COVERING PROPERTIES OF IDEALS

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Abstract. M. Elekes proved that any infinite-fold cover of a σ-finite measure space by a sequence of measurable sets has a subsequence with the same property such that the set of indices of this subsequence has density zero. Applying this theorem he gave a new proof for the random-indestructibility of the density zero ideal. He asked about other variants of this theorem concerning I-almost everywhere infinite-fold covers of Polish spaces where I is a σ-ideal on the space and the set of indices of the required subsequence should be in a fixed ideal $\mathcal{J}$ on ω.

We introduce the notion of the $\mathcal{J}$-covering property of a pair $(\mathcal{A}, I)$ where $\mathcal{A}$ is a σ-algebra on a set X and $I \subseteq \mathcal{P}(X)$ is an ideal. We present some counterexamples, discuss the category case and the Fubini product of the null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$. We investigate connections between this property and forcing-indestructibility of ideals. We show that the family of all Borel ideals $\mathcal{J}$ on ω such that $\mathcal{M}$ has the $\mathcal{J}$-covering property consists exactly of non weak Q-ideals. We also study the existence of smallest elements, with respect to Katětov-Blass order, in the family of those ideals $\mathcal{J}$ on ω such that $\mathcal{N}$ or $\mathcal{M}$ has the $\mathcal{J}$-covering property. Furthermore, we prove a general result about the cases when the covering property "strongly" fails.

1. Introduction

We will discuss the following result due to Elekes [8].

Theorem 1.1. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $(A_n)_{n\in\omega}$ be a sequence of sets from $\mathcal{A}$ that covers μ-almost every $x \in X$ infinitely many times. Then there exists a set $M \subseteq \omega$ of asymptotic density zero such that $(A_n)_{n\in M}$ also covers μ-almost every $x \in X$ infinitely mamy times.*

Applying this result, Elekes gave a nice new proof for the fact that the density zero ideal is random-indestructible. He asked about other variants of this theorem, in particular, he asked whether the measure case could be replaced by the Baire category case.

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For an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, we assume that $\omega \notin \mathcal{I}$ and $\mathrm{Fin} \subseteq \mathcal{I}$ where Fin stands for the ideal of finite subsets of ω. An ideal $\mathcal{I}$ is *tall* if each infinite subset of ω contains an infinite element of $\mathcal{I}$. Clearly, an ideal $\mathcal{J}$ on ω is tall iff its dual filter $\mathcal{J}^*$ does not have a *pseudointersection*, that is a set $X \in [\omega]^\omega$ such that $X \subseteq^* Y$ for each $Y \in \mathcal{J}^*$ where $X \subseteq^* Y$ means that $X \setminus Y$ is finite.

An ideal $\mathcal{J}$ on ω is called a *P-ideal* whenever for every sequence of sets $E_n \in \mathcal{I}$ $(n \in \omega)$, there is a set $E \in \mathcal{I}$ such that $E_n \subseteq^* E$ for each n.

Each ideal on ω can be treated as a subset of the Cantor space 2^ω via the standard bijection between 2^ω and $\mathcal{P}(\omega)$, so we can talk about *Borel, F_σ, analytic, meager...* ideals.

We will need the following very useful characterization of meager ideals:

Theorem 1.2. ([16], [6, Theorem 4.1.2]) *An ideal $\mathcal{J}$ on ω is meager if, and only if there is an partition $(P_n)_{n\in\omega}$ of ω into finite sets (or even intervals) such that each element of $\mathcal{J}$ contains only finitely many P_n's.*

If I is an ideal on a set X then let $I^* = \{X \setminus A : A \in I\}$ its dual filter and let $I^+ = \mathcal{P}(X) \setminus I$ the set of *I-positive* subsets of X. If $Y \subseteq X$ and $Y \in I^+$, then the *restriction of I to Y* is the following ideal on Y: $I \restriction Y = \{Y \cap A : A \in I\}$.

More informations about ideals on ω can be found e.g. in [13].

Elekes discovered that these covering properties have an effect on forcing indestructibility of ideals. Assume $\mathcal{J}$ is a tall ideal on ω and $\mathbb{P}$ is a forcing notion. We say that $\mathcal{J}$ *is $\mathbb{P}$-indestructible* if $\Vdash_{\mathbb{P}} \exists\, A \in \mathcal{J}\ |\dot{X} \cap A| = \aleph_0$ for each $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω, i.e. in $V^{\mathbb{P}}$ the ideal generated by $\mathcal{J}$ is tall. This property has been widely studied for years. For general characterization theorems about forcing indestructibility of ideals see [9] and [7]. In these papers the authors studied a very general class of forcing notions, namely forcing notions of the form $\mathbb{P}_I = \mathrm{Borel}(X) \setminus I$ ordered by inclusion where X is a Polish space and I is a σ-ideal on X (with Borel base). For instance, the Cohen and the random forcing can be represented of this form by $\mathbb{C} = \mathbb{P}_{\mathcal{M}}$ and $\mathbb{B} = \mathbb{P}_{\mathcal{N}}$ where $\mathcal{M}$ is the σ-ideal of meager subsets and $\mathcal{N}$ is the σ-ideal of null subsets (with respect to the Lebesgue measure) of the real line (or 2^ω or ω^ω).

We will need the following classical Borel ideals on ω:

The *density zero ideal*,

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}$$

is a tall $F_{\sigma\delta}$ P-ideal.

The *summable ideal*,

$$\mathcal{I}_{1/n} = \left\{ A \subseteq \omega \colon \sum_{n\in A} \frac{1}{n+1} < \infty \right\}$$

is a tall F_σ P-ideal and clearly $\mathcal{I}_{1/n} \subsetneq \mathcal{Z}$.

Let $\mathcal{ED}$ be the *eventually different* ideal, that is

$$\mathcal{ED} = \Big\{A \subseteq \omega \times \omega : \limsup_{n\to\infty} |(A)_n| < \infty\Big\}$$

where $(A)_n = \{m \in \omega : (n,m) \in A\}$. Let $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$ and $\mathcal{ED}_{\text{fin}} = \mathcal{ED} \restriction \Delta$. These two ideals are tall F_σ non P-ideals.

At last, the Fubini-product of Fin by itself

$$\text{Fin} \otimes \text{Fin} = \big\{A \subseteq \omega \times \omega : \forall^\infty\, n\ |(A)_n| < \infty\big\}.$$

This is a tall $F_{\sigma\delta\sigma}$ non P-ideal.

Furthermore, we will use the *Katětov-Blass (KB) order* on ideals: $\mathcal{J}_0 \leq_{\text{KB}} \mathcal{J}_1$ iff there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}_1$ for each $A \in \mathcal{J}_0$.

In Section 2 we introduce a general covering property of a pair $(\mathcal{A}, I)$ with respect to an ideal on ω where $\mathcal{A}$ is a σ-algebra and I is an ideal on its underlying set. This property is a natural generalization of the interaction between the pair (measurable sets, ideal of measure zero sets) and the density zero ideal proved by Elekes. We discuss this notion, its connection to the star-uniformity of ideals on ω and to the Katětov-Blass order. We give some negative results showing that in certain cases the respective a.e.-subcovers do not exist on ω^ω, and in general, we show that ideals with property (M) cannot have any $\mathcal{J}$-covering properties. We investigate covering properties of $\mathcal{N} \otimes \mathcal{M}$. And at last, we present the general effect of the covering property on forcing indestructibility of ideals.

