frac{1}{k_0 N_0 + i} \leq \frac{1}{k_0^2} \quad \text{and} \quad f(N_0) > k_0 N_0.$$

Assume that we have already defined $N_0, \dots, N_{k-1}$ and $r_0, \dots, r_{k-1}$. By Lemma 17 there are $N_k > N_{k-1} + r_{k-1}$ and r_k such that

$$\sum_{i=0}^{r_k} \frac{1}{N_k + i} \geq \frac{1}{2(k_0+k)}, \quad \sum_{i=0}^{r_k} \frac{1}{(k_0+k)N_k + i} \leq \frac{1}{(k_0+k)^2} \quad \text{and} \quad f(N_k) > (k_0+k)N_k.$$

Define an increasing injection $g : \omega \to \omega$ as follows: $g(n) := n + (k_0 + k - 1)N_k$ for $N_k \leq n < N_{k+1}$ where $N_{-1} := 0$. Note that $g(N_k) = (k_0+k)N_k < f(N_k)$. Since f and g are increasing, we have $g(n) \leq f(n)$ for every $n \in \omega$. Therefore

$$\sum_{n \in A} \frac{1}{g(n)} < \infty \quad \text{implies} \quad \sum_{n \in A} \frac{1}{f(n)} < \infty$$

for any $A \subseteq \omega$. Thus $g[A] \in \mathcal{I}_{(1/n)}$ implies $f[A] \in \mathcal{I}_{(1/n)}$. Let $A := \bigcup_{k=1}^{\infty} [N_k, N_k + r_k]$. By our inductive definition,

$$\sum_{n \in A} \frac{1}{n} = \sum_{k=0}^{\infty} \sum_{i=0}^{r_k} \frac{1}{N_k + i} \geq \sum_{k=0}^{\infty} \frac{1}{2(k_0+k)} = \infty$$

and

$$\sum_{n \in g[A]} \frac{1}{n} = \sum_{k=0}^{\infty} \sum_{i=0}^{r_k} \frac{1}{(k_0+k)N_k + i} \leq \sum_{k=0}^{\infty} \frac{1}{(k+k_0)^2} < \infty.$$

Therefore f is not bi-$\mathcal{I}_{(1/n)}$-invariant. □

Proposition 19. *Let $f : \omega \to \omega$ be an increasing injection. There is $C \in \omega$ with $f(n) \leq Cn$ for every $n \geq 1$ if and only if $\underline{d}(f[\omega]) > 0$.*

Proof. At the beginning, note that in the definitions of $\underline{d}(\cdot)$ and $\overline{d}(\cdot)$, one can use $\{1, \dots, n\}$ instead of $\{0, \dots, n-1\}$. Assume first that $f(n) \leq Cn$ for each $n \geq 1$. This means that, for each $n \geq 1$, there are at least n elements from $f[\omega]$ in the set $\{1, 2, \dots, Cn\}$. Thus

$$\frac{|f[\omega] \cap \{1, 2, \dots, Cn + r\}|}{Cn + r} \geq \frac{n}{Cn + r}$$

for any $r = 0, 1, \dots, C-1$. Hence $\underline{d}(f[\omega]) \geq 1/C$.

Now assume that for any $C \in \omega$ there is $n \in \omega$ with $f(n) > Cn$. Then we can find a sequence of positive integers $N_2 < N_3 < \dots$ such that $f(N_k) > kN_k$. Since f is increasing, $f(n) > kN_k$ for every $n > N_k$. Therefore the set $f[\omega] \cap \{1, 2, \dots, kN_k\}$ has at most N_k elements. Thus

$$\frac{|f[\omega] \cap \{1, 2, \dots, kN_k\}|}{kN_k} \leq \frac{N_k}{kN_k} = \frac{1}{k}.$$

Hence $\underline{d}(f[\omega]) = 0$. □

Now, putting together Theorems 16, 18 and Proposition 19, we obtain

Corollary 20. *Let $f : \omega \to \omega$ be an increasing injection. The following conditions are equivalent:*

(i) *f is bi-$\mathcal{I}_d$-invariant;*
(ii) *$\underline{d}(f[\omega]) > 0$;*
(iii) *there is $C \in \omega$ such that $f(n) \leq Cn$ for every $n \geq 1$;*
(iv) *f is bi-$\mathcal{I}_{(1/n)}$-invariant.*

Question 2. Let $g : \omega \to [0, \infty)$ be increasing and let $\mathcal{J}$ denote the EU ideal associated with g. Is it true that the class of all increasing functions $f : \omega \to \omega$ which are bi-$\mathcal{J}$-invariant equals the class of all increasing injections $f : \omega \to \omega$ which are bi-$\mathcal{I}_{(1/g(n))}$-invariant (where $\mathcal{I}_{(1/g(n))}$ is the respective summable ideal)? Corollary 20 says that this is true for $g(n) := n$, $n \in \omega$.

