INDEPENDENT BERNSTEIN SETS AND ALGEBRAIC CONSTRUCTIONS

ARTUR BARTOSZEWICZ, MAREK BIENIAS, AND SZYMON GŁĄB

Abstract. We present the method of constructing algebras and linear spaces of $2^{\mathfrak{c}}$ generators using independent Bernstein sets. As an application we obtain large algebras of special functions: (i) Strongly everywhere surjective functions which are not perfectly everywhere surjective. (ii) Nowhere continuous Darboux functions. (iii) Nowhere continuous compact to compact functions. (iv) Functions which are continuous precisely on a fixed closed proper subset of $\mathbb{R}$. Most conclusions obtained in this paper are improvements of some already known results.

1. Introduction and notation

Recently, looking for large algebraic structures (infinite dimensional vector spaces, closed infinite dimensional vector spaces, algebras) of functions on $\mathbb{R}$ or $\mathbb{C}$ that have certain properties has become a trend in Mathematical Analysis. A nice result of this sort is that of Rodriguez-Piazza (see [16]) which says that every separable Banach space can be isometrically embedded into a space $X \subseteq C[0,1]$ consisting of nowhere differentiable functions and zero.

Let κ be a cardinal number. Let us recall that the set M of functions satisfying some special property is called κ-lineable if $M \cup \{0\}$ contains a vector space of dimension κ, and is κ-spaceable if $M \cup \{0\}$ contains a closed

2010 *Mathematics Subject Classification.* Primary: 15A03; Secondary: 28A20, 26A15.
Key words and phrases. algebrability; perfectly everywhere surjective functions; strongly everywhere surjective functions; nowhere continuous Darboux functions; nowhere continuous compact to compact functions.

(in some given space $X \supseteq M$) vector space of dimension κ. This notion of lineability was coined by Gurariy and first introduced in [3].

One can go further and not just consider linear spaces but, instead, larger or more complex structures. For instance, in [2] the authors showed that there exists an uncountably generated algebra every non-zero element of which is an everywhere surjective function on $\mathbb{C}$. More in [10] it was shown that there exists an $\mathfrak{c}$-generated algebra every non-zero element of which is a perfectly everywhere surjective function on $\mathbb{C}$ ($\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$).

The notion of algebrability has its origin in works by Aron, Pérez-García and Seoane-Sepúlveda [4, 5] and the following is a slightly simplified version of their definition.

Definition 1. [4, 5] *Let $\mathcal{L}$ be an algebra. A set $A \subseteq \mathcal{L}$ is said to be β-algebrable if there exists an algebra $\mathcal{B}$ so that $\mathcal{B} \subseteq A \cup \{0\}$ and $card(Z) = \beta$, where β is cardinal number and Z is a minimal system of generators of $\mathcal{B}$. Here, by $Z = \{z_\alpha : \alpha \in \Lambda\}$ is a minimal system of generators of $\mathcal{B}$, we mean that $\mathcal{B} = \mathcal{A}(Z)$ is the algebra generated by Z, and for every $\alpha_0 \in \Lambda, z_{\alpha_0} \notin \mathcal{A}(Z \backslash \{z_{\alpha_0}\})$. We also say that A is algebrable if A is β-algebrable for some infinite β.*

Remark 2.
- Notice that if Z is a minimal infinite system of generators of $\mathcal{B}$, then $\mathcal{A}(Z') \neq \mathcal{B}$ for any $Z' \subseteq \mathcal{B}$ such that $card(Z') < card(Z)$.
- Clearly algebrability implies lineability, but the converse (in general) does not hold. For instance García-Pacheco, Palmberg and Seoane-Sepulv́eda [12] proved that, given any unbounded interval I, the set of Riemann-integrable functions on I that are not Lebesgue-integrable is lineable and not algebrable.

Given a cardinality κ, we say that A is a κ-generated free algebra, if there exists a subset $X = \{x_\alpha : \alpha < \kappa\}$ of A such that any function f from X

to some algebra A', can be uniquely extended to a homomorphism from A into A'. Then X is called a set of free generators of the algebra A. A subset $X = \{x_\alpha : \alpha < \kappa\}$ of a commutative algebra B generates a free sub-algebra A if and only if for each polynomial P and any $x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}$ we have $P(x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}) = 0$ if and only if $P = 0$. Also, let us recall that $X = \{x_\alpha : \alpha < \kappa\} \subseteq B$ is a set of free generators of a free algebra $A \subseteq B$ if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $\tilde{X}$ are in $B \cup \{0\}$. With this at hand, let us recall the following definition (see [8]).

Definition 3. *We say that a subset E of a commutative linear algebra B is strongly κ-algebrable if there exists a κ-generated free algebra A contained in $E \cup \{0\}$.*

The notion of strong algebrability is essentially stronger than the notion of algebrability, for instance, c_{00} is algebrable in c_0 but it is not strongly 1-algebrable, [8].