In Section 3, applying a result of C. Laflamme on filter games, we characterize the category case, namely we prove that if $\mathcal{J}$ is Borel, then $\mathcal{M}$ has the $\mathcal{J}$-covering property iff $\mathcal{ED}_{\text{fin}} \leq_{\text{KB}} \mathcal{J}$. In particular, we answer the question from [8] about the category case of Elekes' theorem. We also present some examples which show that our implications are not reversible.

In Section 4 we discuss the existence of KB-smallest elements in the family of those ideals $\mathcal{J}$ on ω such that $\mathcal{N}$ or $\mathcal{M}$ has the $\mathcal{J}$-covering property. We show that $\mathcal{Z}$ is not the KB-smallest in the measure case even among the Borel ideals. We prove that if $\mathfrak{t} = \mathfrak{c}$, then there is no KB-smallest element of these families. At last, we investigate the generic ideal on ω in the Cohen model.

In Section 5 we describe a class of ideals I on $\mathbb{R}$ (in general, on Polish groups) for which the $\mathcal{J}$-covering property fails in a strong fashion.

2. The $\mathcal{J}$-covering property

We can consider the following abstract setting.

Definition 2.1. Let X be an arbitrary set and $I \subseteq \mathcal{P}(X)$ be an ideal of subsets of X. We say that a sequence $(A_n)_{n\in\omega}$ of subsets of X is an *I-a.e. infinite-fold cover of* X if

$$\big\{x \in X : \{n \in \omega : x \in A_n\} \text{ is finite}\big\} \in I \ , \quad \text{i.e.} \quad \limsup_{n\in\omega} A_n \in I^*.$$

Of course, the sequence (A_n) above can be indexed by any countable infinite set. Assume furthermore that given a σ-algebra $\mathcal{A}$ of subsets of X and an ideal $\mathcal{J}$ on ω. We say that the pair $(\mathcal{A}, I)$ has the *$\mathcal{J}$-covering property* if for every I-a.e. infinite-fold cover $(A_n)_{n\in\omega}$ of X by sets from $\mathcal{A}$, there is a set $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is also an I-a.e. infinite-fold cover of X.

If $\mathcal{A}$ is clear from the context (e.g. $\mathcal{A} = \text{Borel}(X)$ if X is a Polish space), then we will write simply "I has the $\mathcal{J}$-covering property" instead of "$(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property". Furthermore, we will use the following notation:

$$\text{CP}(I) = \text{CP}(\mathcal{A}, I) = \big\{\mathcal{J} : (\mathcal{A}, I) \text{ has the } \mathcal{J}\text{-covering property}\big\},$$

and if Γ is a class (or property) of ideals then let $\text{CP}_\Gamma(I) = \text{CP}(I) \cap \Gamma$. For example, we can talk about $\text{CP}_{\text{Borel}}(I)$, that is, the family of those Borel ideals $\mathcal{J}$ such that I has the $\mathcal{J}$-covering property.

First of all, we list some easy observations on this definition:

Observations 2.2.

(1) Clearly, in the previous definition it is enough to check infinite-fold covers instead of I-a.e. infinite-fold covers. Observe that, if $I_1 \subseteq I_2$ and $(\mathcal{A}, I_1)$ possesses the $\mathcal{J}$-covering property, then $(\mathcal{A}, I_2)$ also possesses it. In other words, $I_1 \subseteq I_2$ implies $\text{CP}(I_1) \subseteq \text{CP}(I_2)$.

(2) In this context, Elekes' theorem says that, if $(X, \mathcal{A}, \mu)$ is a σ-finite (complete) measure space, then $(\mathcal{A}, \mathcal{N}_\mu)$ has the $\mathcal{Z}$-covering property (i.e. $\mathcal{Z} \in \text{CP}(\mathcal{N}_\mu)$) where $\mathcal{N}_\mu = \{H \in \mathcal{A} : \mu(H) = 0\}$.

(3) It is trivial that, if a pair $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then $\mathcal{J}$ must be tall. In the study of covering properties of pairs $(\text{Borel}(X), I)$ where I is a "nice" σ-ideal on the Polish space X, interesting ideals on ω must have a bit stronger property than tallness. If $\mathcal{J}$ is a tall ideal on ω then the *star-uniformity of* $\mathcal{J}$ is the following cardinal:

$$\text{non}^*(\mathcal{J}) = \min\big\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \text{ and } \nexists\ A \in \mathcal{J}\ \forall\ H \in \mathcal{H}\ |A \cap H| = \omega\big\}.$$

We claim that, if $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property and $\mathcal{A} \setminus I$ contains infinite antichains (in the forcing sense), then $\text{non}^*(\mathcal{J}) > \omega$.

Simply let $(B_k)_{k\in\omega}$ be a partition of $X = \bigcup \mathcal{A}$ into I-positive sets from $\mathcal{A}$, and assume on the contrary that $(Y_k)_{k\in\omega}$ is a sequence in $[\omega]^\omega$ such that $\forall\ S \in \mathcal{J}$ $\exists\ k \in \omega\ |S \cap Y_k| < \omega$. W.l.o.g. we can assume that this sequence is a partition of ω. Define $A_n = B_k$ if $n \in Y_k$. Then $(A_n)_{n\in\omega}$ is infinite-fold cover, since any B_k appears infinitely many times in this sequence. Take any $S \in \mathcal{J}$. Then there is k such that $S \cap Y_k$ is finite, in particular B_k appears only finitely many times in $(A_n)_{n\in S}$, and therefore $B_k \cap \limsup_{n\in S} A_n = \emptyset$, so $\limsup_{n\in S} A_n \notin I^*$.

(4) Clearly $(\mathcal{P}(X), \{\emptyset\})$ has the $\mathcal{J}$-covering property iff $|X| < \mathrm{non}^*(\mathcal{J})$. This fact and the following reformulation of the $\mathcal{J}$-covering property shows that covering properties can be seen as "analytic star-uniformity": $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property if and only if, for every $(\mathcal{A}, \mathrm{Borel}([\omega]^\omega))$-measurable function $F\colon X \to [\omega]^\omega$ (where $X = \bigcup \mathcal{A}$), there is an $S \in \mathcal{J}$ such that

$$\big\{x \in X : |F(x) \cap S| < \omega\big\} \in I.$$

(5) Notice that if $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then $(\mathcal{A}[I], I)$ also has this property where $\mathcal{A}[I]$ is the "I-completion of $\mathcal{A}$", that is

$$\mathcal{A}[I] = \big\{B \subseteq X : \exists\, A \in \mathcal{A}\ A \triangle B \in I\big\}.$$

For instance, $\mathrm{Borel}(\mathbb{R})[\mathcal{N}]$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and similarly $\mathrm{Borel}(\mathbb{R})[\mathcal{M}]$ is the σ-algebra of sets with the Baire property.

(6) If $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property then for all $Y \in \mathcal{A} \setminus I$ the pair $(\mathcal{A} \restriction Y, I \restriction Y)$ also has this property where of course, $\mathcal{A} \restriction Y = \{Y \cap A : A \in \mathcal{A}\}$ is the restricted σ-algebra.

(7) If both $(\mathcal{A}, I_1)$ and $(\mathcal{A}, I_2)$ have the $\mathcal{J}$-covering property, then $(\mathcal{A}, I_1 \cap I_2)$ also has this property. Moreover, it is easy to see that $\mathrm{CP}_\Gamma(I_1 \cap I_2) = \mathrm{CP}_\Gamma(I_1) \cap \mathrm{CP}_\Gamma(I_2)$.

