ON THE CONVERSE OF CARISTI'S FIXED POINT THEOREM

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ABSTRACT. Let X be a nonempty set of cardinality at most $2^{\aleph_0}$ and T be a self map of X. Our main theorem says that, if each periodic point of T is a fixed point under T, and T has a fixed point, then there exist a metric d on X and a lower semicontinuous map $\phi : X \to \mathbb{R}_+$ such that $d(x, Tx) \leq \phi(x) - \phi(Tx)$ for all $x \in X$ and (X, d) is separable. Assuming CH (the Continuum Hypothesis), we obtain that (X, d) is compact.

We denote $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{Z} = \{\pm k : k \in \mathbb{N}\}$. Let X be a nonempty set and let $T : X \to X$ be a selfmap. By $\operatorname{Fix} T$ and $\operatorname{Per} T$ we denote the set of all fixed and periodic points of T, respectively (i.e., $\operatorname{Fix} T = \{x \in X : Tx = x\}$ and $\operatorname{Per} T = \{x \in X : \exists n \in \mathbb{N}\ (T^n x = x)\}$). If $\operatorname{Fix} T = \operatorname{Per} T \neq \emptyset$, the abstract dynamical system (X, T) is called a C-system. If (X, T) is a C-system and $\operatorname{Fix} T$ is a singleton, then it is called a B-system. The following theorems show connections between B-systems and Banach's fixed point theorem, and between C-systems and Caristi's [2] fixed point theorem.

Theorem 1. *(Bessaga [1]) Let (X, T) be a B-system and $\alpha \in (0, 1)$. Then there exists a complete metric d on X such that*

$$\forall x, y \in X\ d(Tx, Ty) \leq \alpha\, d(x, y).$$

Theorem 2. *(Janos [5]) Let (X, T) be a B-system of cardinality at most $2^{\aleph_0}$ and $\alpha \in (0, 1)$. Then there exists a separable metric d on X such that*

$$\forall x, y \in X\ d(Tx, Ty) \leq \alpha\, d(x, y).$$

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Theorem 3. *(Jachymski [4]) Let (X,T) be a C-system. Then there exist a complete metric d on X and a lower semicontinuous function $\phi : X \to \mathbb{R}_+$ such that*

$$\forall x \in X \; d(x,Tx) \leq \phi(x) - \phi(Tx). \tag{1}$$

Every α–contraction T satisfies (1) for $\phi(x) = \frac{d(x,Tx)}{1-\alpha}$ (for details see [3, p. 16]). Since there exist non-continuous mappings fulfilling (1), the family of all mappings for which (1) holds is essentially greater than the family of all contractions. The next example shows that in $\mathbb{R}$ these two families have even different cardinalities.

Example 1. Let B be a subset of $\mathbb{R}$ such that $0 \in B$. Consider a map $T_B : \mathbb{R} \to \mathbb{R}$ given by

$$T_B(x) = \begin{cases} x & \text{if } x \notin B, \\ 0 & \text{if } x \in B. \end{cases}$$

and let $\phi(x) = |x|$, $x \in \mathbb{R}$. Then

$$\forall x \in X \; d(x, T_B x) \leq \phi(x) - \phi(T_B x).$$

We have $T_B^{-1}(0) = B$. Since B is an arbitrary subset of real line, then assuming that B is non–Borel (non–measurable, without the Baire property, etc.) we obtain that T_B is non–Borel (non–measurable, without the Baire property, etc.). This shows also that there are $2^{2^{\aleph_0}}$ maps T_B, but only $2^{\aleph_0}$ contractions of $\mathbb{R}$.

Janos [6, Theorem 4.6.] proved that there exists a B-system (X,T) of cardinality $2^{\aleph_0}$ such that there is no metric d on X which would be complete separable and simultaneously T would be a Banach contraction relative to d. This proof is unconstructive and it is based on the comparison of cardinalities. We propose two constructive examples. Example 2 is simple but it assumes ¬CH. Example 3 shows a more involved construction in ZFC.

Example 2. Assume ¬CH. Let $X = 2^{\aleph_0}$ and let $T(\alpha)$ equal ω_1 if $\alpha \leq \omega_1$, and 0 otherwise. Obviously (X,T) is a B-system. Since $T^{-1}(\{\omega_1\})$ is of cardinality $\omega_1 < 2^{\aleph_0}$, there cannot exist a complete separable metric d on X such that T is a Banach contraction relative to d (see [7, Theorem 3.2.7]).