By $\mathbf{Inj}^{\uparrow}$ we denote the space of all increasing injections in $\mathbf{Inj}$. Note that

$$\mathbf{Inj}^{\uparrow} = \bigcap_{n>0} \bigcap_{k<n} \bigcup_{i>0} \bigcup_{j<i} \{f \in \mathbf{Inj} : f(n) = i \text{ and } f(k) = j\}.$$

Thus $\mathbf{Inj}^{\uparrow}$ is a G_δ subset of $\mathbf{Inj}$ and consequently, $\mathbf{Inj}^{\uparrow}$ is a Polish space.

Proposition 21. *Let $\mathcal{I} \in \{\mathcal{I}_d, \mathcal{I}_{(1/n)}\}$. The set $B^{\uparrow}_{\mathcal{I}}$ of all increasing bi-$\mathcal{I}$-invariant injections is a true F_σ meager subset of* $\mathbf{Inj}^{\uparrow}$.

Proof. Using Corollary 20, we choose the description (iii) of $B^{\uparrow}_{\mathcal{I}}$. So, $B^{\uparrow}_{\mathcal{I}} = \bigcup_{C\in\omega} A_C$ where

$$A_C := \bigcap_{n\geq 1} \left\{ f \in \mathbf{Inj}^{\uparrow} : f(n) \leq Cn \right\}.$$

Note that each set A_C is closed. Also, it is meager since its interior is empty. Indeed, it cannot contain any basic open set V of the form $\{f \in \mathbf{Inj}^{\uparrow} \colon f(k_i) = l_i \text{ for } i = 1, \dots, p\}$ since we can pick $f \in V$ such that $f(n) > Cn$ for a sufficiently large n. Hence we have shown that $B^{\uparrow}_{\mathcal{I}}$ is an F_σ meager set. Note that $B^{\uparrow}_{\mathcal{I}}$ cannot be a G_δ set since it is dense (and using the Baire category argument in $\mathbf{Inj}^{\uparrow}$, we are done). Indeed, considering V as above, we can easily find $C \in \omega$ and $f \in V \cap A_C$. □

6. Applications to ideal convergence

Given an ideal $\mathcal{I}$ on ω, we say that a sequence $(x_n)_{n\in\omega}$ in a metric space (X, ρ) is *$\mathcal{I}$-convergent* to $x \in X$ (see e.g. [16]) if $\{n \in \omega \colon \rho(x_n, x) \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. We then write $\mathcal{I}\text{-}\lim_{n\in\omega} x_n = x$ or simply $\mathcal{I}\text{-}\lim_n x_n = x$. Note that $\mathrm{Fin}\text{-}\lim_n x_n = x$ means the usual convergence $\lim_n x_n = x$. If $\mathcal{I} := \mathcal{I}_d$, we deal with *statistical convergence* studied by several authors (see [9, 21, 12] and also [3, 7, 19, 5]). A general case was investigated for instance in [3, 4, 10, 20, 11, 6]. Without loss of generality we will focus on $\mathcal{I}$-convergence for sequences of real numbers.

Consider the following question. Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{I}\text{-}\lim_n x_n = x$. Does there exists a bi-$\mathcal{I}$-invariant injection f such that $\lim_n x_{f(n)} = x$? We propose two results where, for one class of ideals, the answer is yes, and for all ideals being outside a larger class, the answer is no.

