DESCRIPTIVE PROPERTIES OF FAMILIES OF AUTOHOMEOMORPHISMS OF THE UNIT INTERVAL

SZYMON GŁĄB

Abstract. Let $\mathbb{H} \subset C[0,1]$ stand for the Polish space of all increasing autohomeomorphisms of $[0,1]$. We show that the family of all strictly singular autohomeomorphisms is Π^1_1–complete. This confirms a suggestion of Graf, Mauldin and Williams. Some related results are also included.

1. introduction

The aim of this paper is to investigate the descriptive complexity of some special sets of autohomeomorphisms of the unit interval $[0,1]$. The motivation comes from the paper [4] by Graf, Mauldin and Williams where the authors showed that the set of all strictly singular autohomeomorphisms of the unit interval is coanalytic. They remarked [4, Remark 5.3, p. 302] that "very likely this set is not a Borel set in $\mathbb{H}$ but we have not demonstrated this". Theorem 1 in our paper states that this set is $\mathbf{\Pi^1_1}$–complete, hence it is not Borel (even not analytic). Theorem 5 is related statement in which we consider other sets of autohomeomorphisms with given conditions on derivatives. Theorems 1 and 5 resemble the pair of classical facts that the set $DIFF$ of all functions from $C[0,1]$ (the Banach space of real–valued continuous functions on $[0,1]$, with the supremum norm) which are differentiable at every point, and the set $NDIFF$ of all functions from $C[0,1]$ which are nowhere differentiable, are both Π^1_1–complete in $C[0,1]$ (see [5] for details).

We use standard set–theoretic notation. For the descriptive set–theoretical background we refer the reader to [5]. By $\mathbb{H} \subset C[0,1]$ we denote the set of all increasing

1991 *Mathematics Subject Classification.* Primary: 28A05, 03E15; Secondary: 26A30.

Key words and phrases. singular autohomeomorphism, coanalytic set, Π^1_1–complete set.

autohomeomorphisms of $[0,1]$. It is easy to see that $\mathbb{H}$ is a G_δ subset of $C[0,1]$ and hence it is a Polish space. By symbols $D^\pm f(x)$ and $D_\pm f(x)$ we denote the Dini derivatives of f at a point x, the first one is the upper Dini derivative and the second one is the lower Dini derivative, where $+$ and $-$ indicate the right-hand and the left-hand sides, respectively. By $f'_+(x)$ and $f'_-(x)$ we denote the right-hand and the left-hand sided derivatives of f at x, respectively. A monotone function on $[0,1]$ with the derivative vanishing almost everywhere is called *singular*. It is well-known that an increasing continuous function is singular if and only if there is a set of full measure whose image is null. This easily implies that the set S of strictly increasing continuous singular functions equals $\bigcap_{n=1}^\infty S_n$, where S_n denotes the set of those functions $f \in \mathbb{H}$ for which $f(F)$ is of measure less than $1/n$ for a suitable compact set F of measure greater than $1-(1/n)$. It is also easy to see that S_n is relatively open in $\mathbb{H}$ for every n. Therefore, S is a G_δ set. A known example of a strictly increasing, continuous and singular function uses the so-called Cantor function. If a function $f \in \mathbb{H}$ is singular, we say that f is a *singular autohomeomorphism*. We say that $f \in \mathbb{H}$ is a *strictly singular autohomeomorphism*, if f has no positive finite derivative at any point, more exactly, f has no positive finite derivative at any point of $(0,1)$ and no one-sided derivative at 0 and 1. Let $SS\mathbb{H} = \{f \in \mathbb{H} : f \text{ is strictly singular}\}$. Let $SS\mathbb{H}^+$ denote the set of all autohomeomorphisms with no positive finite right–hand sided derivative at any point in $[0,1)$. Analogously we define $SS\mathbb{H}^-$ considering the interval $(0,1]$.

