

LARGE FUNCTION ALGEBRAS WITH CERTAIN TOPOLOGICAL PROPERTIES

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ABSTRACT. Let \mathcal{F} be a family of continuous functions defined on a compact interval. We give a sufficient condition so that $\mathcal{F} \cup \{0\}$ contains a dense \mathfrak{c} -generated free algebra, in other words \mathcal{F} is densely \mathfrak{c} -strong algebrable. As an application we obtain dense \mathfrak{c} -strong algebrability of families of nowhere Hölder functions, Bruckner-Garg functions, functions with a dense set of local maxima and local minima and nowhere monotonous functions differentiable at all but finitely many points.

We also study the problem of the existence of large closed algebras within $\mathcal{F} \cup \{0\}$ where $\mathcal{F} \subset \mathbb{R}^X$ or $\mathcal{F} \subset \mathbb{C}^X$. We prove that the set of perfectly everywhere surjective functions together with the zero function contains a $2^{\mathfrak{c}}$ -generated algebra closed in the topology of uniform convergence while it does not contain a non-trivial algebra closed in the pointwise convergence topology. We prove that an infinitely generated algebra which is closed in the pointwise convergence topology needs to contain two valued functions and infinitely valued functions. We give an example of such an algebra, namely, it was shown that there is a subalgebra of $\mathbb{R}^{\mathbb{R}}$ with $2^{\mathfrak{c}}$ generators which is closed in the pointwise topology and for any function f in this algebra, there is an open set U such that $f^{-1}(U)$ is a Bernstein set.

1. INTRODUCTION

The algebraic properties of sets of functions have been considered in Analysis for many years. One direction of such research is finding the so called maximal (additive, multiplicative, and so on) classes for certain families of functions. For example it was proved in [21] that the maximal additive class for Darboux real functions is the set of all constant functions. Recently, a new point of looking on the largeness of sets of functions has appeared. One can call a set A , contained in some algebraic structure of functions, a big one if A (or $A \cup \{0\}$) contains a large, nice substructure inside. The first papers written in this direction were [20], [18], [19] and then [2], [4], [5]. In these papers, the notions contained in the following definition can be found.

Definition 1.1. *Let κ be a cardinal number.*

1. *Let \mathcal{L} be a vector space and $A \subset \mathcal{L}$. We say that A is κ -lineable if $A \cup \{0\}$ contains a κ -dimensional vector space.*
2. *Let \mathcal{L} be a Banach space and $A \subset \mathcal{L}$. We say that A is spaceable if $A \cup \{0\}$ contains an infinite dimensional closed vector space.*
3. *Let \mathcal{L} be a commutative algebra and $A \subset \mathcal{L}$. We say that A is κ -algebrable if $A \cup \{0\}$ contains a κ -generated algebra \mathcal{B} (i.e., the minimal cardinality of the set generating \mathcal{B} equals κ).*

Bartoszewicz and Głąb in [9] introduced the notion of strong algebrability:

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Definition 1.2. *Let κ be a cardinal number, \mathcal{L} be a commutative algebra, and $A \subset \mathcal{L}$. We say that A is strongly κ -algebrable if $A \cup \{0\}$ contains a κ -generated free algebra.*

Let us observe that the notion of spaceability is not a fully algebraic property but it has a topological ingredient (we ask about the existence of *closed* subspace of given Banach space). Ciesielski, Gámez-Merino, Pellegrino, and Seoane-Sepúlveda in [15] asked about the existence of large linear subspaces, closed in the pointwise or uniform convergence topology in $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$. So, following this way, one can define spaceability in linear topological spaces.

Some authors were interested in searching for a large substructure with some other topological property, namely dense lineability (or algebrability) of some classes of functions. For example, Bayart and Quarta in [12] proved that the set \mathcal{NH} of all nowhere Hölder functions is densely ω -algebrable in $C[0, 1]$. In [11] Bastin, Conejero, Esser, and Seoane-Sepúlveda proved that the set of all nowhere Gevrey functions is densely \mathfrak{c} -algebrable in $C_{\infty}[0, 1]$.

The aim of our paper is to formulate, prove and apply some techniques of constructing dense \mathfrak{c} -generated free algebras in the space of continuous functions on a compact interval, and to consider the possibility of the existence of closed algebras in some sets of real or complex functions.

2. DENSE STRONG \mathfrak{c} -ALGEBRABILITY IN $C[0, 1]$

It is a simple observation that the set $\{x \mapsto \exp(rx) : r \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$. Moreover, if $X \subset \mathbb{R}$ is linearly independent over \mathbb{Q} , then $\{x \mapsto \exp(rx) : r \in X\}$ is the set of free generators. In [16] the authors, using the composition of a function with some needed properties with such an exponential function, proved the \mathfrak{c} -algebrability of the set $CMm([0, 1])$ of continuous functions with dense sets of local extrema. Recently, this idea has been further developed in [6] and [7].

Let us call, after [6], a function $f : \mathbb{R} \rightarrow \mathbb{R}$ exponential-like of rank m whenever f is given by the formula $f(x) = \sum_{i=1}^m a_i \exp(\beta_i x)$ for some pairwise distinct nonzero numbers β_1, \dots, β_m and some nonzero numbers a_1, \dots, a_m . We have

Theorem 2.1 ([6]). *Let $\mathcal{F} \subseteq \mathbb{R}^{[0,1]}$ and assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function $f : \mathbb{R} \rightarrow \mathbb{R}$. Then \mathcal{F} is strongly \mathfrak{c} -algebrable. More exactly, if $H \subseteq \mathbb{R}$ is a set of cardinality \mathfrak{c} and linearly independent over the rationals \mathbb{Q} , then $\exp \circ (rF)$, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

Using Stone–Weierstrass Theorem, it is not difficult to observe that the algebra described in Theorem 2.1 is dense in $C[0, 1]$ if and only if the function F is continuous and strictly monotonic. This argument is described in the last section of [7]. To illustrate this, consider the following two examples. Let \mathcal{F} stand for the set of all continuous functions which are differentiable $n - 1$ times, but not differentiable n times at any point of their domains. Let F be the $(n - 1)^{\text{th}}$ antiderivative of a strictly positive nowhere differentiable function. Then by [7, Theorem 4.5] the family \mathcal{F} is densely \mathfrak{c} -strongly algebrable. In turn, using [7, Theorem 4.9] and a similar argument, one can prove that the set of all functions from \mathcal{C}_1 whose derivative is not α -Hölder (for any $\alpha \in (0, 1]$) at all but finitely many points, is densely \mathfrak{c} -strongly algebrable.