Proposition 22. *Let $\mathcal{I}$ be a P-ideal on ω which is not isomorphic to* $\mathrm{Fin} \oplus \mathcal{P}(\omega)$. *Then for any sequence $(x_n)_{n\in\omega}$ of real numbers which is $\mathcal{I}$-convergent to some x, there exists a bi-$\mathcal{I}$-invariant injection f such that $(x_{f(n)})_{n\in\omega}$ is convergent to x.*

Proof. If $\mathcal{I} = \mathrm{Fin}$, the assertion is trivial. So assume that $\mathcal{I} \neq \mathrm{Fin}$ is not isomorphic to $\mathrm{Fin} \oplus \mathcal{P}(\omega)$. Since $\mathcal{I}\text{-}\lim_{n\in\omega} x_n = x$ and $\mathcal{I}$ is a P-ideal, by [16, Theorem 3.2] there exists a set $A \in \mathcal{I}^\star$ such that the sequence $(x_n)_{n\in A}$ converges to x in the usual sense. (Note that the being of a P-ideal is equivalent to condition (AP) used in [16]; cf. [3].) Pick an infinite $C \subseteq A$ such that $C \in \mathcal{I}$ (such a set C exists since if $\mathcal{P}(A) \cap \mathcal{I} \subseteq \mathrm{Fin}$ then $\mathcal{I}$ would be isomorphic to $\mathrm{Fin} \oplus \mathcal{P}(\omega)$). Now take any $f \in \mathbf{Inj}$ such that $f|_{A\setminus C} = \mathrm{id}$ and $f[(\omega \setminus A) \cup C] \subseteq C$. Then $\mathrm{Fix}(f) \in \mathcal{I}^\star$, so (by Fact 3) f is bi-$\mathcal{I}$-invariant. Since $(x_n)_{n\in A}$ tends to x and $f[\omega] \subseteq A$, we obtain $\lim_{n\in\omega} x_{f(n)} = x$ as desired. □

We say that an ideal $\mathcal{I}$ on ω is a *weak P-ideal* if for any sequence (A_n) of sets in $\mathcal{I}$ there exists a set $A \notin \mathcal{I}^\star$ such that for any $n \in \omega$ we have $A_n \subseteq^\star A$ (cf. [17] where weak P-filters were considered). Clearly, every P-ideal is a weak P-ideal. The ideal $\mathrm{Fin} \times \emptyset$ shows that the converse is false. Note that $\mathrm{Fin} \times \mathrm{Fin}$ is not a weak P-ideal, cf. [17, Example 1.2].

Proposition 23. *Assume that $\mathcal{I}$ is not a weak P-ideal. Then there exists an $\mathcal{I}$-convergent sequence (x_n) such that, for any bi-$\mathcal{I}$-invariant injection f, the sequence $(x_{f(n)})_{n\in\omega}$ is not convergent.*

Proof. Consider a sequence $(A_n)_{n\in\omega}$ of sets in $\mathcal{I}$ which witnesses that $\mathcal{I}$ is not a weak P-ideal. We may assume that $\bigcup_{n\in\omega} A_n = \omega$ and A_n's are pairwise disjoint. Define $x(n) := 1/(m+1)$ for $n \in A_m$ and $m \in \omega$. Then $\mathcal{I}\text{-}\lim_n x_n = 0$. Indeed, for any $\varepsilon > 0$ fix $m_0 \in \omega$ such that $\varepsilon \geq 1/(m_0 + 1)$. Then

$$\{n \in \omega \colon |x_n| \geq \varepsilon\} \subseteq \{n \in \omega \colon |x_n| \geq 1/(1 + m_0)\} = \bigcup_{m\leq m_0} A_m \in \mathcal{I}.$$

Now we will prove that, for any bi-$\mathcal{I}$-invariant injection f, the sequence $(x_{f(n)})_{n\in\omega}$ is not convergent to 0 (it is easy to see that it cannot converge to another x). Take any bi-$\mathcal{I}$-invariant injection f and assume that $\lim_n x_{f(n)} = 0$. This implies that $f[\omega] \cap A_n \in \mathrm{Fin}$ for any $n \in \omega$. But then for any $n \in \omega$ we have $A_n \subseteq^\star \omega \setminus f[\omega]$. Since f is bi-$\mathcal{I}$-invariant, $f[\omega] \notin \mathcal{I}$ and so, $\omega \setminus f[\omega] \notin \mathcal{I}^\star$. This contradicts our assumption that $(A_n)_{n\in\omega}$ witnesses that $\mathcal{I}$ is not a weak P-ideal. □

Question 3. What is an exact characterization of ideals $\mathcal{I}$ such that, for any sequence (x_n) of reals, the convergence $\mathcal{I}\text{-}\lim_n x_n = x$ implies $\lim_n x_{f(n)} = x$, for some bi-$\mathcal{I}$-invariant injection f?