(8) How could we conclude a $\mathcal{J}_1$-covering property from a $\mathcal{J}_0$-covering property? It is easy to see that if $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$ and $(\mathcal{A}, I)$ has the $\mathcal{J}_0$-covering property, then $(\mathcal{A}, I)$ has the $\mathcal{J}_1$-covering property as well. In other words, $\mathrm{CP}(I)$ is KB-upward closed.

Now, let us discuss some negative results. We start from an easy example concerning the measure case.

Example 2.3. We will show that $\mathcal{N}$ does not have the $\mathcal{I}_{1/n}$-covering property in a strong sense. First consider interval $(0,1)$ and a fixed infinite-fold cover $(A_n)_{n\in\omega}$ of $(0,1)$ of the form $A_n = (a_n, b_n)$, $b_n - a_n = \frac{1}{n+1}$. Then for each $S \in \mathcal{I}_{1/n}$ we have $\sum_{n\in S} \lambda(A_n) < \infty$ where λ stands for Lebesgue measure on $\mathbb{R}$. Hence by the Borel-Cantelli lemma, $\lambda(\limsup_{n\in S} A_n) = 0$. Fix a homeomorphism h from $(0,1)$ onto $\mathbb{R}$ of class C^1. Then $(h[A_n])_{n\in\omega}$ is an open infinite-fold cover of $\mathbb{R}$. Since h is absolutely continuous, we have $\lambda(\limsup_{n\in S} h[A_n]) = 0$, which gives the desired claim.

This example motivates the following question which will be discussed in Section 5 (see Example 5.1 and Theorem 5.2 below).

Question 2.4. Assume X is a Polish space, I is a σ-ideal on X, and I does not have the $\mathcal{J}$-covering property. Does there exist an infinite-fold Borel cover $(A_n)_{n\in\omega}$ of X such that $\limsup_{n\in S} A_n \in I$ for all $S \in \mathcal{J}$?

We will show that a wide class of ideals cannot have any $\mathcal{J}$-covering properties.

The following property was introduced in [1]: An ideal I on an uncountable Polish space X has *property (M)* if there is an uncountable Polish space Y and a Borel-measurable $f\colon X \to Y$ such that $f^{-1}[\{y\}] \in I^+$ for each $y \in Y$. This property can be seen as a strong violation of the countable chain condition. Recall that I is said to fulfil this condition (or, that I is a *ccc ideal*) if every disjoint subfamily of $\mathrm{Borel}(X) \setminus I$ is countable.

First of all, it is natural to ask about the existence of an ideal I on an uncountable Polish space X without property (M) such that there is family of $\mathfrak{c}$ pairwise disjoint I-positive Borel sets. Surprisingly, this is still an open question in ZFC. D.H. Fremlin constructed such an example under the assumption $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ (see [1, Prop. 1.5]).

Numerous classical ideals (without ccc) have property (M). Let us see some examples:

Example 2.5.

(1) The ideal of nowhere dense sets has property (M) (see [1, Prop. 1.3]).

(2) The σ-ideal $\mathcal{K}_\sigma$, generated by compact sets on ω^ω, has property (M). Indeed, if $F\colon \omega^\omega \to \omega^\omega$, $F(x)(n) = x(2n)$ for each $x \in \omega^\omega$ and $n \in \omega$, then F is continuous and $F^{-1}[\{y\}] \in \mathcal{K}_\sigma^+$ for each $y \in \omega^\omega$.

(3) The ideal of $\mathcal{H}^s$-null subsets of $\mathbb{R}$ for $s \in (0,1)$ has property (M) where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure (see [11]).

(4) The ideal of so-called *σ-porous* subsets of $\mathbb{R}$ has property (M) (see [17, Prop. 2.1]).

(5) The so-called *Mycielski ideal* has property (M) (see [3, Lemma 0.1]).

(6) The ideal of so-called *nowhere Ramsey* subsets of $[\omega]^\omega$ has property (M) (see [1, Prop. 1.4]).

(7) If $\mathcal{J}$ is a tall non prime ideal on ω, then let $\widehat{\mathcal{J}}$ be the ideal on $[\omega]^\omega$ generated by the sets of the form

$$\widehat{A} = \big\{X \in [\omega]^\omega : |A \cap X| = \omega\big\} \quad \text{for } A \in \mathcal{J}.$$

Clearly $\widehat{\mathcal{J}}$ is a σ-ideal iff $\mathcal{J}$ is a P-ideal. Let $C_0 \cup C_1 = \omega$ be a partition into $\mathcal{J}$-positive sets, and let $F\colon [\omega]^\omega \to 2^\omega$, $F(X)(n) = i$ iff the nth element of X belongs to C_i. It is easy to see that F shows that $\widehat{\mathcal{J}}$ has property (M).

(8) It is easy to see that if I or J have property (M), then $I \otimes J$ also has this property (see definition of the Fubini-product below).

The next simple result says that ideals with property (M) are not interesting in the sense of covering properties.

Proposition 2.6. *If an ideal I on X has property (M), then $\mathrm{CP}(I)=\emptyset$. Moreover, there is an infinite-fold Borel cover $(B_n)_{n\in\omega}$ of X such that for each infinite and co-infinite set $S\subseteq\omega$, $\limsup_{n\in S} B_n\notin I^*$.*

Proof. Fix a Borel function $f\colon X\to\omega^\omega$ which witnesses property (M) of I. Consider the following infinite-fold cover $(A_n)_{n\in\omega}$ of ω^ω by F_σ sets:

$$A_n=\{x\in\omega^\omega : x(n)\neq 0 \text{ or } \forall^\infty k\ x(k)=0\}.$$

It is easy to see that, if $S\subseteq\omega$ is infinite and co-infinite, then $\omega^\omega\setminus\limsup_{n\in S}A_n$ is uncountable (even it is dense and does not belong to $\mathcal{K}_\sigma$). For each n, let $B_n=f^{-1}[A_n]$. We claim that for each infinite and co-infinite set $S\subseteq\omega$, $\limsup_{n\in S}B_n\notin I^*$. Indeed,

$$\limsup_{n\in S} B_n = f^{-1}\Big[\limsup_{n\in S} A_n\Big] \quad \text{is co-uncountable,}$$

hence the preimage of any $x\in\omega^\omega\setminus\limsup_{n\in S}A_n$ shows that $\limsup_{n\in S}B_n\notin I^*$. □

As we will see, if we know that $\mathcal{N}$ has the $\mathcal{J}$-covering property, then we can infer the same for the Fubini product $\mathcal{N}\otimes\mathcal{M}$, provided that $\mathcal{J}$ is a P-ideal on ω. For more informations on the σ-ideals $\mathcal{N}\otimes\mathcal{M}$ and $\mathcal{M}\otimes\mathcal{N}$ on $\mathbb{R}^2$, see [2].