However, for many classes of functions the monotonic representative does not exist. Here we propose some method of construction of a dense algebra even if \mathcal{F} does not contain any monotonic function.

2.1. Nowhere constant continuous functions. Let $F : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then F is called left non-decreasing at $x \in (a, b]$ if there is $\delta > 0$ such that $F(y) \leq F(x)$ for any $y \in (x - \delta, x)$. Analogously we define a left non-increasing function at $x \in (a, b]$, and right non-decreasing (non-increasing) function at $x \in [a, b)$. We say that $x \in (a, b)$ is a point of local monotonicity, provided that F is left non-decreasing or left non-increasing and F is right non-decreasing or right non-increasing, see [17] and [13]. Note that if x is a point of local minimum (local minimizer) or a point of local maximum (local maximizer) of F , then x is a point of local monotonicity. We say that F is nowhere constant, provided that its restriction to any open interval is not constant.

Fix a function $g \in C[\alpha, \beta]$ which is nowhere constant and such that α and β are point of (one-sided) monotonicity of g . For $x \in [\alpha, \beta]$ denote by $H(x)$ the largest possible $y \in [x, \beta]$ such that $g(t)$ is between $g(x)$ and $g(y)$ for every $t \in [x, y]$ (here by $[x, x]$ we mean the singleton $\{x\}$). Such a number $H(x)$ always exists by the continuity of g . Let $x = H^0(x)$ and inductively $H^{n+1}(x) = H(H^n(x))$ for $x \in [\alpha, \beta]$.

Lemma 2.2. *Let $n \in \mathbb{N}$. If $H^{n+1}(\alpha) < \beta$, then*

- (i) $H^{n+1}(\alpha) > H^n(\alpha)$;
- (ii) $H^{n+1}(\alpha)$ is a point of local extremum of g ;
- (iii) $H^{n+1}(\alpha)$ is a local minimizer of g if and only if $H^n(\alpha)$ is a local maximizer of g .

Proof. Since α is a point of right local monotonicity of g , say g is right non-decreasing at α , then there is $\delta > 0$ such that $g(\alpha) \leq g(y)$ for every $y \in (\alpha, \alpha + \delta)$. Let $t \in (\alpha, \alpha + \delta]$ be such that $g(t) = \max\{g(y) : y \in [\alpha, \alpha + \delta]\}$. Since g is nowhere constant, then $g(t) > g(\alpha)$. Hence $H(\alpha) \geq t > \alpha$.

Now, we will show that $H(\alpha)$ is a point of right local monotonicity of g . Suppose not, then by the definition of H , $g(y) \geq g(\alpha)$ for $y \in [\alpha, H(\alpha)]$. Moreover $g(H(\alpha)) > g(\alpha)$. Let $\delta > 0$ be such that $|g(H(\alpha)) - g(y)| < (g(H(\alpha)) - g(\alpha))/2$ for $y \in (H(\alpha), H(\alpha) + \delta]$. Then $g|_{[H(\alpha), H(\alpha) + \delta]}$ attains its maximum at some $w \in [H(\alpha), H(\alpha) + \delta]$. Since g is not right non-increasing at $H(\alpha)$, then $g(H(\alpha)) < g(w)$. Moreover $g(\alpha) \leq g(y) \leq g(w)$ for any $y \in [\alpha, w]$. This contradicts the definition of H .

Proceeding inductively we obtain that $H^{n+1}(\alpha) > H^n(\alpha)$ or $H^{n+1}(\alpha) = \beta$. Note that $H^1(\alpha)$, $H^2(\alpha), \dots, H^n(\alpha)$ are local extrema of g . Moreover if $H^i(\alpha)$ is a local minimizer of g , then $H^{i+1}(\alpha)$ is a local maximizer of g , and vice-versa. \square

Lemma 2.3. *There is $n \in \mathbb{N}$ such that $H^n(\alpha) = \beta$.*

Proof. Suppose that $H^n(\alpha) < \beta$ for every $n \in \mathbb{N}$. By Lemma 2.2, the sequence $(H^n(\alpha))_{n \in \mathbb{N}}$ is strictly increasing. Let $w = \lim_{n \rightarrow \infty} H^n(\alpha)$. If g is left non-decreasing at w , then there is $\delta > 0$ such that $g(y) \leq g(w)$ for every $y \in [w - \delta, w]$. Let n be such that $H^n(\alpha)$ is a local minimizer of g with $H^n(\alpha) > w - \delta$. Then for any $y \in (H^n(\alpha), w)$ we have $g(H^n(\alpha)) \leq g(y) \leq g(w)$ which contradicts the definition of $H^{n+1}(\alpha)$. In the same manner, we show that g is not left non-increasing. Therefore $w < \beta$, since g is left monotonous at β .