Now, we turn to the problem how to characterize the $\mathcal{I}$-convergence of a sequence $(x_n)_{n\in\omega}$ in terms of the $\mathcal{I}$-convergence of $(x_{f(n)})_{n\in\omega}$ for the respectively chosen injections f. Our motivation comes from [1, Theorem 2.3] dealing with a special case. If $(x_n)_{n\in\omega}$ is given, every subsequence is of the form $(x_{f(n)})_{n\in\omega}$ for some increasing $f \in \mathbf{Inj}$. In fact, if we consider the usual limit of $(x_{f(n)})_{n\in\omega}$, only the set $f[\omega]$ is important and its ordering can be ignored. For ideal limits, the situation is different: the $\mathcal{I}$-limit of a sequence can depend on the order of terms.

A family $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$ $(m \in \omega)$ will be called $\mathcal{I}$-*good* if

(i) every f_i is bi-$\mathcal{I}$-invariant;
(ii) $\bigcup_{i\in K} f_i[\omega] \in \mathcal{I}^\star$.

Clearly, $\{\mathrm{id}\}$ is an $\mathcal{I}$-good family for any ideal $\mathcal{I}$ on ω.

Proposition 24. *Let $\mathcal{I}$ be an ideal on ω and consider real numbers x and x_n for $n \in \omega$. The following conditions are equivalent:*

(1) $\mathcal{I}\text{-}\lim_n x_n = x$;
(2) $\mathcal{I}\text{-}\lim_n x_{f(n)} = x$ *for every* $f \in \mathbf{Inj}$ *with* $\mathcal{I}$-*invariant* f^{-1};
(3) $\mathcal{I}\text{-}\lim_n x_{f(n)} = x$ *for every bi-$\mathcal{I}$-invariant* $f \in \mathbf{Inj}$;
(4) $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ *for every finite $\mathcal{I}$-good family* $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$;
(5) $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ *for some finite $\mathcal{I}$-good family* $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$.

Proof. (1)⇒(2) Let $\mathcal{I}\text{-}\lim_n x_n = x$ and for $\varepsilon > 0$ define $A(\varepsilon) := \{n \in \omega \colon |x_n - x| \geq \varepsilon\}$. Then $A(\varepsilon) \in \mathcal{I}$. Fix $f \in \mathbf{Inj}$ with $\mathcal{I}$-invariant f^{-1}. Hence $B(\varepsilon) := f[\omega] \cap A(\varepsilon) \in \mathcal{I}$, that is $\{n \in f[\omega] \colon |x_n - x| \geq \varepsilon\} \in \mathcal{I}$. Then $f^{-1}[B(\varepsilon)] \in \mathcal{I}$, that is $\{n \in \omega \colon |x_{f(n)} - x| \geq \varepsilon\} \in \mathcal{I}$. So $\mathcal{I}\text{-}\lim_n x_{f(n)} = x$.

Implications (2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(1) Fix an $\mathcal{I}$-good family $\{f_i \colon i \in K\} \subseteq \mathbf{Inj}$ such that $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ for each $i \in K$. Let $\varepsilon > 0$. Hence putting $B_i^\star(\varepsilon) := \{n \in \omega \colon |x_{f_i(n)} - x| \geq \varepsilon\}$, we have $B_i^\star(\varepsilon) \in \mathcal{I}$ for all $i \in K$. Then $B_i(\varepsilon) := f_\alpha[B_i^\star(\varepsilon)] \in \mathcal{I}$, that is $\{n \in f_i[\omega] \colon |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for all $i \in K$. Now, by (5) it follows that

$$A(\varepsilon) \subseteq \bigcup_{i\in K} (f_i[\omega] \cap A(\varepsilon)) \cup C = \bigcup_{i\in K} B_i(\varepsilon) \cup C$$

for some $C \in \mathcal{I}$ (where $A(\varepsilon)$ is defined as before). Hence $A(\varepsilon) \in \mathcal{I}$ which gives $\mathcal{I}\text{-}\lim_n x_n = x$. □

Remark 25. Note that $\mathcal{I}$-good families can be chosen so that $\{f_i[\omega] \colon i \in K\}$ forms a partition of ω. For instance, fix an integer $p \geq 2$ and consider $\{f_i \colon 0 \leq i \leq p-1\}$ where $f_i(n) := np + i$ for $n \in \omega$. For $p = 2$, this family was used in [1] to show the equivalence (1) ⇔ (5) in this particular case. (In fact, the result od [1] was an inspiration for Proposition 24.) All the injections f_i are increasing, so (by our remarks in Section 4) this example works also for ideals of the form $\mathcal{Z}_g$ with $g \in G$, and for the summable ideals $\mathcal{I}_{(g(n))}$ whenever $g \colon \omega \to [0,\infty)$ is decreasing.