We recall the definition of *Fubini product* of ideals $I\subseteq\mathcal{P}(X)$ and $K\subseteq\mathcal{P}(Y)$. For $A\subseteq X\times Y$ and $x\in X$ let $(A)_x=\{y\in Y:(x,y)\in A\}$, and let

$$I\otimes K=\big\{A\subseteq X\times Y:\{x\in X:(A)_x\in K\}\in I^*\big\}.$$

It is easy to see that

$$\mathrm{CP}(I\otimes K)\subseteq\mathrm{CP}(I)\cap\mathrm{CP}(K)\big(=\mathrm{CP}(I\cap K)\big)$$

for each I and K: Clearly, $(B_n)_{n\in\omega}$ is an I-a.e. infinite-fold cover of X iff $(B_n\times Y)_{n\in\omega}$ is an $I\otimes K$-a.e. infinite-fold cover of $X\times Y$. Similarly, $(C_n)_{n\in\omega}$ is an K-a.e. infinite-fold cover of Y iff $(X\times C_n)_{n\in\omega}$ is an $I\otimes K$-a.e. infinite-fold cover of $X\times Y$.

Theorem 2.7. $\mathrm{CP}_{\text{P-ideals}}(\mathcal{N}\otimes\mathcal{M})=\mathrm{CP}_{\text{P-ideals}}(\mathcal{N})$.

Proof. To show the nontrivial inclusion "$\supseteq$", assume that $\mathcal{J}$ is a P-ideal such that $\mathcal{N}$ has the $\mathcal{J}$-covering property. Let $(A_n)_{n\in\omega}$ be an $\mathcal{N}\otimes\mathcal{M}$-a.e. infinite-fold Borel cover of $\mathbb{R}^2$. By [2, Prop. 4], for each Borel set G in $\mathbb{R}^2$ there is a Borel set H with open sections such that $G\triangle H\in\mathcal{N}\otimes\mathcal{M}$. So, we may assume that all sections $(A_n)_x$ for $n\in\omega$ and $x\in\mathbb{R}$ are open. There exists a Borel set $B\in\mathcal{N}^*$ such that $\limsup_{n\in\omega}(A_n)_x$ is residual for all $x\in B$. Fix a base $\{U_k:k\in\omega\}$ of open sets in $\mathbb{R}$. For all $n,k\in\omega$ define

$$D_{nk}:=\{x\in\mathbb{R}:(A_n)_x\cap U_k\neq\emptyset\}.$$

Since $(A_n)_x\cap U_k\neq\emptyset$ iff $(A_n)_x\cap U_k\notin\mathcal{M}$, the sets D_{nk} are Borel (see [10, 22.22]). Observe that for all $k\in\omega$ we have $B\subseteq\limsup_{n\in\omega}D_{nk}$ since for each $x\in B$ there are infinitely many A_n's such that $(\{x\}\times U_k)\cap A_n\neq\emptyset$. Since $\mathcal{N}$ has the $\mathcal{J}$-covering property, pick an

$S_k \in \mathcal{J}$ such that $\limsup_{n\in S_k} D_{nk} \in \mathcal{N}^*$. Since $\mathcal{J}$ is a P-ideal, we can pick an $S \in \mathcal{J}$ such that $S_k \subseteq^* S$ for all $k \in \omega$. Then

$$C := \bigcap_{k\in\omega} \limsup_{n\in S} D_{nk} \supseteq \bigcap_{k\in\omega} \limsup_{n\in S_k} D_{nk} \in \mathcal{N}^*.$$

Fix an $x \in C$. Then for all $k \in \omega$ and infinitely many indices $n \in S$, we have $(A_n)_x \cap U_k \neq \emptyset$. It follows that $\limsup_{n\in S}(A_n)_x$ is a residual G_δ set for all $x \in C$. Hence $(A_n)_{n\in S}$ is an $\mathcal{N} \otimes \mathcal{M}$-a.e. infinite-fold cover of $\mathbb{R}^2$. □

Question 2.8. Is the analogous result true for $\mathcal{M}\otimes\mathcal{N}$? Can one prove the same theorem for all (Borel) ideals on ω?

Applying $\mathrm{CP}(I \otimes K) \subseteq \mathrm{CP}(I \cap K)$, we obtain the following

Corollary 2.9. $\mathrm{CP}_{\text{P-ideals}}(\mathcal{N}) \subseteq \mathrm{CP}_{\text{P-ideals}}(\mathcal{M})$.

Finally in this section, we discuss the natural generalization of Elekes' result about random-indestructibility of $\mathcal{Z}$.

Theorem 2.10. *Let I be a σ-ideal on a Polish space X, and assume that $\mathbb{P}_I$ is proper. If I has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.*

Proof. Assume on the contrary that $\dot{Y}$ is a $\mathbb{P}_I$-name for an infinite subset of ω, i.e. $\Vdash_{\mathbb{P}_I} \dot{Y} \in [\omega]^\omega$ and $B \Vdash_{\mathbb{P}_I} \forall\ A \in \mathcal{J}\ |\dot{Y} \cap A| < \omega$ for some $B \in \mathbb{P}_I$. Then there are a $C \in \mathbb{P}_I$, $C \subseteq B$, and a Borel function $f : C \to [\omega]^\omega$ (coded in the ground model) such that $C \Vdash_{\mathbb{P}_I} f(\dot{r}_{\mathrm{gen}}) = \dot{Y}$ where $\dot{r}_{\mathrm{gen}}$ is a name for the generic real (see [18, Prop. 2.3.1]). For each $n \in \omega$ let

$$Y_n = f^{-1}[\{S \in [\omega]^\omega : n \in S\}] \in \mathrm{Borel}(X).$$

Then $(Y_n)_{n\in\omega}$ is an infinite-fold cover of C (by Borel sets) because $x \in Y_n$ iff $n \in f(x)$ and $|f(x)| = \omega$. Using the $\mathcal{J}$-covering property of $(\mathrm{Borel}(X) \restriction C, I \restriction C)$ we can choose an $A \in \mathcal{J}$ such that $(Y_n)_{n\in A}$ is an I-a.e. infinite-fold cover of C, that is $|f(x) \cap A| = \omega$ for I-a.e. $x \in C$, i.e. $\{x \in C : |f(x) \cap A| < \omega\} \in I$, so $C \Vdash_{\mathbb{P}_I} |f(\dot{r}_{\mathrm{gen}}) \cap A| = \omega$, and consequently, $C \Vdash_{\mathbb{P}_I} |\dot{Y} \cap A| = \omega$, a contradiction. □

By the following example, the covering property is, in general, truly stronger than the forcing indestructibility.

Example 2.11. We show that $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$ are Cohen-indestructible but $\mathcal{M}$ does not have the $\mathcal{ED}$- or $\mathrm{Fin} \otimes \mathrm{Fin}$-covering properties.

It is easy to see that $\mathrm{non}^*(\mathcal{ED}) = \mathrm{non}^*(\mathrm{Fin} \otimes \mathrm{Fin}) = \omega$, so because of Observation 2.2(3), $\mathcal{M}$ does not have the corresponding covering properties.

Although $\mathcal{ED} \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$, in particular Cohen-indestructibility of $\mathrm{Fin} \otimes \mathrm{Fin}$ follows from the Cohen-indestructibility of $\mathcal{ED}$, it is easy to see that a forcing notion $\mathbb{P}$ destroys $\mathrm{Fin} \otimes \mathrm{Fin}$ iff $\mathbb{P}$ adds dominating reals.

