ADDITIVITY AND LINEABILITY IN VECTOR SPACES

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Abstract. Gámez-Merino, Munoz-Fernández and Seoane-Sepúlveda proved that if additivity $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable where $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. They asked if $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$ can be weakened. We answer this question in negative. Moreover, we introduce and study the notions of homogeneous lineability number and lineability number of subsets of linear spaces.

1. Introduction

A subset M of a linear space V is κ-lineable if $M \cup \{0\}$ contains a linear subspace of dimension κ (see [1], [2], [3], [22], [23], [26]). If additionally V is a linear algebra, then in a similar way one can define albegrability of subsets of V (see [2], [4], [5], [6], [7], [9], [10], [11], [12], [24], [25], [26]). If V is a linear topological space, then $M \subseteq V$ is called spaceable (dense-lineable) if M contains a closed infinitely dimensional subspace (dense subspace) (see [13], [14], [27]). The lineability problem of subsets of linear spaces of functions or sequences have been studied by many authors. The most common way of proving κ-lineability is to construct a set of cardinality κ of linearly independent elements of V and to show that any linear combination of them is in M.

We will concentrate on a non-constructive method in lineability. Following the paper [21], we will consider a connection between lineability and additivity. This method does not give a specific large linear space, but ensures that such a space exists.

Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. The additivity of $\mathcal{F}$ is defined as the following cardinal number

$$\mathcal{A}(\mathcal{F}) = \min(\{|F| : F \subseteq \mathbb{R}^{\mathbb{R}},\ \varphi + F \nsubseteq \mathcal{F} \text{ for every } \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^{+}\}).$$

The notion of additivity was introduced by Natkaniec in [29] and then studied by several authors [17], [16], [18], [19] and [30].

A family $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ is called star-like if $a\mathcal{F} \subseteq \mathcal{F}$ for all $a \in \mathbb{R} \setminus \{0\}$. Gámez-Merino, Munoz-Fernández and Seoane-Sepúlveda proved the in [21] following result which connects the lineability and the additivity of star-like families.

Theorem 1.1. *Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be star-like. If $\mathfrak{c} < \mathcal{A}(\mathcal{F}) \leq 2^{\mathfrak{c}}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable.*

The authors noted that there is a star-like family $\mathcal{F}$ such that $\mathcal{A}(\mathcal{F}) = 2$ and $\mathcal{F}$ is not 2-lineable. They asked if the above result is true for $2 < \mathcal{A}(\mathcal{F}) \leq \mathfrak{c}$. We will answer this question in negative. This will show that Theorem 1.1 is sharp.

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Let us observe that the notion of additivity can be stated for abelian groups as follows. If $(G,+)$ is an abelian group and $\mathcal{F}\subseteq G$, then the additivity of $\mathcal{F}$ is the cardinal number

$$\mathcal{A}(\mathcal{F})=\min(\{|F|:F\subseteq G \text{ and } \forall\varphi\in G(\varphi+F\nsubseteq\mathcal{F})\}\cup\{|G|^+\}).$$

On the other hand, the lineability is a natural notion in vector spaces. We say that a set $\mathcal{F}\subseteq V$, where V is a linear space, is λ-lineable if there exists a subspace W of V such that $W\subseteq\mathcal{F}\cup\{0\}$ and $\dim W=\lambda$. Now, the lineability $\mathcal{L}(\mathcal{F})$ is the cardinal number

$$\mathcal{L}(\mathcal{F})=\min\{\lambda:\mathcal{F}\text{ is not }\lambda\text{-lineable}\}.$$

Clearly $\mathcal{L}(\mathcal{F})$ is a cardinal number less or equal to $(\dim V)^+$ and it can take any value between 1 and $(\dim V)^+$ – see Proposition 2.4.

2. RESULTS

The following Lemma 2.1 and Theorem 2.2 are generalizations of [21, Lemma 2.2 and Theorem 2.4] in the settings of abelian groups and vector spaces over infinite fields, respectively. Short proofs of this facts are essentially the same as those in [21], but our presentation is more general and from the proof of Theorem 2.2 we extract a new notion of lineability, namely *homogeneus lineability.* Moreover the authors of [21] claimed that Theorem 1.1 held true also in the case $\mathfrak{c}=\mathcal{A}(\mathcal{F})$. However we will show (see Theorem 2.5 and Theorem 2.6) that this is not true. Let us remark that all examples of families $\mathcal{F}\subseteq\mathbb{R}^\mathbb{R}$ discussed in [21] have additivity $\mathcal{A}(\mathcal{F})$ greater than $\mathfrak{c}$, so the described mistake has almost no impact on the value of this nice paper.