This paper is mainly devoted to the thorough study of the following recently considered classes of functions:

- Perfectly everywhere surjective ($\mathcal{PES}$), strongly everywhere surjective ($\mathcal{SES}$) and everywhere discontinuous Darboux ($\mathcal{EDD}$) functions;
- Everywhere discontinuous functions that have finitely many values ($\mathcal{EDF}$) and everywhere discontinuous compact to compact functions ($\mathcal{EDC}$);
- Functions that are continuous in a fixed closed set C.

These latter classes have been considered in the context of lineability and algebrability by many authors (see [2, 3, 9, 10, 11, 12]). In particular, in [10, Theorem 2.6, 2.7, 2.8] it was proved that the set of perfectly everywhere surjective functions on $\mathbb{R}$ is $2^{\mathfrak{c}}$-lineable, the set of strongly everywhere surjective functions on $\mathbb{R}$ that are not perfectly everywhere surjective is also $2^{\mathfrak{c}}$-lineable and that the set of perfectly everywhere surjective functions on

$\mathbb{C}$ is $\mathfrak{c}$-algebrable. Moreover, in [9] the authors showed that the last set is $2^{\mathfrak{c}}$-algebrable. In [11] the authors showed that the sets of everywhere discontinuous Darboux functions and everywhere discontinuous compact to compact functions are $2^{\mathfrak{c}}$-lineable. Similar results were obtained in [9] where authors showed that the first set is $\mathfrak{c}$-algebrable.

The aim of this paper is to improve all the mentioned results to the higher (most often the highest possible) level of algebrability. In particular, and among other results, we shall prove the $2^{\mathfrak{c}}$-algebrability of $\mathcal{SES}(\mathbb{C})\setminus\mathcal{PES}(\mathbb{C})$ (Theorem 7), $\mathcal{EDD}(\mathbb{R})$ (Theorem 8), $\mathcal{EDF}(\mathbb{R})$ and $\mathcal{EDC}(\mathbb{R})$ (Theorem 10 and Corollary 11).

The method we use here is based on independent families of Bernstein sets. This idea was introduced in [8] and [9]. This is a powerful method, since it allows to prove $2^{\mathfrak{c}}$-algebrability of certain sets of functions from $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$. Here we present a general approach of constructing large linear algebras using independent Bernstein sets. Then we present several applications of this method.

We shall use standard set theoretical notions. As usual, ω denotes the cardinality of $\mathbb{N}$ and $\mathfrak{c}$ the cardinality of $\mathbb{R}$. We will identify each cardinal number κ with the set of ordinals less than κ, i.e. $\kappa = \{\xi : \xi < \kappa\}$.

2. Independent families of Bernstein sets

Let X be a nonempty set. For a set $A \subseteq X$ let us denote $A^0 = X\setminus A$ and $A^1 = A$. Let $\mathcal{B}$ be a family of subsets of a set X. We say that the family $\mathcal{A}$ is *$\mathcal{B}$-independent* if and only if $A_1^{\varepsilon_1} \cap ... \cap A_n^{\varepsilon_n} \in \mathcal{B}$ for any distinct $A_i \in \mathcal{A}$, any $\varepsilon_i \in \{0,1\}$ for $i \in \{1,...,n\}$ and $n \in \mathbb{N}$. We call a family $\mathcal{A}$ independent if and only if it is $\mathcal{B}$-independent for $\mathcal{B} = \mathcal{P}(X)\setminus\{\emptyset\}$.

Note ([7]) that for any set of cardinality κ there is an independent family of 2^κ many subsets of this set. Recall that a subset B of $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is called a *Bernstein set* if $B \cap P \neq \emptyset \neq B^0 \cap P$ for every perfect subset P of $\mathbb{K}$. Assume that $\mathcal{B}$ is a family of all Bernstein sets in $\mathbb{R}$ (or $\mathbb{C}$). We say that a family $\mathcal{A}$ is an independent family of Bernstein sets provided that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A}$ is $\mathcal{B}$-independent. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}$ into $\mathfrak{c}$ many disjoint Bernstein sets. Observe that if $s \subseteq \mathfrak{c}$ with $s \neq \emptyset$ and $\mathfrak{c} \backslash s \neq \emptyset$ then $\bigcup_{\alpha \in s} B_\alpha$ is a Bernstein set. Let $\{N_\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family in $\mathfrak{c}$ such that for every $\xi_1 < ... < \xi_n < 2^{\mathfrak{c}}$ and for any $\varepsilon_i \in \{0, 1\}$ ($i \in \{1, ..., n\}$), the set $N_{\xi_1}^{\varepsilon_1} \cap ... \cap N_{\xi_n}^{\varepsilon_n}$ is nonempty and has cardinality $\mathfrak{c}$. That means every N_ξ has also cardinality $\mathfrak{c}$ (to see that such a family exists we should take an independent family of $2^{\mathfrak{c}}$ many sets in $\mathfrak{c}$ and use the fact that $\mathfrak{c} = \mathfrak{c} \times \mathfrak{c}$).