Example 3. Here we use notation from [7]. Fix $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ of cardinality $2^{\aleph_0}$. Enumerate sets in $\mathcal{A}$ as $\{A_\alpha : \alpha < 2^{\aleph_0}\}$. Put $X = A \cup B \cup \{0\}$ where

$$A = \bigcup_{\alpha < 2^{\aleph_0}} ((\mathbb{N} \times \{0\} \times \{\alpha\}) \cup (A_\alpha \times \{1\} \times \{\alpha\})),$$

and

$$B = \{(0, \alpha) : \alpha < 2^{\aleph_0}\}.$$

Define $T : X \to X$ by $T(n, i, \alpha) = (n-1, 0, \alpha)$ for $n > 0$, $T(0, i, \alpha) = (0, \alpha)$, $T(0, \alpha) = 0$ and $T(0) = 0$. Then (X, T) is a B–system.

Suppose there exists a metric d on X such that (X, d) is a complete separable metric space and T is a Banach contraction. Then X is a Polish space and $B \cup \{0\} = T^{-1}(\{0\})$, as a closed subset of X, is also Polish. Recall that $\mathcal{P}(\mathbb{N})$ can be treated as the Cantor space $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$.

We have

$$H \in \mathcal{A} \iff \exists x \in B (H = \{n : \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}).$$

Hence $\mathcal{A}$ is the projection into $\mathcal{P}(\mathbb{N})$ of the set

$$\{(H, x) \in \mathcal{P}(\mathbb{N}) \times B : H = \{n : \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}\} =$$

$$\{(H, x) : \forall n \in \mathbb{N}\, (n \in H \iff \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x))\} =$$

$$\bigcap_{n \in \mathbb{N}} [(\{(H, x) : n \in H\} \cup \{(H, x) : \forall y, z \in X \neg (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}) \cap$$

$$(\{(H, x) : n \notin H\} \cup \{(H, x) : \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\})].$$

Notice that sets $\{(H, x) : n \notin H\}$ and $\{(H, x) : n \in H\}$ are clopen for all $n \in \mathbb{N}$. Since $\{(H, x, y, z) \in \mathcal{P}(\mathbb{N}) \times B \times X^2 : \neg(y \neq z$ and $T^{n+1} y = T^{n+1} z = x)\}$ is Borel, the set $\{(H, x) : \forall y, z \in X \neg (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}$ is coanalytic and $\{(H, x) : \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}$ is analytic. Hence

$$\{(H, x) \in \mathcal{P}(\mathbb{N}) \times B : H = \{n : \exists y, z \in X (y \neq z,\, T^{n+1} y = T^{n+1} z = x)\}\}$$

is in the projective class $\Sigma^1_2(\mathcal{P}(\mathbb{N}) \times B)$. Since class Σ^1_2 is closed under projections, $\mathcal{A}$ is in $\Sigma^1_2(\mathcal{P}(\mathbb{N}))$.

Consider a Π^1_2 set in $\mathcal{P}(\mathbb{N})$ which is not Σ^1_2. (Some natural examples of such sets are known. In [7, Exercise 37.15], an example of a Σ^1_2–complete subset of the hyperspace $\mathcal{K}(\mathcal{C})$ of the Cantor space $\mathcal{C}$, is given. Its complement is Π^1_2–complete, and it can be viewed as a subset of $\mathcal{P}(\mathbb{N}) = \mathcal{C}$, since $\mathcal{K}(\mathcal{C})$ is homeomorphic to $\mathcal{C}$.) Then we obtain a B–system (X, T) for which there is no complete and separable metric d on X such that T is a contraction.

By $O_T(x)$ we denote the orbit of point x in the sense of Kuratowski (i.e., $O_T(x) = \{y \in X : \exists m, n \in \mathbb{N}\ (T^m x = T^n y)\}$). A metric space is called pre-compact if its completion is compact. Since a pre-compact space is a subspace of compact space, it is separable.

Now, we shall prove our main result which is an analogue of Theorem 2 for C-systems.