Suppose now that g is not right non-decreasing at w . Let $v \in [w, \beta]$ be a minimizer of g on $[w, \beta]$. Then $v > w$ and $g(v) < g(w)$. Let $\delta > 0$ be such that $|g(y) - g(w)| < (g(w) - g(v))/2$ for every $y \in (w - \delta, w)$. Then fix $n \in \mathbb{N}$ such that $H^n(\alpha) \in (w - \delta, w)$ is a local maximizer of g and $g(H^k(\alpha)) \leq g(H^n(\alpha))$ for $k \geq n$. This is possible since $\lim_{k \rightarrow \infty} g(H^k(\alpha)) = g(w) < g(H^n(\alpha))$ and g is continuous. Therefore $g(H^n(\alpha)) \geq g(y) \geq g(v)$ for $y \in (H^n(\alpha), v)$, which contradicts the definition of $H^{n+1}(\alpha)$. Similarly one can prove that the assumption that g is not right non-increasing at w also leads to contradiction. Hence g is both right non-decreasing and right non-increasing at w . This means that g is constant on $[w, w + \delta]$ for some positive δ , which contradicts the fact that g is nowhere constant. This shows that $H^n(\alpha) = \beta$ for some $n \in \mathbb{N}$. \square

Lemma 2.4. *Let $F \in C[a, b]$ be nowhere constant and $F(a) \leq F(x) \leq F(b)$ for any $x \in [a, b]$. Let $\varepsilon > 0$. Then there is a partition $a = x_0 < x_1 < x_2 < \dots < x_n = b$ such that*

- (i) $F(x)$ is between $F(x_k)$ and $F(x_{k+1})$ for $x_k \leq x \leq x_{k+1}$ and $k = 0, 1, \dots, n-1$;
- (ii) the mesh $\max\{x_{i+1} - x_i : i = 0, 1, \dots, n-1\}$ of the partition is smaller than ε .

Proof. Let $a = t_0 < t_1 < t_2 < \dots < t_m = b$ be any partition of $[a, b]$ with the mesh smaller than ε . We will find a new partition $a = v_0 < v_1 < v_2 < \dots < v_k = b$ of $[a, b]$ such that each interval $[v_i, v_{i+1})$ contains at most one t_j and each v_i is a point of local monotonicity of F . This new partition will also have a mesh smaller than ε . We construct it in the following way.

If t_i is a point of local monotonicity of F , then t_i remains in the new partition. Otherwise by the fact that F is nowhere constant the restriction $F|_{[t_i - \varepsilon/3, t_i]}$ attains its minimum at some $w_L \in [t_i - \varepsilon/3, t_i]$ and maximum at some $w'_L \in [t_i - \varepsilon/3, t_i]$. If one of the points w_L, w'_L is in $(t_i - \varepsilon/3, t_i)$, then it is a point of local monotonicity and we put it to the new partition. However, it may happen that $\{w_L, w'_L\} = \{t_i - \varepsilon/3, t_i\}$, that is w_L and w'_L are the endpoints of the interval $[t_i - \varepsilon, t_i]$. We may assume that $w_L = t_i - \varepsilon/3$ and $F(w_L) < F(t_i)$. Take any $t \in (t_i - \varepsilon/3, t_i)$. If t is a point of local monotonicity of F , then we are done. Assume now that t is not a point of local monotonicity of F . This means that either t is not a point of left monotonicity of F or it is not a point of right monotonicity of F . We may assume that t is not a point of left monotonicity of F . Then F attains its maximum on $[w_L, t]$ on some $w \in (w_L, t)$ and w is a both-sided monotonicity point of F ; w is between $t_i - \varepsilon/3$ and t_i , and we put it to the new partition. Similarly one can find an appropriate both-sided monotonicity point in $(t_i, t_i + \varepsilon/3)$ which we put it into the new partition.

In the next step we will find a refinement $a = x_0 < x_1 < x_2 < \dots < x_n = b$ of $a = v_0 < v_1 < v_2 < \dots < v_k = b$ for which (i) holds true. To find such a refinement, for every $i < k$ we use Lemma 2.3 for the restriction $g = F|_{[v_i, v_{i+1}]}$, $\alpha = v_i$ and $\beta = v_{i+1}$. \square

The assumption that F is nowhere constant in Lemma 2.4 is essential. To see it, consider a function F given by

$$F(x) = \begin{cases} x, & x \in [1, 2], \\ x \sin(\frac{\pi}{2x}), & x \in (0, 1), \\ 0, & x \in [-1, 0], \\ 2x + 2, & x \in [-2, -1). \end{cases}$$

Note that $F(-2) \leq F(x) \leq F(2)$. For every partition $-2 = x_0 < x_1 < \dots < x_n = 2$ with the mesh smaller than 1, there is the largest k with $x_k \leq 0$. Then $F(x_k) = 0$ and we may assume that

$F(x_{k+1}) \geq 0$. But there is $x \in (0, x_{k+1})$ with $F(x) < 0$, which means that the assertion of Lemma 2.4 does not hold for F . The problem is that F is constant on $[-1, 0]$.

Lemma 2.5. *Let $E \subset \mathbb{R}$ be a finite set which is linearly independent over \mathbb{Q} . Let $F : [a, b] \rightarrow \mathbb{R}$ be a nowhere constant continuous function with $F(a) \leq F(x) \leq F(b)$ for any $x \in [a, b]$. Then for any $\varepsilon > 0$ there are $a = x_0 < x_1 < x_2 < \dots < x_n = b$ and $F_0 \in C[a, b]$ such that*

- (i) $\|F_0 - \text{id}\|_{\text{sup}} < \varepsilon$;
- (ii) $F_0(x) = a_k F(x) + b_k$ for $x_k \leq x \leq x_{k+1}$, $k = 0, 1, \dots, n-1$;
- (iii) the set $\{a_0, a_1, \dots, a_{n-1}\} \cup E$ is linearly independent over \mathbb{Q} .

Proof. By the previous Lemma there are $a = x_0 < x_1 < x_2 < \dots < x_n = b$ such that

- (i) $F(x)$ is between $F(x_k)$ and $F(x_{k+1})$ for $x_k \leq x \leq x_{k+1}$ and $k = 0, 1, \dots, n-1$;
- (ii) $\max\{x_{i+1} - x_i : i = 0, 1, \dots, n-1\} < \varepsilon/3$.

