LEBESGUE DENSITY AND STATISTICAL CONVERGENCE

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Abstract.

Let $A \subseteq \mathbb{R}$ be measurable. $\lambda$ stands for Lebesgue measure on $\mathbb{R}$. We say that 0 is Lebesgue right density point of $A$ if

$$d^+(A, 0) = \lim_{h \to 0^+} \frac{\lambda(A \cap [0, h])}{h} = 1.$$ 

Note that

$$\frac{\lambda(A \cap [0, \frac{1}{n+1}])}{\frac{1}{n+1}} \cdot \frac{n}{n+1} \leq \frac{\lambda(A \cap [0, h])}{h} \leq \frac{\lambda(A \cap [0, \frac{1}{n}])}{\frac{1}{n}} \cdot \frac{n+1}{n}.$$ 

Therefore $d^+(A, 0) = 1$ if and only if

$$\lim_{n \to \infty} n\lambda(A \cap [0, 1/n]) = 1.$$ 

Let $a_n = \lambda(A \cap (1/(n+1), 1/n])$. Then

$$\lambda(A \cap [0, 1/n]) = \sum_{k=n}^{\infty} \lambda(A \cap (1/k+1), 1/k]) = \sum_{k=n}^{\infty} a_k.$$ 

The intuition is the following. If $d^+(A, 0) = 1$, then $\lambda(A \cap (1/(n+1), 1/n])$ should be closed to the length $1/(n(n+1))$ of $(1/(n+1), 1/n]$ if $n$ tends to $\infty$, and thus $a_n/(1/n - 1/(n+1)) = n(n+1)a_n$ should tend to $1$.

In fact the following holds.

Theorem 1. $d^+(A, 0) = 1$ if and only if $n(n+1)a_n$ tends statistically to $1$.

Proof. First note that $d^+(A, 0) = 1$ is equivalent to $\lim_{n \to \infty} n \sum_{k=n}^{\infty} a_k = 1$.

Assume that $n(n+1)a_n$ does not tend statistically to $1$. So that there are $\varepsilon, \delta > 0$ and a sequence $N_1 < N_2 < N_3 < \ldots$ such that

$$\sum_{k=N_i}^{\infty} a_k < \frac{1}{\delta}$$

for every $i \in \mathbb{N}$. Let $m \in \mathbb{N}$ be such that $\delta \geq 2/m$. We will prove that $d^+(A^c, 0) \geq \limsup_{i \to \infty} N_i \lambda(A^c \cap [0, 1/N_i]) > 0$. We may assume that $N_i$ divisible by $m$. We consider the worse possible case when the set $\{k \leq mN_i : k(k+1)a_k < 1 - \varepsilon\} = [1, N_i) \cup [(m-1)N_i, mN_i)$. Then

$$N_i\lambda(A^c \cap [0, 1/N_i]) \geq N_i \sum_{k=N_i}^{mN_i} \left( \frac{1}{k(k+1)} - a_k \right) \geq N_i \sum_{k=mN_i}^{mN_i-1} \frac{\varepsilon}{k(k+1)} =$$

$$= N_i \varepsilon \left( \frac{1}{(m-1)N_i} - \frac{1}{mN_i} \right) = \frac{\varepsilon}{m(m-1)} > 0.$$ 

Now, assume that $n(n+1)a_n$ tends statistically to $1$. Then for every $\varepsilon, \delta > 0$ there is $i(\varepsilon, \delta) \in \mathbb{N}$ such that

$$\frac{\left\{ k \leq N : k(k+1)a_k < 1 + \varepsilon \right\}}{N} \leq \delta$$

for every $N \geq i(\varepsilon, \delta)$.

Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Let $i := i(\varepsilon, 1/m)$. As in the previous part of the proof we consider only the worse case. Let $N = lm \geq i$.

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Consider the family $B$ of all sets $B \subseteq \mathbb{N}$ such that
1. $[1, N] \cap B = \emptyset$ and
2. $\frac{|B \cap [1, n]|}{n} \leq \frac{1}{m}$ for every $n \in \mathbb{N}$.

Define $C = [N + 1, N + l] \cup \{N + l + im : i \in \mathbb{N}\}$. Note that $C \in B$ and $|C \cap [1, n]| \geq |B \cap [1, n]|$ for every $B \in B$ and every $n \in \mathbb{N}$.

In the worse possible case $a_k = 0$ if $k \in C$ and $k(k + 1)a_k = 1 - \varepsilon$ if $k \notin C$. Then

$$(N + 1)\lambda(A \cap [0, 1/(N + 1)]) \geq (N + 1) \sum_{k=N+1}^{\infty} a_k =$$

$$(N + 1) \sum_{i=0}^{m-1} \sum_{j=1}^{\infty} \frac{1 - \varepsilon}{(N + l + im + j)(N + l + im + j + 1)} =$$

$$= (N + 1)(1 - \varepsilon) \sum_{i=0}^{\infty} \left(\frac{1}{N + l + m} - \frac{1}{N + l + (i + 1)m}\right) =$$

$$= (N + 1)(1 - \varepsilon) \left(\frac{1}{N + l + 1} - \int_{1}^{\infty} \frac{1}{(N + l + mx - 1)^2}dx\right) =$$

$$\geq (N + 1)(1 - \varepsilon) \left(\frac{1}{N + l + 1} - \frac{1}{m(N + l + m - 1)}\right).$$

Since $l = N/m$, then tending with $m$ to $\infty$ we obtain that $(N + 1)\lambda(A \cap [0, 1/(N + 1)]) \geq (1 - \varepsilon)$. Therefore $d_+^e(A, 0) = 1$.