We will define an independent family of Bernstein sets of cardinality $2^{\mathfrak{c}}$. For $\xi < 2^{\mathfrak{c}}$ put $B^\xi = \bigcup_{\alpha \in N_\xi} B_\alpha$. Then every set B^ξ is Bernstein. Note that for every $\xi_1 < ... < \xi_n < 2^{\mathfrak{c}}$ and any $\varepsilon_i \in \{0, 1\}$ for $i \in \{1, ..., n\}$ the set

$$(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n} = \bigcup_{\alpha \in N_{\xi_1}^{\varepsilon_1} \cap ... \cap N_{\xi_n}^{\varepsilon_n}} B_\alpha$$

is Bernstein. That means $\{B^\xi : \xi < 2^{\mathfrak{c}}\}$ is an independent family of Bernstein sets.

We will be using the following construction of $2^{\mathfrak{c}}$ many linearly independent functions in almost all theorems in this paper. For $\alpha < \mathfrak{c}$, let $g_\alpha : B_\alpha \to \mathbb{C}$ (or $\mathbb{R}$) be a non-zero function defined on a Bernstein set B_α. Then for every $\xi < 2^{\mathfrak{c}}$ let us put

$$f_\xi(x) = \begin{cases} g_\alpha(x) \text{ when } x \in B_\alpha \text{ and } \alpha \in N_\xi \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

Then the family $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ is linearly independent.

Let P be any non-zero polynomial in n variables without constant term and let $f_{\xi_1}, ..., f_{\xi_n}$ be any functions of the above type. Let us consider the function $P(f_{\xi_1}, ..., f_{\xi_n})$. Also, let $s = (\varepsilon_1, ..., \varepsilon_n)$ where $\varepsilon_i \in \{0, 1\}$, and let P_s denotes the polynomial in one variable defined by

$$P_s(x) = P(\varepsilon_1 \cdot x, ..., \varepsilon_n \cdot x). \tag{2}$$

Let us observe here that the function $P(f_{\xi_1},...,f_{\xi_n})|_{B_\alpha}$ for any $\alpha \in N^{\varepsilon_1}_{\xi_1} \cap ... \cap N^{\varepsilon_n}_{\xi_n}$ is of the form $P(\varepsilon_1 \cdot g_\alpha, ..., \varepsilon_n \cdot g_\alpha) = P_s(g_\alpha)$ for $s = (\varepsilon_1, ..., \varepsilon_n)$. Then we have two possibilities:

(i) At least one of the functions $P_s(x)$ for $s \in \{0,1\}^n$ is a non-zero polynomial in one variable. If P_s is non-zero, where $s = (\varepsilon_1, ..., \varepsilon_n)$, then the function $P(f_{\xi_1},...,f_{\xi_n})$ is non-zero on the Bernstein set of the form $(B^{\xi_1})^{\varepsilon_1} \cap (B^{\xi_2})^{\varepsilon_2} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$.

(ii) Every function $P_s(x)$ is zero, in which case $P(f_{\xi_1},...,f_{\xi_n})$ is the zero function. Note that this case is possible: simply take $P(x_1,...,x_n) := \prod_{i\neq j}(x_i - x_j)\prod_k x_k$.

Finally spanning the algebra by the functions $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ and using the fact that this set is linearly independent we obtain an algebra of $2^{\mathfrak{c}}$ many generators.

3. $\mathcal{PES}$, $\mathcal{SES}$ and $\mathcal{EDD}$

Lebesgue was probably the first who exhibited an example of a function $f : \mathbb{R} \to \mathbb{R}$ with $f(I) = \mathbb{R}$ for every non-trivial interval I. This kind of functions are called everywhere surjective. In this section we will consider also classes of functions which fulfill more stringent conditions. Let $\mathbb{K}$ stand for the field $\mathbb{R}$ or $\mathbb{C}$. A function $f : \mathbb{K} \to \mathbb{K}$ is called:

- perfectly everywhere surjective if and only if for every perfect set $P \subseteq \mathbb{K}, f(P) = \mathbb{K}$ and write $f \in \mathcal{PES}(\mathbb{K})$;
- strongly everywhere surjective if and only if it takes every real or complex value $\mathfrak{c}$ times on any interval and write $f \in \mathcal{SES}(\mathbb{K})$.

More we say that $f : \mathbb{R} \to \mathbb{R}$ is an everywhere discontinuous Darboux function (write $f \in \mathcal{EDD}(\mathbb{R})$) if and only if it is nowhere continuous and maps connected sets to connected sets. In ([9]) Bartoszewicz, Głąb, Pellegrino and Seoane-Sepúlveda showed that set $\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable. The following proposition was proved in [9] for $\mathbb{K} = \mathbb{C}$, but in the case $\mathbb{K} = \mathbb{R}$ the proof also works.

