LEBESGUE DENSITY AND STATISTICAL CONVERGENCE

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Abstract.

Let $A \subset \mathbb{R}$ be measurable. λ stands for Lebesgue measure on $\mathbb{R}$. We say that 0 is Lebesgue right density point of A if

$$d^+(A,0) = \lim_{h\to 0^+} \frac{\lambda(A\cap[0,h])}{h} = 1.$$

Note that

$$\frac{\lambda(A\cap[0,\frac{1}{n+1}])}{\frac{1}{n+1}} \cdot \frac{n}{n+1} \leq \frac{\lambda(A\cap[0,h])}{h} \leq \frac{\lambda(A\cap[0,\frac{1}{n}])}{\frac{1}{n}} \cdot \frac{n+1}{n}.$$

Therefore $d^+(A,0)=1$ if and only if

$$\lim_{n\to\infty} n\lambda\left(A\cap[0,1/n]\right) = 1.$$

Let $a_n = \lambda(A\cap[1/(n+1),1/n])$. Then

$$\lambda(A\cap[0,1/n]) = \sum_{k=n}^{\infty} \lambda(A\cap[1/(k+1),1/k]) = \sum_{k=n}^{\infty} a_k.$$

The intuition is the following. If $d^+(A,0)=1$, then $\lambda(A\cap[1/(n+1),1/n])$ should be closed to the length $1/(n(n+1))$ of $[1/(n+1),1/n]$ if n tends to ∞, and thus $a_n/(1/n-1/(n+1)) = n(n+1)a_n$ should tend to 1.

In fact the following holds.

Theorem 1. $d^+(A,0)=1$ *if and only if* $n(n+1)a_n$ *tends statisticaly to* 1.

Proof. First note that $d^+(A,0)=1$ is equivalent to $\lim_{n\to\infty} n\sum_{k=n}^{\infty} a_k = 1$.

Assume that $n(n+1)a_n$ does not tend statistically to 1. So that there are $\varepsilon,\delta>0$ and a sequence $N_1<N_2<N_3<\dots$ such that

$$\frac{|\{k\leq N_i : k(k+1)a_k < 1-\varepsilon\}|}{N_i} > \delta$$

for every $i\in\mathbb{N}$. Let $m\in\mathbb{N}$ be such that $\delta \geq 2/m$. We will prove that $d^+(A^c,0) \geq \limsup_{i\to\infty} N_i\lambda(A^c\cap[0,1/N_i]) > 0$. We may assume that N_i divisible by m. We consider the worse possible case when the set $\{k\leq mN_i : k(k+1)a_k < 1-\varepsilon\} = [1,N_i)\cup[(m-1)N_i, mN_i)$. Then

$$N_i\lambda(A^c\cap[0,1/N_i]) \geq N_i\sum_{k=N_i}^{mN_i}\left(\frac{1}{k(k+1)} - a_k\right) \geq N_i \sum_{k=m-1N_i}^{mN_i-1} \frac{\varepsilon}{k(k+1)} =$$

$$= N_i\varepsilon\left(\frac{1}{(m-1)N_i} - \frac{1}{mN_i}\right) = \frac{\varepsilon}{m(m-1)} > 0.$$

Now, assume that $n(n+1)a_n$ tends statistically to 1. Then for every $\varepsilon,\delta>0$ there is $i(\varepsilon,\delta)\in\mathbb{N}$ such that

$$\frac{|\{k\leq N : k(k+1)a_k < 1+\varepsilon\}|}{N} \leq \delta$$

for every $N\geq i(\varepsilon,\delta)$.

Let $\varepsilon>0$ and $m\in\mathbb{N}$. Let $i := i(\varepsilon,1/m)$. As in the previous part of the proof we consider only the worse case. Let $N = lm \geq i$.

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Consider the family $\mathcal{B}$ of all sets $B \subseteq \mathbb{N}$ such that
1. $[1, N] \cap B = \emptyset$ and
2. $\frac{|B \cap [1,n]|}{n} \leq \frac{1}{m}$ for every $n \in \mathbb{N}$.

Define $C = [N+1, N+l] \cup \{N+l+im : i \in \mathbb{N}\}$. Note that $C \in \mathcal{B}$ and $|C \cap [1,n]| \geq |B \cap [1,n]|$ for every $B \in \mathcal{B}$ and every $n \in \mathbb{N}$.

In the worse possible case $a_k = 0$ if $k \in C$ and $k(k+1)a_k = 1-\varepsilon$ if $k \notin C$. Then

$$(N+1)\lambda(A \cap [0, 1/(N+1)]) \geq (N+1)\sum_{k=N+1}^{\infty} a_k =$$

$$= (N+1)\sum_{i=0}^{\infty}\sum_{j=1}^{m-1}\frac{1-\varepsilon}{(N+l+im+j)(N+l+im+j+1)} =$$

$$= (N+1)(1-\varepsilon)\sum_{i=0}^{\infty}\left(\frac{1}{N+l+im+1} - \frac{1}{N+l+(i+1)m}\right) =$$

$$= (N+1)(1-\varepsilon)\left(\frac{1}{N+l+1} - \frac{1}{N+l+m} + \frac{1}{N+l+m+1} - \frac{1}{N+l+2m} + \frac{1}{N+l+2m+1} - \dots\right) \geq$$

$$\geq (N+1)(1-\varepsilon)\left(\frac{1}{N+l+1} - \sum_{i=1}^{\infty}\frac{1}{(N+l+im)^2}\right) \geq$$

$$\geq (N+1)(1-\varepsilon)\left(\frac{1}{N+l+1} - \int_1^{\infty}\frac{1}{(N+l+mx-1)^2}dx\right) =$$

$$\geq (N+1)(1-\varepsilon)\left(\frac{1}{N+l+1} - \frac{1}{m(N+l+m-1)}\right).$$