We can find real numbers a_0, b_0 such that the set $E \cup \{a_0\}$ is linearly independent over \mathbb{Q} , $a_0 F(x_0) + b_0 = x_0 (= a)$ and $|a_0 F(x_1) + b_0 - x_1| < \varepsilon/9$. Let $x \in [x_0, x_1]$. Since $F(x)$ is between $F(x_0)$ and $F(x_1)$, $a_0 F(x) + b_0$ is between $a_0 F(x_0) + b_0$ and $a_0 F(x_1) + b_0$. We have

$$\begin{aligned} |a_0 F(x) + b_0 - x| &\leq |(a_0 F(x) + b_0) - (a_0 F(x_0) + b_0)| + |a_0 F(x_0) + b_0 - x_0| + |x_0 - x| \leq \\ |(a_0 F(x_1) + b_0) - (a_0 F(x_0) + b_0)| + 0 + \varepsilon/3 &\leq |a_0 F(x_1) + b_0 - x_1| + |x_1 - x_0| + |x_0 - (a_0 F(x_0) + b_0)| + \varepsilon/3 < \\ \varepsilon/9 + \varepsilon/3 + 0 + \varepsilon/3 &\leq \varepsilon. \end{aligned}$$

In the second step we can find real numbers a_1, b_1 such that the set $E \cup \{a_0, a_1\}$ is linearly independent over \mathbb{Q} , $a_1 F(x_1) + b_1 = a_0 F(x_1) + b_0$ and $|a_1 F(x_2) + b_1 - x_2| < \varepsilon/9$. Let $x \in [x_1, x_2]$. Since $F(x)$ is between $F(x_1)$ and $F(x_2)$, then $a_1 F(x) + b_1$ is between $a_1 F(x_1) + b_1$ and $a_1 F(x_2) + b_1$. We have

$$\begin{aligned} |a_1 F(x) + b_1 - x| &\leq |(a_1 F(x) + b_1) - (a_1 F(x_1) + b_1)| + |a_1 F(x_1) + b_1 - x_1| + |x_1 - x| \leq \\ |(a_1 F(x_2) + b_1) - (a_1 F(x_1) + b_1)| + \varepsilon/9 + \varepsilon/3 &\leq |a_1 F(x_2) + b_1 - x_2| + |x_2 - x_1| + |x_1 - (a_1 F(x_1) + b_1)| + \varepsilon/9 + \varepsilon/3 < \\ \varepsilon/9 + \varepsilon/3 + \varepsilon/9 + \varepsilon/3 + \varepsilon/9 &= \varepsilon. \end{aligned}$$

After n steps the construction is complete. □

2.2. Main theorem. Let $F : [a, b] \rightarrow \mathbb{R}$ be a continuous function. We consider the following operation on F . Let $a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$. Let $E_F : [a, b] \rightarrow \mathbb{R}$ be such that $L_F(x) = f_i(F(x))$ for $x_i \leq x \leq x_{i+1}$, f_i is exponential-like and L_F is continuous. We say that L_F is a continuous piecewise exponential-like transformation of F .

We say that a family \mathcal{F} of continuous functions defined on compact intervals is *flexible*, provided

- (1) \mathcal{F} consists of nowhere constant functions;
- (2) there is $f \in \mathcal{F}$ with $f \in C[0, 1]$ and $f(0) \leq f(x) \leq f(1)$ for $x \in [0, 1]$;
- (3) $E_f \in \mathcal{F}$ for every $f \in \mathcal{F}$ and for any of its continuous piecewise exponential-like transformation E_f .

From now on we assume that \mathcal{F} is flexible.

Theorem 2.6. $\mathcal{F} \cap C[0, 1]$ is densely \mathfrak{c} -strongly algebrable in $C[0, 1]$.

Proof. Let $F \in \mathcal{F}$ be such that $F \in C[0, 1]$ and $F(0) \leq F(x) \leq F(1)$ for any $x \in [0, 1]$. Using Lemma 2.5 for $\varepsilon = 1/2$ and $E = \emptyset$ we find a partition $0 = x_0^1 < x_1^1 < \dots < x_{n_1}^1 = 1$ of the unit interval and a continuous function F_1 such that

- (i) $\|F_1 - \text{id}\|_{\text{sup}} < \varepsilon$;
- (ii) $F_1(x) = a_k^1 F(x) + b_k^1$ for $x_k^1 \leq x \leq x_{k+1}^1$, $k = 0, 1, \dots, n_1 - 1$;
- (iii) the set $\{a_0^1, a_1^1, \dots, a_{n_1-1}^1\}$ is linearly independent over \mathbb{Q} .

In the next step we use Lemma 2.5 for $\varepsilon = 1/4$ and $E = \{a_0^1, a_1^1, \dots, a_{n_1-1}^1\}$, we find a refinement $0 = x_0^2 < x_1^2 < \dots < x_{n_2}^2 = 1$ of the partition $0 = x_0^1 < x_1^1 < \dots < x_{n_1}^1 = 1$ and a continuous function F_2 such that

- (i) $\|F_2 - \text{id}\|_{\text{sup}} < \varepsilon$;
- (ii) $F_2(x) = a_k^2 F(x) + b_k^2$ for $x_k^2 \leq x \leq x_{k+1}^2$, $k = 0, 1, \dots, n_2 - 1$;
- (iii) the set $\{a_0^1, a_1^1, \dots, a_{n_1-1}^1\} \cup \{a_0^2, a_1^2, \dots, a_{n_2-1}^2\}$ is linearly independent over \mathbb{Q} , etc.