Cohen-indestructibility of $\mathcal{ED}$: Let $\mathbb{C} = (2^{<\omega}, \supseteq)$ be the Cohen forcing and assume that $\dot{X}$ is a $\mathbb{C}$-name for an infinite subset of $\omega\times\omega$ such that $\Vdash_{\mathbb{C}} \exists^\infty n\ (\dot{X})_n \neq \emptyset$ (because else $\dot{X}$ cannot destroy $\mathcal{ED}$). Enumerate $\mathbb{C} = \{p_n : n\in\omega\}$. By recursion on $n\in\omega$ we will define $A = \{(m_n, k_n) : n\in\omega\} \subseteq \omega\times\omega$ (in the ground model). Assume (m_ℓ, k_ℓ) is done for $\ell < n$. Then we can choose a $q_n \leq p_n$, an $m_n > m_{n-1}$, and a k_n such that $q_n \Vdash k_n \in (\dot{X})_{m_n}$. Clearly $A\in\mathcal{ED}$.

We claim that $\Vdash_{\mathbb{C}} |A\cap\dot{X}| = \omega$. Assume on the contrary that $p \Vdash \forall\ n \geq N\ (A)_n \cap (\dot{X})_n = \emptyset$ for some $p\in\mathbb{C}$ and $N\in\omega$. Then $p = p_n$ for some n and we can assume that $n \geq N$. Then $m_n \geq n$ and $q_n \Vdash k_n \in (A)_{m_n} \cap (\dot{X})_{m_n}$, a contradiction.

3. Around the category case

In this section we will characterize $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{M})$. We will need the following notion and its characterizations: An ideal $\mathcal{J}$ on ω is called *weak Q-ideal* if for each partition $(P_n)_{n\in\omega}$ of ω into finite sets, there is an $X\in\mathcal{J}^+$ such that $|X\cap P_n| \leq 1$ for each n. It is not hard to see that $\mathcal{J}$ is a weak Q-ideal iff $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$.

Note that $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{J}$ holds for a quite big class of ideals, namely for tall analytic P-ideals. This fact easily follows from Solecki's representation theorem (see [15, Thm. 3.1.]) which says that analytic P-ideals are exactly the ideals of the form

$$\mathrm{Exh}(\varphi) = \Big\{A\subseteq\omega : \lim_{n\to\infty}\varphi(A\setminus n) = 0\Big\}$$

for some lower semicontinuous submeasure φ on ω.

C. Laflamme in [12] proved another nice characterization of weak Q-ideals by using infinite games. In general, he investigated the games of the form $\mathfrak{G}(\mathcal{X}, \omega, \mathcal{Y})$ where $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{P}(\omega)$. The rules are the following: At the kth stage Player I chooses an $X_k\in\mathcal{X}$ and Player II responds with an $n_k\in X_k$. Player II wins if $\{n_k : k\in\omega\}\in\mathcal{Y}$. We will need the characterization of existence of winning strategies in a game of this form.

Before the next theorem, we recall a notion: an ideal $\mathcal{J}$ on ω is *ω-diagonalizable* if there is a countable family $\{Y_n : n\in\omega\}$ of infinite and co-infinite subsets of ω such that for each $A\in\mathcal{J}$ there is an n with $A\subseteq^* Y_n$. Clearly, $\mathcal{J}$ is ω-diagonalizable iff $\mathrm{non}^*(\mathcal{J}) = \omega$, in particular, in this case $\mathcal{M}$ does not have the $\mathcal{J}$-covering property.

Theorem 3.1. ([12, Thm 2.2]) *Let $\mathcal{J}$ be an ideal on ω. Then in the game $\mathfrak{G}(\mathrm{Fin}^*, \omega, \mathcal{J}^+)$*

(1) *Player I has a winning strategy iff $\mathcal{J}$ is not a weak Q-ideal;*
(2) *Player II has a winning strategy iff $\mathcal{J}$ is ω-diagonalizable.*

Theorem 3.2. *Assume $\mathcal{J}$ is a Borel ideal. Then $\mathcal{M}$ has the $\mathcal{J}$-covering property iff $\mathcal{J}$ is not a weak Q-ideal (i.e. $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{J}$). In other words,*

$$\mathrm{CP}_{\mathrm{Borel}}(\mathcal{M}) = \big\{\mathcal{J} : \mathcal{J} \textit{ is a Borel non weak Q-ideal}\big\}.$$

Proof. First we prove that $\mathcal{M}$ has the $\mathcal{ED}_{\text{fin}}$-covering property. It implies that $\text{CP}(\mathcal{M})$ contains all non weak Q-ideals. Let $(A_{(n,m)})_{(n,m)\in\Delta}$ be an infinite-fold cover of $\mathbb{R}$ by Borel sets. Without loss of generality, we can assume that all $A_{(n,m)}$'s are open and nonempty.

Enumerate $\{U_k : k \in \omega\}$ a base of $\mathbb{R}$. We will define by recursion a sequence $(n_k, m_k)_{k\in\omega}$ of elements of Δ. First, pick $(n_0, m_0) \in \Delta$ such that $A_{(n_0,m_0)} \cap U_0 \neq \emptyset$. Assume (n_i, m_i) are done for $i < k$. Then we can choose an $(n_k, m_k) \in \Delta$ such that $n_k \neq n_i$ for $i < k$ and $A_{(n_k,m_k)} \cap U_k \neq \emptyset$. We obtain the desired set $S = \{(n_k, m_k) : k \in \omega\} \in \mathcal{ED}_{\text{fin}}$. For every $k \in \omega$, the set $\bigcup_{i\geq k} A_{(n_i,m_i)}$ is dense and open. Consequently, $\limsup_{(n,m)\in S} A_{(n,m)}$ is a dense G_δ set, hence it is residual.

Conversely, first we show that if $\mathcal{J}$ is Borel, then the game $\mathfrak{G}(\text{Fin}^*, \omega, \mathcal{J}^+)$ is determined. We reformulate this game a bit. The Players are choosing elements from the countable set $\text{Fin}^* \times \omega$. The "set of rules" is the following tree on $\text{Fin}^* \times \omega$: $((X_0, n_0), (X_1, n_1), \dots, (X_{k-1}, n_{k-1})) \in T$ iff $n_\ell \in X_{\ell-1}$ for each odd $\ell \in [1, k)$. Player I wins the game $((X_k, n_k) : k \in \omega) \in [T] = \{\text{branches through } T\}$ if $\{n_k : k \text{ is odd}\} \in \mathcal{J}$. It is easy to see that the function from $[T]$ to $\mathcal{P}(\omega)$, given by $((X_k, n_k) : k \in \omega) \mapsto \{n_k : k$ is odd$\}$, is continuous, and hence if $\mathcal{I}$ is Borel then the game is determined.

Applying Theorem 3.1, if $\mathcal{ED}_{\text{fin}} \not\leq_{\text{KB}} \mathcal{J}$, then Player II has a winning strategy, so $\mathcal{J}$ is ω-diagonalizable, and hence $\mathcal{M}$ does not have the $\mathcal{J}$-covering property. □

Notice that this theorem shows that $\mathcal{M}$ plays an important role in the following sense:

Corollary 3.3. *If $\mathcal{A} \setminus I$ contains infinite antichains, then* $\text{CP}_{\text{Borel}}(\mathcal{A}, I) \subseteq \text{CP}_{\text{Borel}}(\mathcal{M})$. *In particular,* $\text{CP}_{\text{Borel}}(\mathcal{N}) \subseteq \text{CP}_{\text{Borel}}(\mathcal{M})$.