Theorem 1 justifies the following definition. Let $I$ be an ideal of subset of $\mathbb{N}$. We say that $x$ is an $I$-right-density point of a measurable set $A$, in symbols $d_+^I(A, x) = 1$, if $n(n + 1)\lambda(A \cap [x + \frac{1}{n + 1}, x + \frac{1}{n}])$ tends to 1 with respect to $I$. Theorem 1 says that $d_+^I(A, x) = 1 \iff d_+(A, x)$ where $I_d$ stands for the density zero ideal. By $\text{Fin}$ we denote the ideal of finite subsets of $\mathbb{N}$. Note that if $I \subset J$, then $d_+^I(A, x) = 1$ implies $d_+^J(A, x) = 1$.

**Lemma 2** ([1]). Let $G : [a, b] \to \mathbb{R}$ be a continuous function and let $U \subset (a, b)$ be an open set. Then the set

$$U_G := \{x \in U : \text{there is } y > x \text{ with } (x, y) \subseteq U \text{ and } G(x) > G(y)\}$$

is also open. Moreover, if $(c, d)$ is a component of $U_G$, then $G(c) \geq G(d)$.

The following is the strengthening of Lebesgue’s one-dimensional density theorem. We mimic the proof of Lebesgue’s density theorem presented by Faure in [1].

**Theorem 3.** Let $A \subset \mathbb{R}$ be measurable. Then $\lambda(\{x \in A : d_+^\text{Fin}(A, x) \neq 1\}) = 0$.

**Proof.** Note that

$$d_+^\text{Fin}(A, x) \neq 1 \iff \liminf_{n \to \infty} n(n + 1)\lambda(A \cap [x + \frac{1}{n + 1}, x + \frac{1}{n}]) < 1 \iff$$

$$\iff \exists k \exists \infty n \ n(n + 1)\lambda(A \cap [x + \frac{1}{n + 1}, x + \frac{1}{n}]) < \frac{k}{k + 1}.$$

Let $E_{n,k} = \{x \in A : n(n + 1)\lambda(A \cap [x + \frac{1}{n + 1}, x + \frac{1}{n}]) < \frac{k}{k + 1}\}$. Consider the map $G : [-k, k] \to \mathbb{R}$ define by

$$G(x) = \lambda(A \cap (-k, x)) - \frac{k}{k + 1}x.$$

Let $\varepsilon > 0$. There is an open set $U \subset (-k, k)$ such that $E_{n,k} \subseteq U$ and $\lambda(U) < \lambda(E_{n,k}) + \varepsilon$. Let $E_k = \{x \in A : d_+^\text{Fin}(A, x) < \frac{k}{k + 1}\}$. We need to prove that each $E_k$ has measure zero. Note that $E_k = \bigcap_{n \geq 1} E_{n,k} \subseteq \bigcup_{n} E_{n,k}$. Therefore it is enough to show that each $E_{n,k}$ has measure zero.
Let \( x \in E_{n,k} \). Then

\[
n(n + 1) \lambda(A \cap \left[ x + \frac{1}{n+1}, x + \frac{1}{n} \right]) < \frac{k}{k + 1}
\]

\[
\lambda(A \cap \left[ x + \frac{1}{n+1}, x + \frac{1}{n} \right]) (x + \frac{1}{n}) - (x + \frac{1}{n+1}) < \frac{k}{k + 1}
\]

\[
\lambda(A \cap (-k, x + \frac{1}{n}]) - \lambda(A \cap (-k, x + \frac{1}{n+1}]) < \frac{k}{k + 1}(x + \frac{1}{n}) - \frac{k}{k + 1}(x + \frac{1}{n+1})
\]

\[
G(x + \frac{1}{n+1}) > G(x + \frac{1}{n}).
\]

Then \( \frac{1}{n+1} + E_{n,k} \subseteq U \). By Lemma we obtain that \( G(c_l) \geq G(d_l) \) for every component \( (c_l, d_l) \) of \( U \). But the inequality \( G(c_l) \geq G(d_l) \) is equivalent to \( \lambda(A \cap (c_l, d_l)) \leq \frac{k}{k + 1}(d_l - c_l) \). Thus

\[
\lambda(E_{n,k}) = \lambda\left(\frac{1}{n+1} + E_{n,k}\right) \leq \sum_l \lambda\left(\frac{1}{n+1} + E_{n,k} \cap (c_l, d_l)\right) \leq \sum_l \frac{k}{k + 1}(d_l - c_l) =
\]

\[
= \frac{k}{k + 1} \lambda(U) \leq \frac{k}{k + 1}(\lambda(E_{n,k}) + \varepsilon).
\]

Therefore \( \lambda(E_{n,k}) \leq k\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, then \( \lambda(E_{n,k}) = 0 \).

\[\square\]

**References**