Proposition 4. [9] *Let $B \subseteq \mathbb{K}$ be a Bernstein set. There exists a function $f \in \mathcal{PES}(\mathbb{K})$ that is* 0 *on the set B^0.*

The following theorem comes from [9] and we recall its proof for the reader's convenience. It is a good illustration of method of construction of large algebra using a family of independent Bernstein sets.

Theorem 5. [9] *The set $\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable.*

Proof. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a decomposition of $\mathbb{C}$ into $\mathfrak{c}$ many disjoint Bernstein sets. Let $\{N_\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family in $\mathfrak{c}$. Let us define for $\alpha < \mathfrak{c}$ the $\mathcal{PES}$ function $g_\alpha : B_\alpha \to \mathbb{C}$ on a Bernstein set B_α as in Proposition 4. Then for every $\xi < 2^{\mathfrak{c}}$ let us define f_ξ as in (1) of Section 2. Then the family $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ is linearly independent and generates an algebra of $2^{\mathfrak{c}}$ many generators. Since every non-zero polynomial in n complex variables is onto $\mathbb{C}$, by the results of Section 2 the function $P(f_{\xi_1}, ..., f_{\xi_n})$ is onto too (or it is equal to 0). That means $P(f_{\xi_1}, ..., f_{\xi_n}) \in \mathcal{PES}(\mathbb{C})$ or it is zero function and we are done. $\square$

Remark 6. Let X, Y be complete, separable metric spaces without isolated points. A function $\phi : X \to Y$ is called Borel isomorphism if ϕ is bijection, ϕ and ϕ^{-1} are Borel mappings. By [14, Theorem 15.6] for any such X and Y there exists Borel isomorphism $\phi : X \to Y$.

Let $\phi : X \to \mathbb{C}$ be a Borel isomorphism and let $f \in \mathcal{PES}(\mathbb{C})$. Let P be a perfect set in X and $y \in \mathbb{C}$. Then $\phi(P)$ is uncountable Borel set, so it contains a perfect set Q. Since $f \in \mathcal{PES}(\mathbb{C})$ we have $y \in f(Q) \subseteq f(\phi(P))$. Therefore $f \circ \phi : X \to \mathbb{C}$ is perfectly everywhere surjective. Moreover note that if $A \subseteq \mathcal{PES}(\mathbb{C}) \cup \{0\}$ is an algebra, then $\{f \circ \phi : f \in A\}$ is also an algebra. Finally we obtain that the set of perfectly everywhere surjective functions from X onto $\mathbb{C}$ is $2^{\mathfrak{c}}$-algebrable.

One can check that $\mathcal{PES}(\mathbb{C}) \subseteq \mathcal{SES}(\mathbb{C})$. Since there are functions that are strongly but not perfectly everywhere surjective [10, Example 2.3], it makes sense to consider the size of $\mathcal{SES}(\mathbb{C})\setminus\mathcal{PES}(\mathbb{C})$.

Theorem 7. *The set $\mathcal{SES}(\mathbb{C})\setminus\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable.*

Proof. We follow the idea given in [2, Theorem 1.3]. Let $(U_n)_{n<\omega}$ be a countable basis of open sets of $\mathbb{C}$. We can define by induction a sequence $(C_n)_{n<\omega}$ of Cantor-like sets such that for every $n < \omega$

(1) C_n is homeomorphic to the Cantor set;
(2) $C_n \subseteq U_n$;
(3) $C_n \cap \bigcup_{k<n} C_k = \emptyset$.

This can be done since C_n's are nowhere dense so for fixed $n < \omega$ there is an open ball in $U_n\setminus \bigcup_{k<n} C_k$ that contains $\mathfrak{c}$ many disjoint Cantor-like sets. Now for every $n < \omega$, by Remark 6, the set of perfectly everywhere surjective functions on C_n onto $\mathbb{C}$ is $2^{\mathfrak{c}}$-algebrable by Theorem 5. Let $\{f^n_\xi : \xi < 2^{\mathfrak{c}}\}$ be generators witnessing that. Let us define for every $\xi < 2^{\mathfrak{c}}$ a function $g_\xi : \mathbb{C} \to \mathbb{C}$ as follows:

$$g_\xi(x) = \begin{cases} f^n_\xi(x) \text{ if } x \in C_n \\ 0 \text{ otherwise.} \end{cases}$$

Fix a natural number n. Let P be any non-zero polynomial in k variables without constant term, and let $\xi_1 < \xi_2 < ... < \xi_k < 2^{\mathfrak{c}}$. Note that $P(g_{\xi_1}, ..., g_{\xi_k})|_{C_n} = P(f^n_{\xi_1}, ..., f^n_{\xi_k})|_{C_n}$. Therefore $P(g_{\xi_1}, ..., g_{\xi_k})|_{C_n}$ is perfectly everywhere surjective on C_n. Hence $h := P(g_{\xi_1}, ..., g_{\xi_k})$ is strongly everywhere surjective, but it does not belong to $\mathcal{PES}(\mathbb{C})$, for if we consider any perfect set $D \subseteq \mathbb{C}\setminus \bigcup_{n<\omega} C_n$ then it follows from our construction that h is constant and equal to 0 on D.