We also know that $\text{CP}_{\text{P-ideals}}(\mathcal{N}) \subseteq \text{CP}_{\text{P-ideals}}(\mathcal{M})$ (see Corollary 2.9), so it is natural to ask the following:

Question 3.4. Does $\text{CP}(\mathcal{N}) \subseteq \text{CP}(\mathcal{M})$ hold?

4. KB-smallest elements in $\text{CP}(I)$ and in $\text{CP}_{\text{Borel}}(I)$

First we show that $\mathcal{Z}$ is not KB-minimal (in particular, it is not the KB-smallest ideal) in $\text{CP}_{\text{Borel}}(\mathcal{N})$. For $A \subseteq \omega$ and $n \in \omega$ let

$$S_n(A) = \max\left\{|A \cap [k, k+n)| : k \in \omega\right\},$$

and let

$$\mathcal{Z}_u = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{S_n(A)}{n} = 0\right\}.$$

Then $\mathcal{Z}_u$ is an ideal called *the uniform density zero ideal.* It is easy to see that $\mathcal{Z}_u$ is an $F_{\sigma\delta}$ non P-ideal (for more details, see [4]), and that $\mathcal{ED}_{\text{fin}} \leq_{\text{KB}} \mathcal{Z}_u \leq_{\text{KB}} \mathcal{Z}$.

Proposition 4.1. $\mathcal{Z} \not\leq_{\text{KB}} \mathcal{Z}_u$.

Proof. Let $f\colon \omega \to \omega$ be finite-to-one. We will inductively define a sequence $(b_n)_{n\geq 1}$ of natural numbers such that

$$A = \bigcup \big\{ f(b_n + k) : n \in \omega \text{ and } k \in [0, n) \big\} \in \mathcal{Z}.$$

Then $f^{-1}[A]$ contains blocks of natural numbers of arbitrary length. Therefore $f^{-1}[A] \notin \mathcal{Z}_u$ so f cannot witness that $\mathcal{Z} \leq_{\mathrm{KB}} \mathcal{Z}_u$.

Let b_1 be such that $f(b_1) > 2 = 2^1$. Let b_2 be such that $f(b_2), f(b_2 + 1) > f(b_1) + 2^2$. Assume we have already defined $b_1, \ldots, b_n$. Let b_{n+1} be such that

$$\min \big\{ f(b_{n+1} + k) : k \in [0, n+1) \big\} > \max \big\{ f(b_n + k) : k \in [0, n) \big\} + 2^{n+1}.$$

Clearly, $A = \bigcup \big\{ f(b_n + k) : n \in \omega \text{ and } k \in [0, n) \big\}$ has asymptotic density zero, that is, $A \in \mathcal{Z}$. □

The proof of the following theorem is Elekes' original proof, simply it works for $\mathcal{Z}_u$ as well.

Theorem 4.2. *$\mathcal{N}$ has the $\mathcal{Z}_u$-covering property.*

Proof. Let $(A_n)_{n\in\omega}$ be an infinite-fold Borel cover of 2^ω. Let $N_0 = 0$. By continuity of measure we can find $A_0, A_1, \ldots, A_{N_1 - 1}$ such that $\lambda(2^\omega \setminus (A_0 \cup A_1 \cup \cdots \cup A_{N_1 - 1})) < 1/2$ (where λ is the Lebesgue measure on 2^ω). Note that if we remove finitely many elements from an infinite-fold cover, then it is still an infinite-fold cover. Therefore we can inductively define numbers $0 = N_0 < N_1 < N_2 < \ldots$ such that $\lambda(2^\omega \setminus (A_{N_k} \cup A_{N_k + 1} \cup \cdots \cup A_{N_{k+1} - 1})) < 2^{-k-1}$ and k divides $N_k - N_{k-1}$ for any $k > 0$.

Since $N_k - N_{k-1} = d_k k$ for some $d_k \in \omega$, we can define

$$W_i^k = \{N_{k-1} + i, N_{k-1} + k + i, \ldots, N_{k-1} + (d_k - 1)k + i\}$$

for $i \in [0, k)$, that is, the partition of $[N_{k-1}, N_k)$ into k many arithmetical progressions of length d_k and difference k.

Consider $X = \prod_{k\geq 1} [0, k)$ the product space of the uniformly distributed probability spaces on $k \geq 1$ with the discrete topology, and denote by μ the product probability measure on this Polish space.

For each $\bar{x} = (x_k)_{k\geq 1} \in X$ let $Z(\bar{x}) = \bigcup_{k\geq 1} W_{x_k}^k$. Clearly $Z(\bar{x}) \in \mathcal{Z}_u$ for any $\bar{x} \in X$. To end the proof we claim that $(A_n)_{n\in Z(\bar{x})}$ is a λ-a.e. infinite-fold cover of 2^ω for μ-a.e. $\bar{x} \in X$.

For a fixed $f \in 2^\omega$ let

$$E_k^f = \Big\{ \bar{x} \in X : f \in \bigcup \Big\{ A_n : n \in W_{x_k}^k \Big\} \Big\} \in \mathrm{Borel}(X).$$

Then $\mu(E_k^f) \geq 1/k$ provided $f \in A_{N_{k-1}} \cup \cdots \cup A_{N_k - 1}$. But for λ-a.e. $f \in 2^\omega$ there is $k_0(f)$ such that $f \in A_{N_{k-1}} \cup \cdots \cup A_{N_k - 1}$ for all $k \geq k_0(f)$. Why? Simply because of the construction of $(N_k)_{k\in\omega}$ we have

$$\lambda\left(\bigcup_{k_0\geq 1}\bigcap_{k\geq k_0}\left(A_{N_{k-1}}\cup\cdots\cup A_{N_k-1}\right)\right)=1.$$

Therefore $\sum_{k=1}^{\infty}\mu(E_k^f)=\infty$ for λ-a.e. $f\in 2^\omega$. Since $(E_k^f)_{k\in\omega}$ are independent, applying the second Borel-Cantelli Lemma we obtain that

$$\lambda\left(\left\{f\in 2^\omega:\mu\left(\limsup_{k\to\omega}E_k^f\right)=1\right\}\right)=1.$$

And hence by Fubini's theorem we are done. □

Unfortunately, despite of the category case, the following question is still open.

Question 4.3. Does there exist a KB-smallest (or at least KB-minimal) element in $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{N})$?

Recall that a sequence $\mathcal{T}=(T_\alpha)_{\alpha<\kappa}$ in $[\omega]^\omega$ is a *tower* if κ is regular, $\mathcal{T}$ is $\subseteq^*$-descending (i.e. $T_\beta\subseteq^* T_\alpha$ if $\alpha<\beta<\gamma$), and it has no pseudointersection. The *tower number* $\mathfrak{t}$ is the smallest cardinality of a tower, and $\mathfrak{c}$ stands for the continuum.

Theorem 4.4. *Assume $\mathfrak{t}=\mathfrak{c}$ and $|\mathcal{A}|\leq\mathfrak{c}$. Then there is no KB-smallest element in $\mathrm{CP}(\mathcal{A},I)$.*

Proof. If $\mathrm{CP}(\mathcal{A},I)=\emptyset$ then we are done. If $(\mathcal{A},I)$ has the $\mathcal{J}_0$-covering property then we will construct a $\mathcal{J}$ such that $\mathcal{J}_0\not\leq_{\mathrm{KB}}\mathcal{J}$ but $(\mathcal{A},I)$ has the $\mathcal{J}$-covering property.