Inductively we define F_1, F_2, \dots . Let $E = \bigcup_{k=1}^{\infty} \{a_0^k, a_1^k, \dots, a_{n_k-1}^k\}$. By the construction, E is linearly independent over \mathbb{Q} . We extend E to a linearly independent set H over \mathbb{Q} of cardinality \mathfrak{c} . We may assume that there is $\{h_n : n \in \mathbb{N}\} \subset H \setminus E$ with $h_n \rightarrow 0$. By the assumption $\{\exp \circ F_p : p \in \mathbb{N}\} \cup \{\exp \circ (rF) : r \in H \setminus E\} \subseteq \mathcal{F}$. Let P be a polynomial in m variables without a constant term. Consider a function $g = P(e^{F_1}, \dots, e^{F_p}, e^{r_{p+1}F}, \dots, e^{r_n F})$. Then g restricted to $[x_l^p, x_{l+1}^p]$ is of the form

$$\sum_{i=1}^m c_i \exp(F(x)(d_1 k_{i1} + d_2 k_{i2} + \dots + d_p k_{ip} + r_{p+1} k_{ip+1} + \dots + r_n k_{in}))$$

where $d_1, \dots, d_p \in E$, $r_{p+1}, \dots, r_n \in H \setminus E$ are pairwise distinct and the vectors of integers $[k_{i1}, k_{i2}, \dots, k_{in}]$ are pairwise distinct. Therefore the numbers $d_1 k_{i1} + d_2 k_{i2} + \dots + d_p k_{ip} + r_{p+1} k_{ip+1} + \dots + r_n k_{in}$, $i = 1, \dots, m$, are distinct as well. Thus the mapping

$$x \mapsto \sum_{i=1}^m c_i \exp(F(x)(d_1 k_{i1} + d_2 k_{i2} + \dots + d_p k_{ip} + r_{p+1} k_{ip+1} + \dots + r_n k_{in}))$$

is a continuous exponential-like transformation of F on $[x_l^p, x_{l+1}^p]$. Since \mathcal{F} is closed under continuous piecewise exponential-like transformations, $g \in \mathcal{F}$.

This shows that the algebra \mathcal{A} generated by $\{\exp \circ F_p : p \in \mathbb{N}\} \cup \{\exp \circ (rF) : r \in H \setminus E\}$ is a free algebra of \mathfrak{c} generators. To see that \mathcal{A} is dense in $C[0, 1]$ note that the sequence $\exp(F_1), \exp(F_2), \dots$ tends to $x \mapsto \exp(x)$, and therefore \mathcal{A} separates the points of $[0, 1]$. Moreover, note that $\lim_{n \rightarrow \infty} \exp(h_n F) = 1$, which means that the closure of \mathcal{A} contains all constant functions. Using Stone–Weierstrass Theorem we obtain the assertion. \square

2.3. Applications. 1. We say that a continuous function $F : [a, b] \rightarrow \mathbb{R}$ is nowhere Hölder, provided that for any $x \in [a, b]$ and any $\alpha \in (0, 1)$

$$\limsup_{y \rightarrow x} \frac{|F(x) - F(y)|}{|x - y|^\alpha} = \infty.$$

Let us denote the set of all nowhere Hölder functions by \mathcal{NH} . It was proved in [7] that $f \circ F \in \mathcal{NH}$ for any nonconstant analytic function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any $F \in \mathcal{NH}$. It can be easily seen that if $F : [a, b] \rightarrow \mathbb{R}$ and $F' : [b, c] \rightarrow \mathbb{R}$ are nowhere Hölder with $F(b) = F'(b)$, then $F \cup F' : [a, c] \rightarrow \mathbb{R}$

is also nowhere Hölder. Therefore \mathcal{NH} is closed under taking continuous piecewise exponential-like transformations. Clearly \mathcal{NH} does not contain a function which is constant on some open interval.

Now, we prove that condition (2) in definition of flexibility is fulfilled. Let $F \in \mathcal{NH} \cap C[0, 1]$. We may assume that $F(0) \leq F(1)$ (otherwise, consider $-F$ which is also nowhere Hölder). If $F(0) \leq F(x) \leq F(1)$ for $x \in [0, 1]$, then we are done. Otherwise, find a maximizer $x_0 \in (0, 1)$ of F . Then $F(x) \leq F(x_0)$ for $x \in [0, x_0]$. If $F(0) \leq F(x) \leq F(x_0)$ for $x \in [0, x_0]$, then an affine transformation $t \mapsto F(t/x_0)$ of $F|_{[0, x_0]}$ fulfills the condition (2) in the definition of a flexible family. Otherwise, find a minimizer $x_1 \in (0, x_0)$ of $F|_{[0, x_0]}$. Then $F(x_1) \leq F(x) \leq F(x_0)$ for $x \in [x_1, x_0]$. Then an affine transformation $t \mapsto F(t/(x_0 - x_1) - x_1/(x_0 - x_1))$ of $F|_{[x_1, x_0]}$ fulfills the condition (2) in the definition of a flexible family. This argument will hold also for the next families.

Finally, by Theorem 2.6, the set of all nowhere Hölder functions in $C[0, 1]$ is densely \mathfrak{c} -strongly algebrable.

2. We say that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is Bruckner-Garg of rank $k \in \mathbb{N}$ (shortly $f \in \mathcal{BG}_k$), provided that there exists a countable set $A \subseteq (\min f, \max f)$ with the property that for all $x \in A$ the preimage $f^{-1}(\{x\})$ is a union of a Cantor set with at most k many isolated points, and for all $x \in (\min f, \max f) \setminus A$ the preimage $f^{-1}(\{x\})$ is a Cantor set. A function f is Bruckner-Garg (shortly $f \in \mathcal{BG}_\omega$), provided it is Bruckner-Garg of rank k for some $k \in \mathbb{N}$. Bruckner-Garg functions of rank 1 were investigated in [14], where it was shown that \mathcal{BG}_1 is residual in $C[0, 1]$. By [7, Theorem 4.13] we can easily conclude that \mathcal{BG}_ω is flexible and hence it is densely \mathfrak{c} -strongly algebrable.