□

It easy to see that the set $\mathcal{PES}(\mathbb{R})$ is not algebrable because $f^2 \notin \mathcal{PES}(\mathbb{R})$ for any $f \in \mathcal{PES}(\mathbb{R})$. In the positive direction, the same argument as in Theorem 5 gives us an algebra of everywhere discontinuous Darboux functions. Indeed, for every non-zero polynomial P in n variables without constant term, any functions $f_{\xi_1}, ..., f_{\xi_n} \in \mathcal{PES}(\mathbb{R}) \subseteq \mathcal{EDD}(\mathbb{R})$ and any real numbers $a < b$, the image $P(f_{\xi_1}, ..., f_{\xi_n})((a,b))$ is an union of sets of the type $\mathbb{R}$ or $\mathbb{R}_+ \cup \{0\}$ or $\mathbb{R}_- \cup \{0\}$, so it is connected. Hence the following holds.

Theorem 8. *The set $\mathcal{EDD}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable.*

It is easy to verify that every function $f \in \mathcal{PES}(\mathbb{C})$ from the construction in Theorem 5 is a generator of free algebra, so the set $\mathcal{PES}(\mathbb{C})$ is strongly 1-algebrable. But the answer to the following question remains unknown.

Question 9. *Are the sets $\mathcal{PES}(\mathbb{C}), \mathcal{EDD}(\mathbb{R})$ strongly $2^{\mathfrak{c}}$-algebrable?*

4. $\mathcal{EDF}$ and $\mathcal{EDC}$

Velleman in [17] proved that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is Darboux (i.e. maps connected sets to connected sets) and compact to compact (i.e. maps compact sets to compact sets). Gámez-Merino, Muñoz-Fernández and Seoane-Sepúlveda [11] proved that the family of Darboux nowhere continuous functions and the family of compact to compact nowhere continuous functions are $2^{\mathfrak{c}}$-lineable. Theorem 8 extends the first of these results to $2^{\mathfrak{c}}$-algebrability. Now we will also extend the second of them in the same way.

Let us denote by the $\mathcal{EDF}(\mathbb{R})$ ($\mathcal{EDC}(\mathbb{R})$, resp.) the set of all nowhere continuous functions having finitely many values (mapping compact sets to compact sets, resp.).

Theorem 10. *The set $\mathcal{EDF}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable but not strongly 1-algebrable.*

Proof. Let $\{B^\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family of Bernstein sets. For $\xi < 2^{\mathfrak{c}}$ define the function f_ξ as the characteristic function of a set B^ξ.

Let P be any non-zero polynomial in n variables without constant term and let $\xi_1 < \xi_2 < ... < \xi_n < 2^{\mathfrak{c}}$. Assume that $P(f_{\xi_1}, ..., f_{\xi_n})$ is non-zero and note it is constant on every set of the form $(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$, where $\varepsilon_i \in \{0,1\}$ for $i \in \{1,...,n\}$. Since each f_{ξ_i} is constant on every set $(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$ so is $P(f_{\xi_1}, ..., f_{\xi_n})$ and it has at most 2^n many values. Since $P(f_{\xi_1}, ..., f_{\xi_n})$ is non-zero so there exist $\varepsilon_i \in \{0,1\}$ for $i \in \{1,...,n\}$ such that $P(f_{\xi_1}, ..., f_{\xi_n})|_{(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}} \neq 0$. But clearly $P(f_{\xi_1}, ..., f_{\xi_n})|_{(B^{\xi_1})^0 \cap ... \cap (B^{\xi_n})^0} = 0$. Since every set of the type $(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$ is Bernstein, it is dense, so $P(f_{\xi_1}, ..., f_{\xi_n})$ is everywhere discontinuous. Hence $\mathcal{EDF}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable.

We will show that $\mathcal{EDF}(\mathbb{R})$ is not strongly 1-algebrable. Let $f \in \mathcal{EDF}(\mathbb{R})$ and put $f(\mathbb{R}) = \{a_1, ..., a_n\}$. Clearly the set

$$\{(a_1, ..., a_n), (a_1^2, ..., a_n^2), ..., (a_1^{n+1}, ..., a_n^{n+1})\}$$

is not linearly independent in $\mathbb{R}^n$. Hence there are numbers (not every zero) $\alpha_1, ..., \alpha_{n+1} \in \mathbb{R}$ such that $\alpha_1 a_k + ... + \alpha_{n+1} a_k^{n+1} = 0$ for every $k \in \{1, ..., n\}$. Therefore $\alpha_1 f + ... + \alpha_{n+1} f^{n+1}$ is the zero function, so $\mathcal{EDF}(\mathbb{R})$ is not strongly 1-algebrable. □

Since finite sets are compact, we have that $\mathcal{EDF}(\mathbb{R}) \subseteq \mathcal{EDC}(\mathbb{R})$ and the following holds.