Enumerate $(f_\alpha)_{\alpha<\mathfrak{c}}$ all finite-to-one functions from ω to ω, and enumerate $((A_n^\alpha)_{n\in\omega}:\alpha<\mathfrak{c})$ the infinite-fold covers of $X=\bigcup\mathcal{A}$ by sets from $\mathcal{A}$. By recursion on $\mathfrak{c}$ we will define a $\subseteq^*$-increasing sequence $(S_\xi)_{\alpha<\mathfrak{c}}$ of infinite and co-infinite subsets of ω and the ideal $\mathcal{J}$ generated by this sequence will be as required.

Assume $(S_\xi)_{\xi<\alpha}$ is done for some $\alpha<\mathfrak{c}$. Because of our assumption on $\mathfrak{t}$, we can choose an infinite and co-infinite S'_α such that $S_\xi\subseteq^* S'_\alpha$ for each $\xi<\alpha$. The set $f_\alpha[\omega\setminus S'_\alpha]$ contains an infinite element E of $\mathcal{J}_0$. We want to guarantee that $f_\alpha^{-1}[E]\notin\mathcal{J}$ because then f_α can not witness $\mathcal{J}_0\leq_{\mathrm{KB}}\mathcal{J}$. Let $H=f_\alpha^{-1}[E]\setminus S'_\alpha\in[\omega]^\omega$.

Consider the αth cover $(A_n^\alpha)_{n\in\omega}$. If $(A_n)_{n\in\omega\setminus H}$ is an I-a.e. infinite-fold cover of X, then let $S_\alpha=S'_\alpha\cup(\omega\setminus H)$.

If not, then

$$C=\left\{x\in X:\{n\in\omega\setminus H:x\in A_n^\alpha\}\text{ is finite}\right\}\notin I.$$

Applying our assumption for $(\mathcal{A}[I]\restriction C, I\restriction C)$ and $(A_n^\alpha\cap C)_{n\in H}$ (with a copy of $\mathcal{J}_0$ on H) we can choose an infinite $H'\subseteq H$ such that $H\setminus H'$ is also infinite and $(A_n^\alpha\cap C)_{n\in H'}$ is an $I\restriction C$-a.e. infinite-fold cover of C.

Finally, let $S_\alpha=S'_\alpha\cup(\omega\setminus H)\cup H'$. It is easy to see from the construction that $\mathcal{J}$ is as required. □

Corollary 4.5. *If* $\mathfrak{t} = \mathfrak{c}$ *then there are ideals* $\mathcal{J}_0 \in \mathrm{CP}(\mathcal{N})$ *and* $\mathcal{J}_1 \in \mathrm{CP}(\mathcal{M})$ *such that* $\mathcal{Z}_u \not\leq_{\mathrm{KB}} \mathcal{J}_0$ *and* $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}_1$.

Without $\mathfrak{t} = \mathfrak{c}$ we can use a simple forcing construction.

Theorem 4.6. *After adding* ω_1 *Cohen-reals, there is an ideal* $\mathcal{J}$ *such that* $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$ *(in particular,* $\mathcal{Z}_u \not\leq_{\mathrm{KB}} \mathcal{J}$*) but* $\mathcal{N}$ *and* $\mathcal{M}$ *have the* $\mathcal{J}$*-covering property (i.e.* $\mathcal{J} \in \mathrm{CP}(\mathcal{N}) \cap \mathrm{CP}(\mathcal{M}) = \mathrm{CP}(\mathcal{N} \cap \mathcal{M})$*).*

Proof. Let $(c_\alpha)_{\alpha<\omega_1}$ be the sequence of generic Cohen-reals in 2^ω, $C_\alpha = c_\alpha^{-1}[\{1\}] \subseteq \omega$, and let $\mathcal{J}$ be the ideal generated by these sets. $\mathcal{J}$ is a proper ideal because it is well-known that $\{C_\alpha : \alpha < \omega_1\}$ is an independent system of subsets of ω.

To show that $\mathcal{N}$ has the $\mathcal{J}$-covering property in the extension, it is enough to see that if $(A_n)_{n\in\omega}$ is an infinite-fold cover of 2^ω by Borel sets in a ground model V, then $(A_n)_{n\in C}$ is an $\mathcal{N}$-a.e. infinite-fold cover of 2^ω in $V[C]$ where $C \subseteq \omega$ is a Cohen-real over V.

Clearly, it is enough to prove that $V[C] \models \lambda\big(\bigcup_{n\in C\setminus k} A_n\big) = 1$ for each $k \in \omega$ (because then $V[C] \models \lambda\big(\limsup_{n\in C} A_n\big) = 1$). Let $p \in \mathbb{C} = (2^{<\omega}, \supseteq)$, $k \in \omega$, and $\varepsilon < 1$. We can assume that $|p| \geq k$. Then there is an $m \geq |p|$ such that $\lambda\big(\bigcup_{n\in m\setminus|p|} A_n\big) > \varepsilon$ so if $q : m \to 2$, $q \restriction |p| = p$, and $q \restriction (m \setminus |p|) \equiv 1$, then $q \leq p$ and $q \Vdash \lambda\big(\bigcup_{n\in \dot{C}\setminus k} A_n\big) > \varepsilon$.

To show that $\mathcal{M}$ has the $\mathcal{J}$-covering property in the extension, it is enough to prove that if $(A_n)_{n\in\omega}$ is an infinite-fold cover of 2^ω by open sets in V, then $(A_n)_{n\in C}$ is an $\mathcal{M}$-a.e. infinite-fold cover of 2^ω in $V[C]$. By a simple density argument $V[C] \models$"$\bigcup_{n\in C\setminus k} A_n$ is dense open" for each $k \in \omega$, so $V[C] \models$"$\limsup_{n\in C} A_n$ is residual."

To show that $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$, it is enough to see that if $f \in \Delta^\omega \cap V[(c_\xi)_{\xi<\alpha}]$ is a finite-to-one function for some $\alpha < \omega_1$, then there is an $A \in \mathcal{ED}_{\mathrm{fin}} \cap V[(c_\xi)_{\xi\leq\alpha}]$ such that $f^{-1}[A]$ cannot be covered by finitely many of C_ξ's ($\xi < \omega_1$). Simply let A be a Cohen function in $\prod_{n\in\omega}(n+1)$, i.e. the graph of a Cohen-function in Δ, for example $A = \big\{(n,k) \in \Delta : c'_\alpha(n) \equiv k \mod (n+1)\big\}$ is suitable. Using the presentation of this iteration by finite partial functions from $\omega_1 \times \omega$ to 2, we are done by a simple density argument. □

Question 4.7. Is it provable in ZFC that there are no KB-smallest elements of $\mathrm{CP}(\mathcal{N})$ and $\mathrm{CP}(\mathcal{M})$? Or at least, is it provable that $\mathcal{Z}_u$ and $\mathcal{ED}_{\mathrm{fin}}$ are not the KB-smallest members of these families?

The next result shows that, to give positive answer to the second question, it would be enough to find a non meager weak Q-ideal. However, at this moment we are unable to construct such an ideal. Discussing the existence of a non meager weak Q-ideal could be interesting on its on right because the so-called *Q-ideals* satisfy these properties (these are the most natural consequences of property Q) but it is consistent that there are no Q-ideals (see [14]).