Let X be a Polish space. A subset A of X is called *analytic* if it is the projection of a Borel subset B of $X \times X$. A subset C of X is called *coanalytic* if $X \setminus C$ is analytic. The pointclasses of analytic and coanalytic sets are denoted by Σ^1_1 and Π^1_1, respectively. A set $C \subset X$ is called Π^1_1*–complete (Borel* Π^1_1*–complete)* if C is coanalytic and for every zero–dimensional Polish space Y and every coanalytic set $B \subset Y$ there is a continuous (Borel) function $f : Y \to X$ such that $f^{-1}(C) = B$.

Let $\mathbb{Z}$ and $\mathbb{N}$ stand for the sets of all integers and of all nonnegative integers, respectively. By $\mathbb{Z}^{<\mathbb{N}}$ we denote the set of all finite sequences of integers. Let $2\mathbb{Z}$ stand for the set of all even integers. For a sequence $s = (s(0), s(1), ..., s(k-1)) \in \mathbb{Z}^{<\mathbb{N}}$ and $m \in \mathbb{Z}$ let $|s| = k$, $2s = (2s(0), 2s(1), ..., 2s(k-1))$ and $s\hat{\ }m = (s(0), s(1), ..., s(k-1), m)$. For a sequence $\alpha \in \mathbb{Z}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\alpha|n = (\alpha(0), \alpha(1), ..., \alpha(n-1)) \in \mathbb{Z}^{\mathbb{N}}$. Similarly for $s \in \mathbb{Z}^{<\mathbb{N}}$ and $n \leq |s|$, let $s|n = (s(0), s(1), ..., s(n-1))$. By Tr we denote the set of all trees on $\mathbb{Z}$ and by $WF \subset Tr$ we denote the set of all well-founded trees on $\mathbb{Z}$ (a tree is said to be *well-founded* if it has no infinite branch). For $T \in Tr$ let $[T] = \{\alpha \in \mathbb{Z}^{\mathbb{N}} : \forall n \in \mathbb{N} \ \ (\alpha|n \in T)\}$. Then $T \in WF$ if and only if $[T] = \emptyset$. It is well known that WF is Π^1_1–complete subset of Tr (cf. [5, 32.B]). To prove the Π^1_1–completeness of a set $A \subset X$ one usually defines a continuous map $f : Tr \to X$ such that $f^{-1}(A) = WF$. A nontrivial part of such a proof is to find a suitable continuous map.

2. STRICTLY SINGULAR AUTOHOMEOMORPHISMS

In this section we prove the following

Theorem 1. *The sets $SS\mathbb{H}^+$, $SS\mathbb{H}^-$ and $SS\mathbb{H}$ are Π^1_1–complete.*

To prove Theorem 1 we will use a function defined on a compact interval and constructed by Cater in [3]. This function has the following property:

(∗) f is continuous strictly increasing, and for every x, either $D^-f(x) = +\infty$ or $D_-f(x) = 0$, and simultaneously, either $D^+f(x) = +\infty$ or $D_+f(x) = 0$.

Given an interval $[a, b]$ and a number $q \in (0, 1)$, by *q–division* of $[a, b]$ we mean a family of intervals $\{I^n_{[a,b]} : n \in \mathbb{Z}\}$ where $I^n_{[a,b]} = [a + \frac{q^{-n+1}}{1+q}(b-a), a + \frac{q^{-n}}{1+q}(b-a)]$ for $n < 0$, $I^0_{[a,b]} = [a + \frac{q}{1+q}(b-a), b - \frac{q}{1+q}(b-a)]$ and $I^n_{[a,b]} = [b - \frac{q^n}{1+q}(b-a), b - \frac{q^{n+1}}{1+q}(b-a)]$ for $n > 0$. Then $(a, b) = \bigcup_{n\in\mathbb{Z}} I^n_{[a,b]}$.