Corollary 11. *The set $\mathcal{EDC}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable.*

Note that if $f \in \mathcal{EDC}(\mathbb{R})$ and f have only finitely many values on some interval I, then using the same reasoning as in Theorem 10 one can find a non-zero polynomial P of one variable such that $P(f)$ is constant on I, and therefore the algebra generated by f is not contained in $\mathcal{EDC}(\mathbb{R}) \cup \{0\}$. Hence to get strong algebrability of $\mathcal{EDC}(\mathbb{R})$ one should first answer the following.

Question 12. *Is there a function $f \in \mathcal{EDC}(\mathbb{R})$ having infinitely many values on each interval?*

If the answer is positive, then one can ask further the following.

Question 13. *Is the set $\mathcal{EDC}(\mathbb{R})$ strongly 1-algebrable (strongly $\mathfrak{c}$-algebrable, strongly $2^{\mathfrak{c}}$-algebrable)?*

5. Functions that are continuous on a fixed closed set C

Let $C \subsetneq \mathbb{R}$ be a fixed closed subset of $\mathbb{R}$. In this section we are going to consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of C. García-Pacheco, Palmberg and Seoane-Sepúlveda in [13, Theorem 5.1] proved the ω-lineability of the set of functions of finitely many points of continuity. Then Aizpuru, Pérez-Eslava, García-Pacheco and Seoane-Sepúlveda established in [1] that the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of U (G) for a fixed open set U (fixed G_δ set G, resp.) is lineable (coneable). In the following we analyze algebrability of set of all functions which are continuous precisely on a fixed closed set.

Theorem 14. *The set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only at the points of C is $2^{\mathfrak{c}}$-algebrable.*

Proof. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}$ into $\mathfrak{c}$ many disjoint Bernstein sets and $\{N_\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family in $\mathfrak{c}$. Let $\{B^\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family of Bernstein sets and enumerate the set $[1,2] = \{r_\alpha : \alpha < \mathfrak{c}\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = d(x, C)$ for $x \in \mathbb{R}$ where d stands for the natural metric in $\mathbb{R}$. Clearly g is zero only on the set C. For every $\alpha < \mathfrak{c}$ define the function $g_\alpha(x) = r_\alpha \cdot g(x)$ and for every $\xi < 2^{\mathfrak{c}}$ define f_ξ as in (1) of Section 2. Let us consider an arbitrary non-zero polynomial P in n variables without constant term and $\xi_1 < ... < \xi_n < 2^{\mathfrak{c}}$. If each function $P_s(x)$ (defined by (2) of Section 2) is zero for every $s \in \{0,1\}^n$, then $P(f_{\xi_1}, ..., f_{\xi_n})$ is zero function by the argument contained in Section 2. Assume that there is $s_0 = (\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n$ such that $P_{s_0}(x)$ is non-zero. We shall show that $P(f_{\xi_1}, ..., f_{\xi_n})$ is continuous at any point of C. Let $x \in C$

and $x_k \to x$. Since $|r_\alpha| \leq 2$, then $0 \leq g_{\alpha_k}(x_k) \leq 2g(x_k) \to 2g(x) = 0$ for any $\alpha_k < \mathfrak{c}$. Hence $P(f_{\xi_1}, ..., f_{\xi_n})(x_k) \to 0 = P(f_{\xi_1}, ..., f_{\xi_n})(x)$.

Suppose now that the function $P(f_{\xi_1}, ..., f_{\xi_n})$ is continuous in a point $x_0 \notin C$. From the construction of the family $\{f_\xi : \xi < 2^\mathfrak{c}\}$ we get that $P(f_{\xi_1}, ..., f_{\xi_n})$ is zero on the Bernstein set

$$\bigcup_{\alpha \in N^0_{\xi_1} \cap N^0_{\xi_2} \cap ... \cap N^0_{\xi_n}} B_\alpha.$$

Since every Bernstein set is dense, we have $P(f_{\xi_1}, ..., f_{\xi_n})(x_0) = 0$ and for every $\beta \in N^{\varepsilon_1}_{\xi_1} \cap N^{\varepsilon_2}_{\xi_2} \cap ... \cap N^{\varepsilon_n}_{\xi_n}$ there exists a sequence $(y_k)_{k\in\mathbb{N}} \subseteq B_\beta$ such that $y_k \to x_0$. Hence by the continuity of $P_s \circ g_\beta$ we get that $P_{s_0}(g_\beta(y_k)) \to P_{s_0}(g_\beta(x_0)) = 0$ for any such β. Since for $\alpha, \beta \in N^{\varepsilon_1}_{\xi_1} \cap N^{\varepsilon_2}_{\xi_2} \cap ... \cap N^{\varepsilon_n}_{\xi_n}$ with $\alpha \neq \beta$ we have that $g_\alpha(x_0) = r_\alpha \cdot g(x_0) \neq r_\beta \cdot g(x_0) = g_\beta(x_0)$. Then $Q(\beta) := P_{s_0}(g_\beta(x_0))$ has infinitely many zeros as a polynomial in β. Hence it is the zero function, and we reach a contradiction. Hence $\{f_\xi : \xi < 2^\mathfrak{c}\}$ span an algebra of functions with the desired property. □