Proposition 4.8. $\mathrm{CP}(\mathcal{M}) \cap \mathrm{CP}(\mathcal{N})$ *contains all non meager ideals.*

Proof. Let $\mathcal{J}$ be non meager, that is (by Theorem 1.2), for each partition $(P_n)_{n\in\omega}$ of ω into finite sets, there is an $S \in \mathcal{J}$ which contains infinite many elements of the partition.

We claim that $\mathcal{J} \in \mathrm{CP}(\mathcal{M})$. Let $(A_n)_{n\in\omega}$ be an $\mathcal{M}$-a.e. infinite-fold Borel cover of $\mathbb{R}$. We can assume that A_n's are open. Fix a base $\{U_k : k \in \omega\}$ of $\mathbb{R}$. By recursion on n we can define a partition $(P_n)_{n\in\omega}$ of ω into finite sets such that for each $k \leq n$

$$U_k \cap \bigcup \big\{A_i : i \in P_n\big\} \neq \emptyset.$$

We can do it because for each N the sequence $(A_n)_{n\geq N}$ is still an $\mathcal{M}$-a.e. infinite-fold cover of $\mathbb{R}$. By our assumption on $\mathcal{J}$, there is an $S \in \mathcal{J}$ such that $P_n \subseteq S$ for infinitely many n's. It yields that $\bigcup \big\{A_n : n \in S \setminus N\big\}$ is a dense open set for each N, hence $\limsup_{n\in S} A_n$ is residual.

We claim that $\mathcal{J} \in \mathrm{CP}(\mathcal{N})$. Let $(A_n)_{n\in\omega}$ be an $\mathcal{N}$-a.e. infinite-fold Borel cover of $\mathbb{R}$. By recursion on n we can define a partition $(P_n)_{n\in\omega}$ of ω such that for each n

$$\lambda\Big([-n, n] \cap \bigcup \big\{A_i : i \in P_n\big\}\Big) > 2n - 2^{-n}.$$

We can do it because for each N the sequence $(A_n)_{n\geq N}$ is still an $\mathcal{N}$-a.e. infinite-fold cover. Applying our assumption on $\mathcal{J}$, there is an $S \in \mathcal{J}$ such that $P_n \subseteq S$ for infinitely many n's. It yields that $\bigcup \big\{A_n : n \in S\setminus N\big\}$ is co-null for each N and hence $\limsup_{n\in S} A_n$ is also co-null. □

5. When the $\mathcal{J}$-covering property "strongly" fails

In this section we give a positive answer to Question 2.4 in a special case. First of all, we present a counterexample:

Example 5.1. Consider $X = (-1, 1)$ and let an ideal I on X consist of sets $A \subseteq X$ such that $A \cap (-1, 0]$ is meager and $A \cap (0, 1)$ is of Lebesgue measure zero. Using Example 2.3 and Theorem 3.2 observe that I yields the negative answer to Question 2.4 with $\mathcal{J} = \mathcal{I}_{1/n}$. However, this question remains interesting if we restrict it to translation invariant ideals on $\mathbb{R}$. In this case, we describe a class of ideas which yields a positive answer to Question 2.4 provided $\mathcal{J}$ is a P-ideal.

Let $\mathbb{Q}$ stand for the set of rational numbers. For $A, B \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ we write $A + x = \{a + x : a \in A\}$ and $A + B = \{a + b : A \in A \text{ and } b \in B\}$.

Theorem 5.2. *Assume that I is a translation invariant ccc σ-ideal on $\mathbb{R}$ fulfilling the condition*

$$\mathbb{Q} + A \in I^* \text{ for each } A \in \mathrm{Borel}(\mathbb{R}) \setminus I. \tag{1}$$

Fix a P-ideal $\mathcal{J}$ on ω. If I does not have the $\mathcal{J}$-covering property, then there exists an infinite-fold Borel cover $(A'_n)_{n\in\omega}$ of $\mathbb{R}$ with $\limsup_{n\in S} A'_n \in I$ for all $S \in \mathcal{J}$.

Proof. Fix an infinite-fold Borel cover $(A_n)_{n\in\omega}$ of $\mathbb{R}$ such that $\limsup_{n\in S} A_n \notin I^*$ for all $S \in \mathcal{J}$. We will show that there is a Borel set $B \subseteq \mathbb{R}$ with $B \notin I$ and $(\limsup_{n\in S} A_n)\cap B \in I$ for all $S \in \mathcal{J}$. Suppose it is not the case. So, in particular (when $B = \mathbb{R}$), we find $S_0 \in \mathcal{J}$ with $X_0 := \limsup_{n\in S_0} A_n \notin I$. Then by transfinite recursion we define sequences $(S_\alpha)_{\alpha<\gamma}$ and $(X_\alpha)_{\alpha<\gamma}$ with $S_\alpha \in \mathcal{J}$ and $X_\alpha := (\limsup_{n\in S_\alpha} A_n) \setminus \bigcup_{\beta<\alpha} X_\beta \notin I$ (when $B = \mathbb{R} \setminus \bigcup_{\beta<\alpha} X_\beta \notin I$). Since I is ccc, this construction stops at a stage $\gamma < \omega_1$ with $\bigcup_{\alpha<\gamma} \limsup_{n\in S_\alpha} A_n = \bigcup_{\alpha<\gamma} X_\alpha \in I^*$. Since I is a P-ideal, there is $S \in \mathcal{J}$ which almost contains each S_α for $\alpha < \gamma$. Then $\limsup_{n\in S} A_n \in I^*$ which contradics our supposition.

So, fix a Borel set $B \notin I$ such that $(\limsup_{n\in S} A_n) \cap B \in I$ for all $S \in \mathcal{J}$. Let $\mathbb{Q} = \{q_k : k \in \omega\}$. Define $B_0 := B$ and $B_k := (q_k + B) \setminus \bigcup_{i<k} B_i$ for $k \in \omega$. Then put $A'_n := \bigcup_{k\in\omega}((q_k + A_n) \cap B_k)$ for $n \in \omega$. Since $(A_n)_{n\in\omega}$ is an infinite-fold cover of $\mathbb{R}$, we have $\limsup_{n\in\omega}((q_k + A_n) \cap B_k) = B_k$ for all $k \in \omega$. Note that $(A'_n)_{n\in\omega}$ is an I-a.e. infinite-fold cover of $\mathbb{R}$ since

$$\limsup_{n\in\omega} A'_n \supseteq \bigcup_{k\in\omega} \limsup_{n\in\omega}((q_k + A_n) \cap B_k) = \mathbb{Q} + B$$

and $\mathbb{Q} + B \in I^*$ by (1). Finally, let $S \in \mathcal{J}$. Since I is translation invariant and $(\limsup_{n\in S} A_n) \cap B \in I$, we have $\limsup_{n\in S}((q_k + A_n) \cap B_k) \in I$ for all $k \in \omega$. Since I is a σ-ideal and B_k's are pairwise disjoint, it follows that

$$\limsup_{n\in S} A'_n = \bigcup_{k\in\omega} \limsup_{n\in S}((q_k + A_n) \cap B_k) \in I.$$

Of course, we can modify (A'_n) to be an infinite-fold cover of $\mathbb{R}$. □

Theorem 5.2 can be generalized to any Polish group G with $\mathbb{Q}$ replaced by a countable dense subset of G. Condition (1) is related to the Steinhaus property, for details see [5]. Note that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy (1) with $\mathbb{Q}$ replaced by any dense subset of $\mathbb{R}$ (resp. $\mathbb{R}^2$).

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