Lemma 2. *Let $\{I^n_{[a,b]} : n \in \mathbb{Z}\}$ be a q–division of $[a, b]$. Then each square $I^n_{[a,b]} \times I^n_{[a,b]}$ lies under lines l_2 and l_3 and above lines l_1 and l_4 where*

1. *(a, a) is a common point of lines l_1 and l_2;*
2. *(b, b) is a common point of lines l_3 and l_4;*

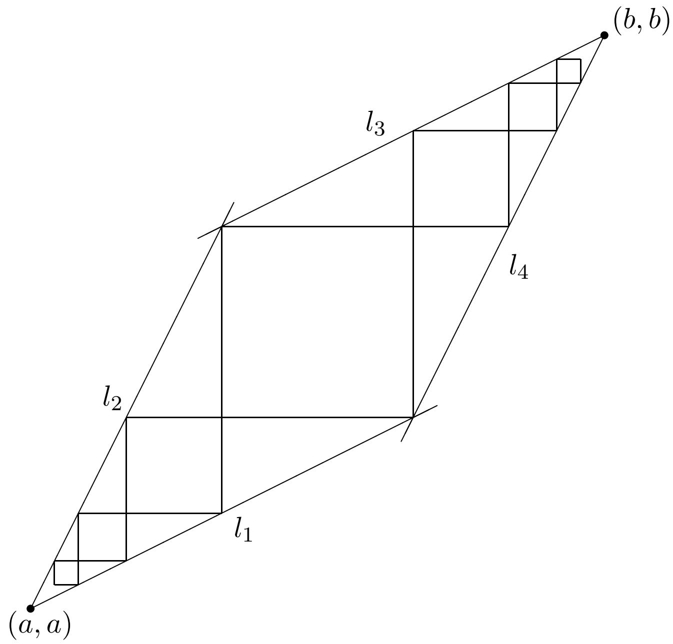

FIGURE 1. Mutual position of squares $I^n_{[a,b]} \times I^n_{[a,b]}$ and lines l_1, l_2, l_3 and l_4

3. *the slopes of* l_1 *and* l_3 *are equal to* q*;*
4. *the slopes of* l_2 *and* l_4 *are equal to* $1/q$*.*

Proof. It is enough to have a look at Figure 1. □

Now we define a family $\{I_s : s \in \mathbb{Z}^{<\mathbb{N}}\}$ of closed intervals contained in $[0,1]$ by the induction on the length of s. Put $I_\emptyset = [0,1]$. Suppose that we have already defined the intervals I_s for $|s| \leq k$. Let $s \in \mathbb{Z}^k$ and let $\{I_{s\hat{}n} : n \in \mathbb{Z}\}$ be the $\frac{k+1}{k+2}$–division of I_s. If $|s| = k$ then we say that I_s is an interval of the k-th generation.

Lemma 3. *Let* $s \in \mathbb{Z}^k$*, and* $n, n', m, m' \in \mathbb{Z}$ *be such that* $n \neq m$*. Suppose that* $x, f(x) \in I_{s\hat{}n\hat{}n'}$ *and* $y, f(y) \in I_{s\hat{}m\hat{}m'}$*. Then*

$$\frac{k+2}{k+3} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{k+3}{k+2}.$$

Proof. Without loss of generality we may assume that $n < m$. Since $(x, f(x)) \in I_{s\hat{}n\hat{}n'} \times I_{s\hat{}n\hat{}n'}$ then by Lemma 2 the point $(x, f(x))$ lies under the line l^x_3 and above the line l^x_4; the slopes of l^x_3 and l^x_4 are $\frac{k+2}{k+3}$ and $\frac{k+3}{k+2}$, respectively, and $(\max(I_{s\hat{}n}), \max(I_{s\hat{}n}))$

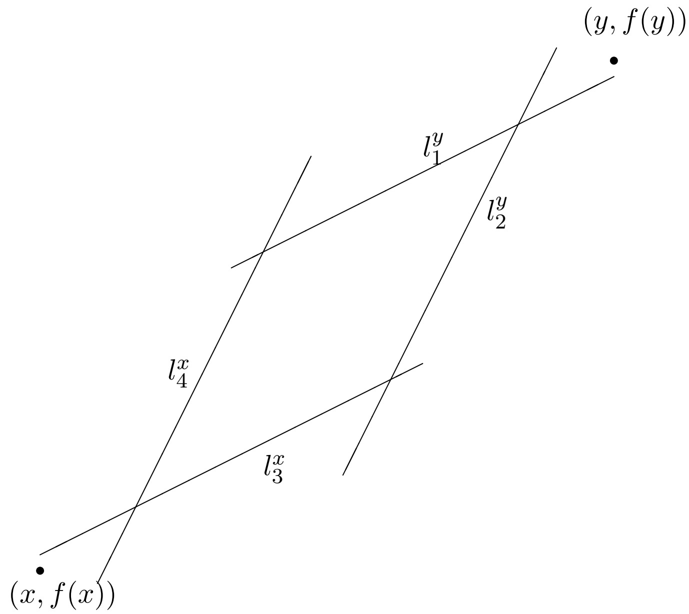

FIGURE 2. Mutual position of points $(x, f(x))$, $(y, f(y))$ and lines l_1^y, l_2^y, l_3^x and l_4^x