It is natural to ask whether an infinite countable $\mathcal{I}$-good family can be used in statement (5) of Proposition 24. A partial answer is the following.

Proposition 26. *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Let φ be an lsc submeasure on ω and $\mathcal{I} := \mathrm{Exh}(\varphi)$. Assume that $\{f_i \colon i \in \omega\}$ is an $\mathcal{I}$-good family such that $f_0[\omega], f_1[\omega], \ldots$ are pairwise disjoint and $\sum_{i\in\omega} \varphi(f_i[\omega]) < \infty$. If $\mathcal{I}\text{-}\lim_n x_{f_i(n)} = x$ for every $i \in \omega$ then $\mathcal{I}\text{-}\lim_n x_n = x$.*

Proof. Let $\varepsilon > 0$, $A := \{n \in \omega \colon |x_n - x| \geq \varepsilon\}$ and $\delta > 0$. Pick $i_0 \in \omega$ such that $\sum_{i>i_0} \varphi(f_i[\omega]) < \delta/2$. As in the proof of (5)⇒(1) of Proposition 24, we infer that $A \subseteq \bigcup_{i\in\omega} B_i \cup C$ where

$$B_i := \{n \in f_i[\omega] \colon |x_n - x| \geq \varepsilon\} \in \mathcal{I} \text{ and } C \in \mathcal{I}.$$

Hence $A \setminus C \subseteq \bigcup_{i\leq i_0} B_i \cup \bigcup_{i>i_0} f_i[\omega]$. Since $B_i \in \mathcal{I} = \mathrm{Exh}(\varphi)$, pick $k_i \in \omega$ such that

$$\varphi(B_i \cap \{k_i, k_i+1, \ldots\}) \leq \frac{\delta}{2(i_0+1)}.$$

Let $k := \max_{i\le i_0} k_i$. Then

$$\varphi((A\setminus C)\cap\{k,k+1,\dots\}) \le \sum_{i\le i_0}\varphi(B_i\cap\{k_i,k_i+1,\dots\}) + \sum_{i>i_0}\varphi(f_i[\omega]) < \delta.$$

Therefore $A\setminus C \in \mathrm{Exh}(\varphi)$ and consequently, $A\in\mathcal{I}$. □

We propose an application of Proposition 26 dealing with the classical density. Note that $\{f_i\colon i\in\omega\}$, with $f_i(n) := 2^i(2n+1)$ for $n\in\omega$, forms an $\mathcal{I}_d$-good family and $f_0[\omega], f_1[\omega],\dots$ are pairwise disjoint. Also $\sum_{i\in\omega} d(f_i[\omega]) = 1$. Hence, by Proposition 26, if $\mathcal{I}_d$-$\lim_n x_{f_i(n)} = x$ for every $i\in\omega$ then $\mathcal{I}_d$-$\lim_n x_n = x$.

One cannot simply omit the assumption $\sum_{i\in\omega}\varphi(f_i[\omega]) < \infty$ in Proposition 26 which is shown in the following example.

Example 27. Set $x_n := 1$ if $n\in\omega$ is even, and $x_n := 0$ if $n\in\omega$ is odd. Define $f_k(0) := 2k+1$ and $f_k(n) := 2^{k+1}-2+2^{k+2}(n-1)$ for $n,k\in\omega$, $n>0$. Note that f_0 and f_1 are increasing except for the first two terms, and the remaining f_k's are increasing. It is easy to see that $d(f_k[\omega]) = 1/2^{k+2}$, hence, by Theorem 16, all f_k's are bi-$\mathcal{I}_d$-invariant. Moreover, $\bigcup_{k\in\omega} f_k[\omega] = \omega$, so $\{f_k\colon k\in\omega\}$ is an $\mathcal{I}_d$-good family. Also $\mathcal{I}$-$\lim_n x_{f_k(n)} = 1$ for every $k\in\omega$ but obviously (x_n) is not $\mathcal{I}_d$-convergent.

The next proposition implies that we can reduce considerations to countable $\mathcal{I}$-good families, and condition (ii) in the definition of an $\mathcal{I}$-good family may be replaced by $\bigcup_{i\in K} f_i[\omega] = \omega$.