Recall (see [15]) that the set of continuity points of an arbitrary function $\mathbb{R} \to \mathbb{R}$ is of type G_δ. Conversely, for each G_δ set C there is a function f whose set of continuity points is exactly C. Therefore one can ask the following.

Question 15. *Fix a G_δ set $C \subseteq \mathbb{R}$. What can be said about the algebrability of set of all function whose set of continuity points is exactly C?*

6. Measurability of the composition

It is well known that the composition of any two continuous functions is also continuous and therefore measurable. Moreover if f is continuous on the interval $[a, b]$ and g is measurable on the interval $[\alpha, \beta]$ with $g([\alpha, \beta]) \subseteq [a, b]$, then the composition $f \circ g$ is measurable on $[\alpha, \beta]$. On the other hand it is not always true that $f \circ g$ is measurable when f is measurable and g is continuous. Azagra, Muñoz-Fernández, Sánchez and Seoane-Sepúlveda proved in [6] that there is a $\mathfrak{c}$-dimensional vector space $W \subseteq \mathbb{R}^{\mathbb{R}}$ of continuous functions such that for every $g \in W \setminus \{0\}$ there is a $\mathfrak{c}$-dimensional vector space $V \subseteq \mathbb{R}^{\mathbb{R}}$ of measurable functions such that $f \circ g$ is non-measurable for any $f \in V \setminus \{0\}$. In this section we will improve this result combining the method of independent Bernstein sets and ideas from [6]. We will need in the sequel the following.

Proposition 16 ([6])**.** *There is a $\mathfrak{c}-$dimensional linear space $W \subseteq \mathbb{R}^{\mathbb{R}}$ of continuous functions such that if $g \in W \setminus \{0\}$ then there is a closed set D of measure zero such that $g|_{g^{-1}(D)}$ is one-to-one and $g^{-1}(D)$ is of positive measure.*

Theorem 17. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that there is a perfect set $D \subseteq \mathbb{R}$ of measure zero such that $g|_{g^{-1}(D)}$ is one-to-one and $g^{-1}(D)$ is of positive measure. Then there exists a linear space V with $dim V = 2^{\mathfrak{c}}$ of measurable functions such that $f \circ g$ is non-measurable for any $f \in V \setminus \{0\}$.*

Proof. It is easy to see that if $\{B_0^{\xi} : \xi < 2^{\mathfrak{c}}\}$ is an independent family of Bernstein sets and D is perfect, the family $\{B_0^{\xi} \cap g^{-1}(D) : \xi < 2^{\mathfrak{c}}\}$ is also an independent family of Bernstein sets in $g^{-1}(D)$. Assume that $\{B^{\xi} : \xi < 2^{\mathfrak{c}}\}$ is an independent family of Bernstein sets in $g^{-1}(D)$. For $\xi < 2^{\mathfrak{c}}$ let f_{ξ} be a characteristic function of a set $g(B^{\xi})$. Note that for every $\xi < 2^{\mathfrak{c}}$ the set $g(B^{\xi}) \subseteq D$ is of measure zero, hence the function f_{ξ} is measurable. Let $V \subseteq \mathbb{R}^{\mathbb{R}}$ be the linear subspace spanned by the family $\{f_{\xi} : \xi < 2^{\mathfrak{c}}\}$.

Let $f \in V$. There are $\xi_1 < ... < \xi_n < 2^{\mathfrak{c}}$ and $\alpha_i \neq 0$ with $f = \alpha_1 f_{\xi_1} + ... + \alpha_n f_{\xi_n}$. Note that f maps $\mathbb{R}$ into $A = \left\{ \sum_{i \in I} \alpha_i : \emptyset \neq I \subseteq \{1, ..., n\} \right\}$. We will prove that $(f \circ g)^{-1}(\{a\})$ is non-measurable for any $a \in A \setminus \{0\}$. Note that the set $f^{-1}(\{a\})$ equals

$$\bigcup \left\{ \bigcap_{i \in I} g(B^{\xi_i}) \cap \bigcap_{i \in \{1,...,n\} \setminus I} g((B^{\xi_i})^{-1}) : I \subseteq \{1, ..., n\}, \sum_{i \in I} \alpha_i = a \right\} =$$

$$= g\left(\bigcup\left\{\bigcap_{i\in I} B^{\xi_i} \cap \bigcap_{i\in\{1,...,n\}\setminus I} (B^{\xi_i})^{-1} : I \subseteq \{1,...,n\}, \sum_{i\in I}\alpha_i = a\right\}\right).$$

The last equality follows from the fact that $g|_{g^{-1}(D)}$ is one-to-one. Clearly $(f\circ g)^{-1}(\{a\})$ is Bernstein set in $g^{-1}(D)$. □

Using Theorem 17 and Proposition 16 we obtain the following.