is a common point of l_3^x and l_4^x. Similarly $(y, f(y))$ lies above the line l_1^y and under the line l_2^y; the slopes of l_1^y and l_2^y are $\frac{k+2}{k+3}$ and $\frac{k+3}{k+2}$, respectively, and $(\min(I_{s\hat{\ }m}), \min(I_{s\hat{\ }m}))$ is the common point of l_1^y and l_2^y. Since $(\max(I_{s\hat{\ }n}), \max(I_{s\hat{\ }n}))$ and $(\min(I_{s\hat{\ }m}), \min(I_{s\hat{\ }m}))$ lie on the graph of the indentity function, and $\max(I_{s\hat{\ }n}) < \min(I_{s\hat{\ }m})$ then the mutual position of the points $(x, f(x))$ and $(y, f(y))$ and lines l_1^y, l_2^y, l_3^x and l_4^x looks like in the Figure 2, and the assertion easily follows. $\square$

Lemma 4. *The following sets are Borel:*

(a) $\{(f, x) \in \mathbb{H} \times [0, 1) : D^+f(x) < a\}$ *for* $a \in (0, \infty]$*;*

(a') $\{(f, x) \in \mathbb{H} \times [0, 1) : D^+f(x) > a\}$ *for* $a \in [0, \infty)$*;*

(b) $\{(f, x) \in \mathbb{H} \times [0, 1) : D^+f(x) = D_+f(x)\}$*;*

(c) $\{(f, x) \in \mathbb{H} \times [0, 1] : f'(x)$ *exists*$\}$*.*

The sets analogous to those described in (a) *and* (a') *are Borel if we replace* D^+ *by* D_+*, or if we replace "*$<$*" by "*$\leq$*" or by "*$=$*" in* (a)*, and "*$>$*" by "*$\geq$*" in* (a')*, respectively.*

The sets analogous to those described in (a), (a'), (b) *are Borel if we consider* D^-, D_- *instead of* D^+, D_+ *for* $x \in (0,1]$.

Proof. It is a routine calculation. □

Now we are ready to prove the main result.

Proof of Theorem 1. At first we show that $SS\mathbb{H}^+$ is coanalytic. To see this, let us consider the complement of $SS\mathbb{H}^+$. By Lemma 4, the set

$$\{(f,x) \in \mathbb{H} \times [0,1) : f'_+(x) \text{ exists and } 0 < f'_+(x) < \infty\}$$

is Borel. Then the set

$$\mathbb{H} \setminus SS\mathbb{H}^+ = \{f \in \mathbb{H} : \exists x \in [0,1) : \ \ (f'_+(x) \text{ exists and } 0 < f'_+(x) < \infty)\}$$

is analytic as the projection of a Borel set. Finally, $SS\mathbb{H}^+$ is coanalytic. Analogously one can show that $SS\mathbb{H}^-$ is Π^1_1. Since $SS\mathbb{H} = SS\mathbb{H}^- \cap SS\mathbb{H}^+$, the set $SS\mathbb{H}$ is also coanalytic.

To show that $SS\mathbb{H}$, $SS\mathbb{H}^+$ and $SS\mathbb{H}^-$ are Π^1_1–complete, we will reduce WF to them by a continuous function. Fix a function $g \in \mathbb{H}$ with property $(*)$. We say that a function h on $[a,b]$ is an *affine copy* of g if $h = (b-a)(g \circ \beta) + a$ on $[a,b]$, where $\beta : [a,b] \to [0,1]$ is an increasing affine bijection.