Theorem 4. *Let (X, T) be a C-system of cardinality at most $2^{\aleph_0}$. There exist a pre-compact metric d on X and a lower semicontinuous function $\phi : X \to \mathbb{R}_+$ (relative to d) such that*

$$\forall x \in X\ d(x, Tx) \leq \phi(x) - \phi(Tx). \tag{2}$$

Proof. Fix $x_0 \in \operatorname{Fix} T$. Put $\mathcal{A} = \{O_T(x) : x \in X\}$. Let $A \subset X$ be a selector of $\mathcal{A}$, i.e. for all $x \in X$ the set $A \cap O_T(x)$ is a singleton. Let $x \in X \setminus \{x_0\}$. Pick $a \in A$ such that $x \in O_T(a)$. Let $m = \min\{k \in \mathbb{N} : \exists n \in \mathbb{N}\ (T^n x = T^k a)\}$ and let $n = \min\{k \in \mathbb{N} :\ (T^k x = T^m a)\}$. Put $h(x) = n - m$, and $h(x_0) = -\infty$.

Since $\operatorname{Fix} T = \operatorname{Per} T$, the map $h : X \to \mathbb{Z} \cup \{-\infty\}$ has the following property:

$$\forall x \in X (x \neq Tx \Rightarrow h(x) > h(Tx)).$$

Let $g_k : h^{-1}(\{k\}) \to [\pi/2^{|k|} - \pi/2^{|k|+2}, \pi/2^{|k|} + \pi/2^{|k|+2}]$ be a one-to-one map, for every $k \in \mathbb{Z}$. Define $G : X \to \mathbb{C}$ by

$$G(x) = \begin{cases} \exp(-ig_k(x)) & \text{if } x \in h^{-1}(\{k\}) \text{ and } k < 0, \\ \exp(ig_k(x)) & \text{if } x \in h^{-1}(\{k\}) \text{ and } k \geq 0, \\ 1 & \text{if } h(x) = -\infty. \end{cases}$$

Next define $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} (4+k)\pi & \text{if } x \in h^{-1}(\{k\}) \text{ and } k > 0, \\ 4\pi/2^{|k|} & \text{if } x \in h^{-1}(\{k\}) \text{ and } k \leq 0, \\ 0 & \text{if } x = x_0. \end{cases}$$

Since $G : X \to \mathbb{C}$ is one-to-one, we may consider X as a subspace of $\{\exp(ix) : x \in [-\pi, \pi]\}$ with a metric given by the length of a shorter arc joining two points.

Let $x \in X$. We shall show that (1) holds.

Case 1. $x = x_0$. Then $Tx_0 = x_0$ and (1) obviously holds.

Case 2. $k = h(x) \leq 0$ and $x \neq x_0$. Then $d(x, Tx) \leq (\pi/2^{|k|} + \pi/2^{|k|+2})$ and $\phi(x) - \phi(Tx) = 4\pi/2^{|k|} - 4\pi/2^{|k|+1} = 4\pi/2^{|k|+1} = \pi/2^{|k|-1}$. Hence (1) holds.

Case 3. $h(x) \geq 1$. Then $d(x, Tx) \leq \pi$, $\phi(x) - \phi(Tx) = \pi$ and (1) holds.

We now show that ϕ is lower semicontinuous. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X convergent to $y \in X$.

Case 1. $y = x_0$. Then $\phi(x_n) \geq 0$, for all $n \in \mathbb{N}$, and $\phi(y) = 0$. Hence $\phi(y) \leq \liminf \phi(x_n)$.

Case 2. $y \neq x_0$. Then there exists n_0 such that $\phi(y) = \phi(x_m)$ for all $m \geq n_0$. Hence $\phi(y) = \lim \phi(x_n)$. □

Under CH we can improve the assertion of Theorem 4.

Corollary 5. *(CH) Let (X, T) be a C-system of cardinality at most $2^{\aleph_0}$. There exist a metric d on X and a lower semicontinuous function $\phi : X \to \mathbb{R}_+$ (relative to this metric) such that*

$$\forall x \in X \; d(x, Tx) \leq \phi(x) - \phi(Tx),$$

and (X, d) is compact and zero-dimensional.