Since $l = N/m$, then tending with m to ∞ we obtain that $(N+1)\lambda(A \cap [0, 1/(N+1)]) \geq (1-\varepsilon)$. Therefore $d^+(A, 0) = 1$. □

Theorem 1 justifies the following definition. Let $\mathcal{I}$ be an ideal of subset of $\mathbb{N}$. We say that x is an $\mathcal{I}$-right-density point of a measurable set A, in symbols $d_+^{\mathcal{I}}(A, x) = 1$, if $n(n+1)\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}])$ tends to 1 with respect to $\mathcal{I}$. Theorem 1 says that $d_+^{\mathcal{I}_d}(A, x) = 1 \iff d_+(A, x)$ where $\mathcal{I}_d$ stands for the density zero ideal. By Fin we denote the ideal of finite subsets of $\mathbb{N}$. Note that if $\mathcal{I} \subset \mathcal{J}$, then $d_+^{\mathcal{I}}(A, x) = 1$ implies $d_+^{\mathcal{J}}(A, x) = 1$.

Lemma 2 ([1])**.** *Let $G : [a, b] \to \mathbb{R}$ be a continuous function and let $U \subset (a, b)$ be an open set. Then the set*

$$U_G := \{x \in U : \text{ there is } y > x \text{ with } (x, y) \subseteq U \text{ and } G(x) > G(y)\}$$

is also open. Moreover, if (c, d) is a component of U_G, then $G(c) \geq G(d)$.

The following is the strengthening of Lebesgue's one-dimensional density theorem. We mimic the proof of Lebesgue's density theorem presented by Faure in [1].

Theorem 3. *Let $A \subset \mathbb{R}$ be measurable. Then $\lambda(\{x \in A : d_+^{\mathrm{Fin}}(A, x) \neq 1\}) = 0$.*

Proof. Note that

$$d_+^{\mathrm{Fin}}(A, x) \neq 1 \iff \liminf_{n\to\infty} n(n+1)\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]) < 1 \iff$$

$$\iff \exists k \exists^{\infty} n \quad n(n+1)\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]) < \frac{k}{k+1}.$$

Let $E_{n,k} = \{x \in A : n(n+1)\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]) < \frac{k}{k+1}\}$. Consider the map $G : [-k, k] \to \mathbb{R}$ define by

$$G(x) = \lambda(A \cap (-k, x)) - \frac{k}{k+1}x.$$

Let $\varepsilon > 0$. There is an open set $U \subset (-k, k)$ such that $E_{n,k} \subseteq U$ and $\lambda(U) < \lambda(E_{n,k}) + \varepsilon$. Let $E_k = \{x \in A : \underline{d}_+^{\mathrm{Fin}}(A, x) < \frac{k}{k+1}\}$. We need to prove that each E_k has measure zero. Note that $E_k = \bigcap_l \bigcup_{n \geq l} E_{n,k} \subseteq \bigcup_n E_{n,k}$. Therefore it is enough to show that each $E_{n,k}$ has measure zero.

Let $x \in E_{n,k}$. Then

$$n(n+1)\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}]) < \frac{k}{k+1}$$

$$\frac{\lambda(A \cap [x + \frac{1}{n+1}, x + \frac{1}{n}])}{(x + \frac{1}{n}) - (x + \frac{1}{n+1})} < \frac{k}{k+1}$$

$$\lambda(A \cap (-k, x + \frac{1}{n}]) - \lambda(A \cap (-k, x + \frac{1}{n+1}]) < \frac{k}{k+1}(x + \frac{1}{n}) - \frac{k}{k+1}(x + \frac{1}{n+1})$$

$$G(x + \frac{1}{n+1}) > G(x + \frac{1}{n}).$$

Then $\frac{1}{n+1} + E_{n,k} \subseteq U_G$. By Lemma we obtain that $G(c_l) \geq G(d_l)$ for every component (c_l, d_l) of U_G. But the inequality $G(c_l) \geq G(d_l)$ is equivalent to $\lambda(A \cap (c_l, d_l)) \leq \frac{k}{k+1}(d_l - c_l)$. Thus

$$\lambda(E_{n,k}) = \lambda(\frac{1}{n+1} + E_{n,k}) \leq \sum_l \lambda(\frac{1}{n+1} + E_{n,k} \cap (c_l, d_l)) \leq \sum_l \frac{k}{k+1}(d_l - c_l) =$$

$$= \frac{k}{k+1}\lambda(U_G) \leq \frac{k}{k+1}(\lambda(E_{n,k}) + \varepsilon).$$

Therefore $\lambda(E_{n,k}) \leq k\varepsilon$. Since $\varepsilon > 0$ is arbitrary, then $\lambda(E_{n,k}) = 0$. □

References

[1] C. A. Faure, A short proof of Lebesgue's density theorem. Amer. Math. Monthly 109 (2002), no. 2, 194–196.

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