Let $T \in Tr$. For $n \in \mathbb{Z}$ let f^T_0 on $I_{(n)}$ be an affine copy of g. Additionally define $f^T(0) = 0$ and $f^T(1) = 1$. Next we define inductively the functions f^T_n for $n \in \mathbb{N}$. To obtain f^T_{n+1} we modify f^T_n an each interval I_{2s} for $|s| = n$ and $s \in T$ putting on each interval $I_{(2s)^\frown k^\frown l}$, $k, l \in \mathbb{Z}$, an affine copy of g. On the rest of $[0,1]$, the function f^T_n remains unchanged, i.e. $f^T_{n+1}(x) = f^T_n(x)$. Note that $f^T_n \in \mathbb{H}$ implies that $f^T_{n+1} \in \mathbb{H}$. By an easy induction it follows that $f^T_n \in \mathbb{H}$ for all $n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ and $0 < m < n$. By the construction, the supremum norm $||f^T_m - f^T_n||$ in $C[0,1]$ is less than the length of an interval I_t where t is the sequence of m zeros, hence it is less than 2^{-m}. Indeed, the graphs of f^T_m and f^T_n may differ at least in the squares $I_s \times I_s$ for $|s| > m$, and each of these squares is smaller than the square

$I_t \times I_t$. Hence $(f_n^T)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C[0,1]$, and then it converges to some function $f^T \in C[0,1]$ such that $f^T(0) = 0$ and $f^T(1) = 1$. It is clear that f^T is strictly increasing. Then $f^T \in \mathbb{H}$.

To show that $T \mapsto f^T$ is a continuous map from Tr to $\mathbb{H}$, let $n \in \mathbb{N}$, fix $s \mapsto [s]$, a bijection from $\mathbb{Z}^{<\mathbb{N}}$ onto $\mathbb{N}$, a number $N \in \mathbb{N}$ such that the length of I_s is less than 2^{-n} if $[s] \geq N$, and let S and T be two trees on $\mathbb{Z}$ such that

$$T \cap \{s \in \mathbb{Z}^{<\mathbb{N}} : [s] < N\} = S \cap \{s \in \mathbb{Z}^{<\mathbb{N}} : [s] < N\}.$$

Then $||f^T - f^S|| < 2^{-n}$ which proves the continuity of $T \mapsto f^T$.

To finish the proof, it remains to verify that

$$T \in WF \iff f^T \in SS\mathbb{H}^+ \iff f^T \in SS\mathbb{H}^- \iff f^T \in SS\mathbb{H}.$$

Let $T \in WF$. We have to check that f^T has no right-hand sided positive derivative at any $x \in [0,1)$ and has no left-hand sided positive derivative at any $x \in (0,1]$. Let $x \in [0,1]$. We have the following cases:

(a) $\exists s \in T \;\; \exists n \notin 2\mathbb{Z} \;\; (x \in int(I_{(2s)\hat{}n}))$;

(b) $\exists s \in T \;\; \exists n \in \mathbb{Z} \;\; (s\hat{}n \notin T$ and $x \in I_{2s\hat{}n})$;

(c) x is a common point of two intervals of the type I_s described in (a) and (b);

(d) $\exists s \in T \;\; [(\exists n \in \mathbb{Z} \;\; s\hat{}n \in T)$ and $(x = \min(I_{2s})$ or $x = \max(I_{2s}))]$.

In cases (a) and (b), in some open neighborhood of x the function f^T is an affine copy of g, and so it has no one-sided finite and positive derivatives at x. In case (c), the point x connects two affine copies of g. Hence f^T has no one-sided finite positive derivatives at x. In case (d) assume that $x = \min(I_{2s})$ (for $x = \max(I_{2s})$ the proof is similar). At first we see that $f^T(x) = x$. Since g is not the indentity function, there is x_0 such that $g(x_0) \neq x_0$. It is clear from the construction that there is a sequence (x_k) converging from the right to x such that $f^T(x_k) = x_k$ for every k. It is enough to consider endpoints of the intervals $I_{(2s)\hat{}k}$. Since f^T equals an affine copy of g in the intervals $I_{(2s)\hat{}k\hat{}l}$ for every l and every even k, there is a sequence (y_k) converging from the right to x such that $(f^T(y_k) - f^T(x))/(y_k - x) = g(x_0)/x_0$ for every k, and such that all points x and each y_k lie on one line. The slope of this line is strictly between $g(x_0)/x_0$ and 1. Therefore, f^T has neither finite nor infinite right hand side derivative at x.