Corollary 18. *There is a $\mathfrak{c}$-dimensional vector space $W \subseteq \mathbb{R}^{\mathbb{R}}$ of continuous functions such that for every $g \in W\setminus\{0\}$ there is a $2^{\mathfrak{c}}$-dimensional vector space $V \subseteq \mathbb{R}^{\mathbb{R}}$ of measurable functions such that $f\circ g$ is non-measurable for any $f \in V\setminus\{0\}$.*

In [6] it is asked if there are two $\mathfrak{c}-$dimensional subspaces W and V of $\mathbb{R}^{\mathbb{R}}$ such that any $g \in W$ is continuous, any $f \in V$ is measurable and $f\circ g$ is non-measurable. It seems to be a hard problem. Even the problem of finding *one* measurable function f and W as before with $f\circ g$ non-measurable (for all $g \in W\setminus\{0\}$) is interesting.

Acknowledgement. We are thankful to the Referee for helpful remarks and suggestions. The first and the last authors have been supported by the Polish Ministry of Science and Higher Education Grant No. N N201 414939 (2010-2013).

References

[1] A. Aizpuru, C. Pérez-Eslava, F. J. García-Pacheco, J. B. Seoane-Sepúlveda, Lineability and coneability of discontinuous functions on $\mathbb{R}$. Publ. Math. Debrecen 72 (2008), no. 1-2, 129–139.

[2] R.M. Aron, J.A. Conejero, A. Peris, J.B. Seoane-Sepúlveda, Uncountably generated algebras of everywhere surjective functions, Bull. Belg. Math. Soc. Simon Stevin 17 (2010), no. 3, 571–575.

[3] R.M. Aron, V.I. Gurariy, J.B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on $\mathbb{R}$, Proc. Amer. Math. Soc. 133 (2005), no. 3, 795–803.

[4] R.M. Aron, D. Pérez-García, J.B. Seoane-Sepúlveda, Algebrability of the set of non-convergent Fourier series, Studia Math. 175 (2006), no. 1, 83–90.

[5] R.M. Aron, J.B. Seoane-Sepúlveda, Algebrability of the set of everywhere surjective functions on $\mathbb{C}$, Bull. Belg. Math. Soc. Simon Stevin 14 (2007), no. 1, 25–31.

[6] D. Azagra, G.A. Muñoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, Riemann integrability and Lebesgue measurability of the composite functions, J. Math. Anal. Appl. 354 (2009), no. 1, 229–233.

[7] B. Balcar, F. Franěk, Independent families in complete Boolean algebras, Trans. Amer. Math. Soc. 274 (1982), no. 2, 607–618.

[8] A. Bartoszewicz, S. Głąb, Strong algebrability of sets of sequences and functions, Proc. Amer. Math. Soc., Accepted.

[9] A. Bartoszewicz, S. Głąb, D. Pellegrino, J.B. Seoane-Sepúlveda, Algebrability, non-linear properties and special functions, Proc. Amer. Math. Soc., Accepted.

[10] J.L. Gámez-Merino, G.A. Munoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, Sierpiński-Zygmund functions and other problems on lineability, Proc. Amer. Math. Soc. 138 (11) (2010) 3863–3876.

[11] J.L. Gámez-Merino, G.A. Munoz-Fernández, J.B. Seoane-Sepúlveda, A characterization of continuity revisited, Amer. Math. Monthly 118 (2011), no. 2, 167–170.

[12] F.J. García-Pacheco, M. Martín, J.B Seoane-Sepúlveda, Lineability, spaceability and algebrability of certain subsets of function spaces, Taiwanese J. Math. 13 (2009), no. 4, 1257–1269.

[13] F.J. García-Pacheco, N. Palmberg, J.B Seoane-Sepúlveda, Lineability and algebrability of pathological phenomena in analysis, J. Math. Anal. Appl. 326 (2007) 929–939.

[14] A. Kechris, Classical descriptive set theory, Springer-Verlag, New York, 1994.

[15] J.C. Oxtoby, Measure and Category, Springer-Verlag, New York, 1980.

[16] L. Rodríguez-Piazza, Every separable Banach space is isometric to a space of continuous nowhere differentiable functions, Proc. Amer. Math. Soc. 123, 3649–3654 (1995).

[17] D. Velleman, Characterizing continuity, Amer. Math. Monthly 104 (1997), no. 4, 318–322.

Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: arturbar@p.lodz.pl

Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: marek.bienias88@gmail.com

Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: szymon.glab@p.lodz.pl