3. Let $CMm([0, 1])$ be the set of all continuous functions such that both sets of their proper local minima and maxima are dense in $[0, 1]$. Using a similar argument to that in [16] one can prove that the set of all functions from $CMm([0, 1])$ is flexible and thereby it is densely \mathfrak{c} -strongly algebrable.

4. Denote by DNM the set of all functions in $C[0, 1]$ which are nowhere monotonic and differentiable in all but finitely many points, see [3]. It can be shown in a standard way that DNM is flexible; thus it is densely \mathfrak{c} -strongly algebrable.

3. CLOSED ALGEBRABILITY

Aron, Conejero, Peris and Seoane-Sepúlveda posed the following problem [1, Problem 4.1]: *Characterize when there exists a closed infinite dimensional algebra of functions with a particular "strange" property.* Among the classes considered by the authors, there was the family of everywhere surjective function $f : \mathbb{C} \rightarrow \mathbb{C}$. In the space \mathbb{C}^X or \mathbb{R}^X , $X \neq \emptyset$, we consider two natural topologies, namely the topology τ_p of pointwise convergence – the weakest topology in which each projection is continuous – and the topology τ_u of uniform convergence. We will show that the τ_p -closure of any non-trivial algebra contains a two-valued function (some characteristic function). Moreover, we will give a sufficient condition for the existence of a closed algebra inside $\mathcal{F} \cup \{0\}$ of $2^{\mathfrak{c}}$ generators.

The following proposition shows that if \mathcal{A} is a τ_p -closed nontrivial algebra, then \mathcal{A} contains a two-valued function.

Proposition 3.1. *Let \mathcal{A} be a subalgebra of \mathbb{C}^X or \mathbb{R}^X . Then for any $f \in \mathcal{A}$ the characteristic function χ_S of $S := \{x \in X : f(x) \neq 0\}$ is in $\text{cl}_{\tau_p}(\mathcal{A})$.*

Proof. Let $f \in \mathcal{A} \subset \mathbb{R}^X$. Let $g = \chi_S$ be the characteristic function of S . Take any $x_1, \dots, x_n \in X$ and $\varepsilon > 0$. Let $V = \{h \in \mathbb{R}^X : |h(x_i) - g(x_i)| < \varepsilon \text{ for } i = 1, \dots, n\}$. We need to show that $\mathcal{A} \cap V \neq \emptyset$.

Let $Y = \{f(x_i) : f(x_i) \neq 0, i = 1, \dots, n\} = \{y_1, \dots, y_k\}$. Put

$$P(y) = \sum_{j=1}^k \frac{y}{y_j} \prod_{i \neq j} \frac{y - y_i}{y_j - y_i}.$$

Then P is a polynomial without a constant term such that $P(y_j) = 1$ for any $j = 1, \dots, k$. If $f(x_i) = 0$, then $P(f)(x_i) = 0$. Since $x_i \notin S$, $f(x_i) = g(x_i)$. If $f(x_i) \neq 0$, then $y_j = f(x_i)$ for some j and $P(f)(x_i) = P(y_j) = 1 = g(x_i)$. This shows that $P(f) \in \mathcal{A} \cap V$. \square

By $\mathcal{ES}(\mathbb{C})$ we denote the family of all everywhere surjective functions $f : \mathbb{C} \rightarrow \mathbb{C}$, i.e. functions which map any nonempty open subset of \mathbb{C} onto \mathbb{C} . This family appeared at first in terms of algebrability in [5]. By $\mathcal{PES}(\mathbb{C})$ we denote the family of all perfectly everywhere surjective functions $f : \mathbb{C} \rightarrow \mathbb{C}$, i.e. functions which map any perfect subset of \mathbb{C} onto \mathbb{C} . It was proved in [10] that $\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$ -strongly algebrable. Since $\mathcal{PES}(\mathbb{C}) \subset \mathcal{ES}(\mathbb{C})$, $\mathcal{ES}(\mathbb{C})$ is $2^{\mathfrak{c}}$ -strongly algebrable too. Let \mathcal{D} stand for the family of all non-constant Darboux functions. Since any non-constant Darboux function attains \mathfrak{c} many values, we obtain the following.

Corollary 3.2. $\mathcal{D} \cup \{0\}$ does not contain a nontrivial closed algebra. In particular, the set $\mathcal{ES}(\mathbb{C})$ of all everywhere surjective functions is not 1 - τ_p -closely algebrable.

Proposition 3.1 says that any τ_p -closed algebra contains two-valued functions. The next step is searching for large τ_p -closed algebras in those consisting of functions with a finite range. Note that $\{f \in \mathbb{R}^{\mathbb{R}} : f \text{ has a finite range}\}$ is an algebra of cardinality $2^{\mathfrak{c}}$. However, the following shows that it does not contain a large τ_p -closed (even τ_u -closed) algebra.

Theorem 3.3. Let \mathcal{A} be an algebra consisting of functions with finite ranges. Then

- (i) if \mathcal{A} is finitely generated, then \mathcal{A} is τ_p -closed;
- (ii) if \mathcal{A} is not finitely generated, then \mathcal{A} is not τ_u -closed (in particular, it is not τ_p -closed).

Proof. (i) Assume that \mathcal{A} is generated by f_1, \dots, f_n . Since each f_i has a finite range, we can write

$$f_i = \sum_{j=1}^{k_i} c_{ij} \chi_{A_{ij}}$$

where c_{i1}, \dots, c_{ik_i} are distinct and A_{i1}, \dots, A_{ik_i} is a partition of \mathbb{R} . Let \mathcal{B} stand for all finite Boolean combinations of $\{A_{ij} : i = 1, \dots, n, j = 1, \dots, k_i\}$. Clearly, any member of \mathcal{A} is \mathcal{B} -measurable. Let $A \in \mathcal{B}$ be a nonempty atom of the algebra \mathcal{B} . Then there are j_1, \dots, j_n such that $A = A_{1j_1} \cap \dots \cap A_{nj_n}$. For any $i = 1, \dots, n$ there is a polynomial P_i such that $P_i(c_{ij_i}) = 1$ and $P_i(c_{ij}) \leq 0$ for $j \neq j_i$. Then

$$P_1(f_1(x)) + \dots + P_n(f_n(x)) > 0 \iff x \in A.$$

Since $P_1(f_1) + \dots + P_n(f_n)$ is constant on A and has finitely many values, there is a polynomial P such that $P(P_1(f_1) + \dots + P_n(f_n))$ is a characteristic function of A . Therefore any \mathcal{B} -measurable function is in \mathcal{A} . Since \mathcal{B} is a σ -algebra of sets, the family of all \mathcal{B} -measurable functions is τ_p -closed (a pointwise limit of \mathcal{B} -measurable functions is \mathcal{B} -measurable).