Assume now, that $T \notin WF$. Then there exists $\alpha \in [T]$. Let x be the unique point of $\bigcap_{n\in\mathbb{N}} I_{(2\alpha)|n}$. We will show that $(f^T)'(x) = 1$. Let $z_m \to x$ where $z_m \in [0,1]$ for $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ denote by k_m the maximal number of generation in which there is an interval containing x and z_m. Then clearly $k_m \to \infty$. Since x is not the end point of any interval of type I_s, then by Lemma 3 we obtain

$$\frac{k_m+2}{k_m+3} \le \frac{f^T(z_m)-f^T(x)}{z_m-x} \le \frac{k_m+3}{k_m+2}.$$

Tending with m to ∞ we have $(f^T)'(x) = 1$. □

3. Other sets of autohomeomorphisms

In this section, Tr denotes the set of all trees on $\mathbb{N}$ and WF – the set of all well-founded trees on $\mathbb{N}$. We will use notation for sequences in $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ similar to that used for sequences in $\mathbb{Z}^{<\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$. We will consider the set

$$\overline{\Delta}_{<\infty} = \{f \in \mathbb{H} : \forall x \in [0,1)(D^+f(x) < \infty) \text{ and } \forall x \in (0,1](D^-f(x) < \infty)\},$$

of all autohomeomorphisms with finite Dini derivatives, and the set

$$\underline{\Delta}_{>0} = \{f \in \mathbb{H} : \forall x \in [0,1)(D_+f(x) > 0) \text{ and } \forall x \in (0,1](D_-f(x) > 0)\}.$$

of all autohomeomorphisms with positive Dini derivatives.

Theorem 5. *The families $\overline{\Delta}_{<\infty}$, $\underline{\Delta}_{>0}$ and $\overline{\Delta}_{<\infty} \cap \underline{\Delta}_{>0}$ are Π^1_1–complete subsets of $\mathbb{H}$.*

Proof. By Lemma 4 and the definitions of sets $\overline{\Delta}_{<\infty}$ and $\underline{\Delta}_{>0}$ it is easy to see that they are coanalytic. The set $\overline{\Delta}_{<\infty} \cap \underline{\Delta}_{>0}$ is also coanalytic as the intersection of two coanalytic sets.

First we show that $\overline{\Delta}_{<\infty}$ is Π^1_1–complete. Let $I_{(n)} = [2^{-n}, 2^{-n-1}]$ for $n \in \mathbb{N}$. We define inductively closed intervals I_s and J_s for $s \in \mathbb{N}^{<\mathbb{N}}$, $|s| \ge 1$ such that

1. $|J_s| = \frac{1}{4^{1/|I_s|}}|I_s|$;
2. J_s is concentric in I_s.
3. $\forall n \in \mathbb{N}\ (I_{s\hat{}n} = [\min(J_s) + 2^{-n-1}|J_s|, \min(J_s) + 2^{-n}|J_s|]$;

Fix $T \in Tr$. Let $f_0^T = id_{[0,1]}$. We define the functions f_n^T, $n \geq 1$, inductively in the following way: To obtain f_{n+1}^T we modify f_n^T on each interval I_s where $s \in T$ and $|s| = n+1$. On J_s we define f_{n+1}^T as an affine function with slope 2^{n+1} and such that $f_{n+1}^T(center(J_s)) = f_n^T(center(I_s))$. On $I_s \setminus J_s$ we define f_{n+1}^T as a piecewise affine function such that $f_{n+1}^T(\min I_s) = f_n^T(\min I_s)$, $f_{n+1}^T(\max I_s) = f_n^T(\max I_s)$ and f_{n+1}^T is continuous on I_s.