Proof. Since (under CH) sets $h^{-1}(\{k\})$ are countable or of cardinality $2^{\aleph_0}$, we may assume that for all $k \in \mathbb{Z}$, g_k is one-to-one function, whose range is a countable compact or Cantor–like subset of $[-\pi/2^{|k|+1}, \pi/2^{|k|+1}]$. The result follows. □

Example 4. Corollary 5 can not be improved to get a function ϕ which is continuous even if (X, T) is B-system. To see this let $X = \mathbb{N}^{\mathbb{N}}$ and let

$\alpha \in X$. If $\alpha = (0,0,0,...)$, then put $T(\alpha) = \alpha$. If $\alpha \neq (0,0,0,...)$, then put $T(\alpha) = (\alpha(0), \alpha(1), ..., \alpha(n-1), \alpha(n)-1, \alpha(n+1), ...)$ where $n = \min\{k \in \mathbb{N} : \alpha(k) > 0\}$. (X,T) is a B-system. Suppose that there exist a compact metric d on X and a continuous function $\phi : X \to \mathbb{R}_+$ such that

$$\forall x \in X \; d(x,Tx) \leq \phi(x) - \phi(Tx).$$

We shall define inductively a sequence α_ξ, $\xi < \omega_1$. Let $\alpha_0 = (0,0,0,...)$. Assume that $0 < \gamma < \omega_1$ and that we have defined α_ξ for all $\xi < \gamma < \omega_1$, with $\phi(\alpha_\xi) < \phi(\alpha_\eta)$ for all ξ, η such that $\xi < \eta < \gamma$. If $\gamma = \eta + 1$ and $\alpha_\eta = (n_0, n_1, n_2, ...)$ then put $\alpha_\gamma = (n_0+1, n_1, n_2, ...)$. Since $T\alpha_\gamma = T\alpha_\eta$ then $\phi(\alpha_\eta) < \phi(\alpha_\gamma)$. If γ is a limit ordinal, pick an increasing sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of ordinals less than γ, with $\bigcup\{\eta_n : n \in \mathbb{N}\} = \gamma$. By the compactness of X we choose a convergent subsequence $\{\alpha_{\eta_{n_k}}\}_{k\in\mathbb{N}}$ of $\{\alpha_{\eta_n}\}_{n\in\mathbb{N}}$. Let $\alpha_\gamma = \lim_{k\to\infty} \alpha_{\eta_{n_k}}$. By inductive assumption and continuity of ϕ we easily obtain that $\phi(\alpha_\eta) < \phi(\alpha_\gamma)$ for all $\eta < \gamma$. Thus by transfinite induction we have defined an uncountable and strictly increasing sequence $\{\phi(\alpha_\gamma) : \gamma < \omega_1\}$ in $\mathbb{R}$ which yields a contradiction.

The following example shows that Theorem 2 cannot be improved to get metric d being pre-compact.

Example 5. Let (X,T) be a B–system such that T is a surjection. Suppose that there is a bounded metric d making T an α–contraction. Then $\operatorname{diam} T^n(X) \leq \alpha^n \operatorname{diam} X$. Hence $\bigcap T^n(X)$ is a singleton, but since T is a surjection we have $\bigcap T^n(X) = X$, which is true only when X is singleton. Since a pre-compact metric is bounded, d cannot be pre-compact.

Now, we give variants of Theorem 4 and Corollary 5 with slightly modified assertions.

Theorem 6. *Let (X,T) be a C-system of cardinality at most $2^{\aleph_0}$. There exist a separable metric d on X and a continuous function $\phi : X \to \mathbb{R}_+$ (relative to d) such that*

$$\forall x \in X \; d(x,Tx) \leq \phi(x) - \phi(Tx).$$

Proof. Let $h : X \to \mathbb{Z} \cup \{-\infty\}$ be a function defined as in Theorem 4. Let $g_k : h^{-1}(\{k\}) \to [2^{-k} - 2^{-k-2}, 2^{-k} + 2^{-k-2}]$ be a one-to-one map. Define $G : X \to \mathbb{R}$ by

$$G(x) = \begin{cases} g_k(x) & \text{if } x \in h^{-1}(\{k\}), \\ 0 & \text{if } h(x) = -\infty \end{cases}$$

and $\phi : X \to \mathbb{R}_+$ by

$$\phi(x) = \begin{cases} 2^{-k} & \text{if } x \in h^{-1}(\{k\}), \\ 0 & \text{if } h(x) = -\infty. \end{cases}$$

Since G is one-to-one, we may consider X as a subspace of $\mathbb{R}$ with the Euclidean metric. □

Corollary 7. *(CH) Let (X, T) be a C-system of cardinality at most $2^{\aleph_0}$. There exist a metric d on X and a continuous function $\phi : X \to \mathbb{R}_+$ (relative to this metric) such that*

$$\forall x \in X \; d(x, Tx) \leq \phi(x) - \phi(Tx),$$

and (X, d) is complete and zero-dimensional.

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