(ii) Assume now, that \mathcal{A} is not finitely generated. There are $f_1, f_2, \dots \in \mathcal{A}$ which are algebraically independent. As before, $f_i = \sum_{j=1}^{k_i} c_{ij} \chi_{A_{ij}}$ and let \mathcal{B} stand for the set of all finite Boolean combinations of $\{A_{ij} : i \in \mathbb{N}, j = 1, \dots, k_i\}$. Suppose that \mathcal{B} is finite. Again, any characteristic function of an

atom in \mathcal{B} is an algebraic combination of finitely many f_i 's. Therefore there is $n \in \mathbb{N}$ such that any \mathcal{B} -measurable function f is an algebraic combination of f_1, \dots, f_n . This yields a contradiction. Therefore \mathcal{B} is infinite. Hence we can find pairwise disjoint sets $A_1, A_2, \dots \in \mathcal{B}$. Define $f_n = \sum_{k=1}^n \frac{1}{k} \chi_{A_k}$. Since $\chi_{A_i} \in \mathcal{A}$, each f_n is in \mathcal{A} . Clearly, f_n tends uniformly to $f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k} \notin \mathcal{A}$. \square

By \mathcal{EDF} , denote the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are everywhere discontinuous and $f(\mathbb{R})$ is finite. It was proved in [8] that \mathcal{EDF} is $2^{\mathfrak{c}}$ -algebrable. Immediately we obtain the following.

Corollary 3.4. $\mathcal{EDF} \cup \{0\}$ does not contain an infinitely generated τ_u -closed algebra.

By Proposition 3.1 and Theorem 3.3 any infinitely generated τ_p -closed algebra contains finite valued and countably valued functions. It turns out that there are large τ_p -closed algebras of countably valued functions. Such construction, using the existence of large σ -independent family, will be used in the next theorem.

A family $\{A_\alpha : \alpha < \kappa\}$ of subsets of Y is called σ -independent, if for every countable set $X \subset \kappa$ and every $\varepsilon : X \rightarrow \{0, 1\}$

$$\bigcap_{\alpha \in X} A_\alpha^{\varepsilon(\alpha)} \neq \emptyset$$

where $A^0 = A$ and $A^1 = Y \setminus A$. By the Tarski theorem [22] there exists a σ -independent family on \mathfrak{c} of cardinality $2^{\mathfrak{c}}$.

Theorem 3.5. *There is a linear algebra $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$ of $2^{\mathfrak{c}}$ generators such that for any function $f \in \text{cl}_{\tau_p}(\mathcal{A}) \setminus \{0\}$ there is open set U such that $f^{-1}(U)$ is a Bernstein set. In particular, if \mathcal{F} is the family of all non-measurable functions (having no Baire property, non-measurable in the sense of Marczewski), then $\mathcal{F} \cup \{0\}$ contains a τ_p -closed algebra of $2^{\mathfrak{c}}$ generators.*

Proof. We use the method of independent Bernstein sets which was introduced in [8]. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a partition of \mathbb{R} into \mathfrak{c} many pairwise disjoint Bernstein sets. Let $\{A_\xi : \xi < 2^{\mathfrak{c}}\}$ be a σ -independent family on \mathfrak{c} . For any $\xi < 2^{\mathfrak{c}}$ put $C_\xi = \bigcup \{B_\alpha : \alpha \in A_\xi\}$. Let \mathcal{B} be the σ -algebra generated by $\{C_\xi : \xi < 2^{\mathfrak{c}}\}$.

Let \mathcal{A} be the linear algebra generated by $\{\chi_{C_\xi} : \xi < 2^{\mathfrak{c}}\}$. Then each function in \mathcal{A} is a simple function of the form $\sum_{k=1}^{2^n} c_k \chi_{D_k}$ where D_k are Boolean combinations of $C_{\xi_1}, \dots, C_{\xi_n}$ for some distinct $\xi_1, \dots, \xi_n < 2^{\mathfrak{c}}$. If $f \in \text{cl}_{\tau_p}(\mathcal{A}) \setminus \{0\}$, then there are $f_n \in \mathcal{A}$ which tend pointwisely to f . Let $X \subset 2^{\mathfrak{c}}$ be the smallest set such that each f_n is measurable with respect to σ -algebra \mathcal{B}_X generated by $\{C_\xi : \xi \in X\}$. Clearly X is countable. There is $\alpha < \mathfrak{c}$ which does not belong to any A_ξ , $\xi \in X$. Consequently, $B_\alpha \subset \bigcap_{\xi \in X} \mathbb{R} \setminus C_\xi$. Therefore $f_n|_{B_\alpha} = 0$ and $f|_{B_\alpha} = 0$. Since f is not the zero function, $f(x) \neq 0$ for some $x \in \mathbb{R}$. There is $\delta > 0$ such that $f^{-1}(f(x) - \delta, f(x) + \delta)$ is disjoint with $f^{-1}(0)$. Since f is \mathcal{B}_X -measurable, $f^{-1}(f(x) - \delta, f(x) + \delta)$ contains a Bernstein set of the form $\bigcap_{\xi \in X} C_\xi^{\varepsilon(\xi)}$ for some $\varepsilon : X \rightarrow \{0, 1\}$. Finally, a set which contains a Bernstein set and is disjoint with some other Bernstein set is also a Bernstein set. \square