Note that $f_n^T \in \mathbb{H}$ for $n \in \mathbb{N}$. Since $||f_{n+1}^T - f_n^T|| \leq \frac{1}{2^{n+1}}$ then for $m > n$ we have

$$||f_m^T - f_n^T|| \leq ||f_m^T - f_{m-1}^T|| + ... + ||f_{n+1}^T - f_n^T|| \leq \frac{1}{2^m} + ... + \frac{1}{2^{n+1}} < \frac{1}{2^n}.$$

Hence f_n^T tends to some f^T in $C[0,1]$ and so f^T is continuous. Moreover $f^T(0) = 0$ and $f^T(1) = 1$. By the construction, f^T is strictly increasing. Hence $f^T \in \mathbb{H}$. One can show that $T \mapsto f^T$ is a continuous map from Tr to $\mathbb{H}$ in the same way as in the proof of the Theorem 1.

Our proof will be complete if we show that for all $T \in Tr$,

$$T \in WF \iff f^T \in \overline{\Delta}_{<\infty}.$$

Let $T \in WF$ and $x \in [0,1]$. If $x \neq 0$ and $x \neq \min(J_s)$ for any $s \in T$, there are $n \in \mathbb{N}$ and an open neighborhood U of x such that $f_n^T|_U = f^T|_U$ and we have $D^+f^T(x) < +\infty$ and $D^-f^T(x) < +\infty$. If $x = \min(J_s)$ for some $s \in T$, then the graph of f^T on small enough right hand side neighbourhood of x is contained in the union of rectangles R_n converging to $(x, f^T(x))$ such that R_n has the width equal to $|I_{s\hat{}n}|$, the length equal to $2^{n+1}|I_{s\hat{}n}|$, and the centers of R_n and $(x, f^T(x))$ lie on a line with slope 2^{n+1}. Hence $D^+f^T(x) \leq 2^{n+2}$. There are $n \in \mathbb{N}$ and a right hand side neighborhood U of x such that $f_n^T|_U = f^T|_U$. Hence $D^-f^T(x) < +\infty$. Since the graph of f^T on $(0,1]$ is contained in $\bigcup_{n\in\mathbb{N}} I_{(n)} \times I_{(n)}$, then $D^+f^T(0) \leq 2$.

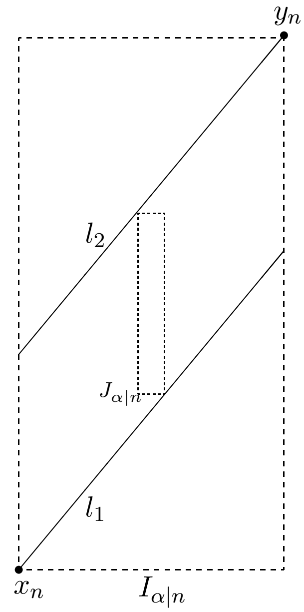

FIGURE 3.

If $T \notin WF$, there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha|n \in T$ for all $n \in \mathbb{N}$. Let $x_n = \min I_{\alpha|n}$, $y_n = \max I_{\alpha|n}$ and $x \in \bigcap_{n\in\mathbb{N}} I_{\alpha|n} = \bigcap_{n\in\mathbb{N}} J_{\alpha|n}$. For all $n \in \mathbb{N}$, x is in the subinterval $J_{\alpha|n}$ of $I_{\alpha|n}$ with $|J_{\alpha|n}| = 4^{-1/|I_{\alpha|n}|}|I_{\alpha|n}|$. On Figure 3, the big dash rectangle has the width equal to $|I_{\alpha|n}|$, the length equal to $2^n|I_{\alpha|n}|$ and the graph of the function $f^T|_{I_{\alpha|n}}$ is contained in it. The small dash rectangle has the width equal to $|J_{\alpha|n}|$, the length equal to $2^{n+1}|J_{\alpha|n}|$, it is concentric in the big rectangle and it contains the graph of $f^T|_{J_{\alpha|n}}$. In particular, the small rectangle contains the point $(x, f^T(x))$. Hence the numbers $\frac{f^T(y_n)-f^T(x)}{y_n-x}$ and $\frac{f^T(x_n)-f^T(x)}{x_n-x}$ are not greater than the slopes of lines l_1 and l_2. We see also that the slopes of l_1 and l_2 are the same and they are equal to