Proposition 28. *Let $\mathcal{I}$ be an ideal on ω and let (x_i) be a sequence of reals. Let f be a bi-$\mathcal{I}$-invariant injection such that $(x_{f(i)})$ is $\mathcal{I}$-convergent to some x. Assume that card$(\omega\setminus f[\omega])\ge 2$. Then for any distinct $n,k\in\omega\setminus f[\omega]$ there exists a bi-$\mathcal{I}$-invariant injection f' such that $(x_{f'(i)})$ is $\mathcal{I}$-convergent to x and $f'[\omega] = f[\omega]\cup\{n,k\}$.*

Proof. Since $n,k\notin f[\omega]$, they must be initial points of some orbits. Define $f'\colon\omega\to\omega$ as follows:

- if $m\notin O_f(n)$ then $f'(m) := f(m)$,
- if $m\in O_f(n)\setminus\{n\}$ then $f'(m) := f^{-1}(m)$,
- $f'(n) := k$.

One can easily check that f' satisfies the requested conditions. □

To show the promised application, assume that $\{f_\alpha\colon \alpha<\kappa\}$ with $\kappa\ge\omega$, is a family of bi-$\mathcal{I}$-invariant injections such that $\bigcup_{\alpha<\kappa} f_\alpha[\omega]\notin\mathcal{I}^\star$. Arrange the infinite set $\omega\setminus\bigcup_{\alpha<\omega} f_\alpha[\omega]$ into a sequence $m_0,k_0,m_1,k_1,\dots$ of distinct numbers. Then applying Proposition 28 to f_α and m_α,k_α for $\alpha<\omega$, we obtain an $\mathcal{I}$-good family $\{f'_\alpha\colon\alpha<\omega\}$ with $\bigcup_{\alpha<\omega} f'_\alpha[\omega] = \omega$.

In the case of usual convergence, it happens that, for several subsequences of the sequence $0,1,2,\dots$, one chooses a common subsequence which leads to some desired effect. For ideal convergence, a similar role is played by bi-$\mathcal{I}$-invariant injections, as the following proposition shows.

Proposition 29. *Assume that $\mathcal{I}$ is a P-ideal on ω which is not isomorphic to* $\mathrm{Fin}\oplus\mathcal{P}(\omega)$. *Let $\{f_k\colon k\in\omega\}$ be a family of bi-$\mathcal{I}$-invariant injections such that $\mathcal{I}$-$\lim_n x_{f_k(n)} = y_k$ for each $k\in\omega$. Then there is a bi-$\mathcal{I}$-invariant injection h such that $\lim_n x_{h(f_k(n))} = y_k$ for each $k\in\omega$.*

Proof. Since $\mathcal{I}$ is a P-ideal and $\mathcal{I}$-$\lim_n x_{f_k(n)} = y_k$ for each $k\in\omega$, there is $E_k\in\mathcal{I}^\star$ such that $(x_n)_{n\in f_k[E_k]}$ tends to y_k in the usual sense, for each $k\in\omega$. Using again the fact that $\mathcal{I}$ is a P-ideal, we find $E\in\mathcal{I}^\star$ such that $E\subseteq^\star E_k$ for every $k\in\omega$. Then $(x_n)_{n\in f_k[E]}$ tends to y_k in the usual sense, as well. Using the same technique as in the proof of Proposition 22, we find an injection $h\colon\omega\to\omega$ such that $\mathrm{Fix}(h)\in\mathcal{I}^\star$ and $h[\omega]\subseteq E$. This yields the assertion. □

Question 4. Let (x_n) be a sequence of reals. Consider the following simple fact. If any sequence (n_k) of indices contains a subsequence (n_{k_l}) such that $(x_{n_{k_l}})$ is convergent to x, then (x_n) is convergent to x. Is the ideal version of this fact true? Namely, assume that for any bi-$\mathcal{I}$-invariant $f\colon\omega\to\omega$ there is a bi-$\mathcal{I}$-invariant $g\colon\omega\to\omega$ such that $(x_{g(f(n))})$ is $\mathcal{I}$-convergent to x. Does it imply that (x_n) is $\mathcal{I}$-convergent to x? The answer is positive for every ideal such that, for each $f\in\mathbf{Inj}$, f bi-$\mathcal{I}$-invariant iff $\mathrm{Fix}(f)\in\mathcal{I}^*$; in particular it is true for maximal ideals. We do not know the answer even in a special case of the classical density ideal $\mathcal{I}_d$.

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