Let $f \in \mathbb{R}^X$ (or $f \in \mathbb{C}^X$). Fix the partition $\{B_\xi : \xi < \kappa\}$ of \mathbb{R} (or \mathbb{C}). By $V(f)$ we define the set

$$\left\{ \bigcup_{\xi < \kappa} t(\xi) f|_{B_\xi} : t \in \mathbb{R}^\kappa \right\} \quad \left(\left\{ \bigcup_{\xi < \kappa} t(\xi) f|_{B_\xi} : t \in \mathbb{C}^\kappa \right\}, \text{resp.} \right).$$

Let $g_1, \dots, g_n \in V(f)$ and let $P(y_1, \dots, y_n)$ be a polynomial in n variables. Let $t_i(\xi)$ be such that $g_i|_{B_\xi} = t_i(\xi)f|_{B_\xi}$. Then

$$P(g_1, \dots, g_n)|_{B_\xi} = P(g_1|_{B_\xi}, \dots, g_n|_{B_\xi}) = P(t_1(\xi)f|_{B_\xi}, \dots, t_n(\xi)f|_{B_\xi}) = P'(f)|_{B_\xi}$$

where $P'(y) = P(t_1(\xi)y, \dots, t_n(\xi)y)$. Therefore the algebra $\mathcal{A}(f)$ generated by $V(f)$ is of form

$$\mathcal{A}(f) = \left\{ \bigcup_{\xi < \kappa} f_\xi : f_\xi \in \mathcal{A}_\xi \right\}$$

where \mathcal{A}_ξ is a subalgebra of \mathbb{C}^{B_ξ} generated by $f|_{B_\xi}$.

Theorem 3.6. *Assume that $f|_{B_\xi}$ is unbounded for every $\xi < \kappa$. Then $\mathcal{A}(f)$ is τ_u -closed.*

Proof. Note that τ_u is metrizable by the metric $d(g, h) = \min\{1, \sup\{|g(x) - h(x)| : x \in \mathbb{C}\}\}$. To prove that $\mathcal{A}(f)$ is τ_u -closed, take a sequence (g_n) in $\mathcal{A}(f)$ tending with respect to d to some function g . Fix $\xi < \kappa$. If g is zero on B_ξ , then obviously $g \in \mathcal{A}_\xi$. Otherwise $g|_{B_\xi}$ is nonzero. Then the sequence $(g_n|_{B_\xi})_{n \in \mathbb{N}}$ eventually consists of nonzero functions. Note that $g_n|_{B_\xi} = P_n(f)|_{B_\xi}$ for some nonzero polynomials P_n in one variable. By the assumption $f|_{B_\xi}$ is unbounded. Note that the sequence $P_n : f(B_\xi) \rightarrow \mathbb{C}$ is a Cauchy sequence with respect to $d(g, h) = \min\{1, \sup\{|g(y) - h(y)| : y \in f(B_\xi)\}\}$ for $g, h \in \mathbb{C}^{f(B_\xi)}$. Since $f(B_\xi)$ is unbounded, then for distinct polynomials in one variable $P, Q : f(B_\xi) \rightarrow \mathbb{C}$ without constant term, we have $\sup\{|P(y) - Q(y)| : y \in f(B_\xi)\} = \infty$. Therefore, the sequence (P_n) is eventually constant and equal to some polynomial P . Thus $g|_{B_\xi} = P(f)|_{B_\xi} \in \mathcal{A}_\xi$. \square

Corollary 3.7. *There exists a τ_u -closed algebra \mathcal{A} of cardinality $2^\mathfrak{c}$, and hence $2^\mathfrak{c}$ -generated, such that $\mathcal{A} \setminus \{0\}$ consists of perfectly everywhere surjective functions.*

Proof. Let $\{B_\xi : \xi < \mathfrak{c}\}$ be a decomposition of \mathbb{C} into \mathfrak{c} many Bernstein sets. For any $\xi < \mathfrak{c}$ let $f_\xi : B_\xi \rightarrow \mathbb{C}$ be a free generator such that algebra generated by f_ξ consists of perfectly everywhere surjective functions; the existence of such a function was proved in [10]. Put $f = \bigcup_{\xi < \mathfrak{c}} f_\xi : \mathbb{C} \rightarrow \mathbb{C}$. Then $\mathcal{A}(f)$ is the desired algebra. \square

For a sequence $x \in \ell_\infty$ put $\text{LIM}(x) = \{y \in \mathbb{R} : x(n_k) \rightarrow y \text{ for some increasing } (n_k)_{k \in \mathbb{N}}\}$. It was proved in [9] that the set of $x \in \ell_\infty$ for which $\text{LIM}(x)$ is homeomorphic to the Cantor set is strongly \mathfrak{c} -algebrable and comeager. We complete this result with the following.

Theorem 3.8. *The set of those $x \in \ell_\infty$ for which $\text{LIM}(x)$ is homeomorphic to the Cantor set, does not contain any nontrivial closed algebra.*

Proof. Let \mathcal{A} be an algebra such that for any $x \in \mathcal{A} \setminus \{0\}$ the set of limit points $\text{LIM}(x)$ is homeomorphic to the Cantor set. Fix nonzero $x \in \mathcal{A}$ and let $C = \text{LIM}(x)$. There is a continuous function $f : [\min C, \max C] \rightarrow [0, 1]$ such that $f(C) = [0, 1]$. Let (P_n) be a sequence of polynomials, tending uniformly to f . It is evident that $P_n(x)$ tends in ℓ_∞ to some y with $\text{LIM}(y) = [0, 1]$. Since $[0, 1]$ is not homeomorphic to C , the algebra \mathcal{A} cannot be closed. \square

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