$$\frac{\frac{1}{2}2^n|I_{\alpha|n}| - \frac{1}{2}2^{n+1}|J_{\alpha|n}|}{\frac{1}{2}|I_{\alpha|n}| + \frac{1}{2}|J_{\alpha|n}|} = \frac{2^n|I_{\alpha|n}| - 2^{n+1}4^{-1/|I_{\alpha|n}|}|I_{\alpha|n}|}{|I_{\alpha|n}| + 4^{-1/|I_{\alpha|n}|}|I_{\alpha|n}|} = 2^n\frac{1-2\cdot 4^{-1/|I_{\alpha|n}|}}{1+4^{-1/|I_{\alpha|n}|}} \geq$$

$$\geq 2^n\frac{1-2\cdot 4^{-n}}{2} \geq 2^{n-2}.$$

Since $x_n \to x$ and $y_n \to y$, we obtain $D^+f^T(x) = +\infty$ and $D^-f^T(x) = +\infty$.

By the construction of f^T (for any tree $T \in Tr$) it follows that $D_+ f^T(x) \geq \frac{1}{2}$ for $x \in [0,1)$ and $D_- f^T(x) \geq \frac{1}{2}$ for $x \in (0,1]$. Hence for each $T \in Tr$ we obtain

$$T \in WF \iff f^T \in \overline{\Delta}_{<\infty} \cap \underline{\Delta}_{>0}.$$

This proves the Π^1_1–completeness of $\overline{\Delta}_{<\infty} \cap \underline{\Delta}_{>0}$.

Since $f \mapsto f^{-1}$ is a homeomorphism between $\overline{\Delta}_{<\infty}$ and $\underline{\Delta}_{>0}$, then $\underline{\Delta}_{>0}$ is Π^1_1–complete. □

Remark 6. Sets A and B are said to be *Borel–inseparable* if there is no Borel set C such that $A \subset C$ and $B \cap C = \emptyset$. Let UB be the set of all trees on $\mathbb{N}$ with a unique infinite branch. It is known that WF and UB is a Borel–inseparable pair of coanalytic sets (see [5, Exercise 35.2], for other examples of Borel–inseparable pair of coanalytic sets, see [1]). Denote by $SS\mathbb{H}_1$ the set of all autohomeomorphisms with an exactly one point in $[0,1]$ at which the derivative exists, and is finite and positive. Note that for any $T \in Tr$

$$T \in WF \iff f^T \in SS\mathbb{H} \quad \text{and} \quad T \in UB \iff f^T \in SS\mathbb{H}_1,$$

where f^T is the function defined in the proof of Theorem 1. This shows that $SS\mathbb{H}$ and $SS\mathbb{H}_1$ are Borel–inseparable. One can prove the analogous facts for $SS\mathbb{H}^+$, $\Delta_{<\infty}$, $\Delta_{>0}$ and $\Delta_{<\infty} \cap \Delta_{>0}$. □

This work has been supported by the Polish Ministery of Science and Higher Education Grant No. 1P03A02330. This is a part of the author's PhD thesis prepared in the Mathematical Institute of the Polish Academy of Science.

Author's Acknowledgements. I would like to thank my advisor Prof. Marek Balcerzak for his help in preparing this paper and all his input on my thesis project. I would like also to thank the anonymous referee for a suggestion of simplifying the proof of Theorem 1.

References

[1] Becker, H.: Some examples of Borel-inseparable pairs of coanalytic sets, Mathematika **33** (1986), no. 1, 72–79.

[2] Billingsley, P.; Probability and Measure, Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[3] Cater, F. S.; On the Dini derivates of a particular function, Real Anal. Exchange **25** (1999/00), no. 2, 943–946.

[4] Graf, S.; Mauldin, R. Daniel; Williams, S. C.: Random homeomorphisms, Adv. in Math. **60** (1986), no. 3, 239–359.

[5] Kechris, A.S.: Classical Descriptive Set Theory, Springer, New York 1995.

Mathematical Institute, Polish Academy of Science, Śniadeckich 8, 00-956 Warszawa, Poland; current address: Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: szymon_glab@yahoo.com