Lemma 2.1. *Let $(G,+)$ be an abelian group. Assume that F is a subgroup of G and $\mathcal{F}\subseteq G$ is such that*

$$2|F|<\mathcal{A}(\mathcal{F}). \tag{1}$$

Then there is $g\in\mathcal{F}\setminus F$ with $g+F\subseteq\mathcal{F}$. That means actually that some coset of F different from F is contained in $\mathcal{F}$.

Proof. Let $h\in G\setminus F$ and put $F_h=(h+F)\cup F$. Then $|F_h|=2|F|$. By (1) there is $g\in G$ such that $g+F_h\subseteq\mathcal{F}$. Thus $g+F\subseteq\mathcal{F}$, $(g+h)+F\subseteq\mathcal{F}$ and $0\in F$, and consequently $g\in\mathcal{F}$ and $g+h\in\mathcal{F}$. It is enough to show that $g\notin F$ or $g+h\notin F$. Suppose to the contrary that $g,g+h\in F$. Then $h=(g+h)-g\in F$ which is a contradiction. □

Let us remark that if $\mathcal{A}(\mathcal{F})$ is an infinite cardinal, then the condition $|F|<\mathcal{A}(\mathcal{F})$ implies condition (1).

Assume that V is a vector space, $A\subseteq V$ and $f_1,\dots,f_n\in V$. Fix the following notation $[A]=\operatorname{span}(A)$ and $[f_1,\dots,f_n]=\operatorname{span}(\{f_1,\dots,f_n\})$.

Theorem 2.2. *Let V be a vector space over a field $\mathbb{K}$ with $\omega\le|\mathbb{K}|=\mu<\dim V$. Assume that $\mathcal{F}\subseteq V$ is star-like, that is $a\mathcal{F}\subseteq\mathcal{F}$ for every $a\in\mathbb{K}\setminus\{0\}$, and*

$$\mu<\mathcal{A}(\mathcal{F})\le\dim V. \tag{2}$$

Then $\mathcal{F}\cup\{0\}$ is $\mathcal{A}(\mathcal{F})$-lineable, in symbols $\mathcal{L}(\mathcal{F})>\mathcal{A}(\mathcal{F})$. Moreover, any linear subspace Y of V contained in $\mathcal{F}$ of dimension less than $\mathcal{A}(\mathcal{F})$ can be extended to $\mathcal{A}(\mathcal{F})$-dimensional subspace also contained in $\mathcal{F}$.

Proof. Let Y be a linear subspace of V with $Y\subseteq\mathcal{F}\cup\{0\}$. Let X be a maximal element, with respect to inclusion, of the family

$$\{X : Y\subseteq X\subseteq \mathcal{F}\cup\{0\},\ X \text{ is a linear subspace of } V\}.$$

Suppose to the contrary that $|X|<\mathcal{A}(\mathcal{F})$. Then by Lemma 2.1 there is $g\in\mathcal{F}\setminus X$ with $g+X\subseteq\mathcal{F}$. Let $Z=[g]+X$ and take any $z\in Z\setminus X$. Then there is $x\in X$ and nonzero $a\in\mathbb{K}$ with $z=ag+x=a(g+x/a)\in\mathcal{F}$. Since $X\subseteq\mathcal{F}$ we obtain that $Z\subseteq\mathcal{F}$. This contradicts the maximality of X. Therefore $\mathcal{A}(\mathcal{F})\leq|X|<\mathcal{L}(\mathcal{F})$. □

The assertion of Theorem 2.2 leads us to the following definition of a new cardinal function. We define the **homogeneous lineability number** of $\mathcal{F}$ as the following cardinal number

$$\mathcal{HL}(\mathcal{F})=\min(\{\lambda : \text{there is linear space } Y\subseteq\mathcal{F}\cup\{0\} \text{ with } \dim Y<\lambda$$

$$\text{which cannot be extended to a linear space } X\subseteq\mathcal{F}\cup\{0\} \text{ with } \dim X=\lambda\}\cup\{|V|^+\}).$$

Now the assertion of Theorem 2.2 can be stated briefly.

Corollary 2.3. *Let V be a vector space over a field $\mathbb{K}$ with $\omega\leq|\mathbb{K}|=\mu<\dim V$. Assume that $\mathcal{F}$ is star-like and $\mu<\mathcal{A}(\mathcal{F})\leq\dim V$. Then $\mathcal{HL}(\mathcal{F})>\mathcal{A}(\mathcal{F})$.*

Now we will show basic connections between $\mathcal{HL}(\mathcal{F})$ and $\mathcal{L}(\mathcal{F})$.

Proposition 2.4. *Let V be a vector space. Then*

(i) *$\mathcal{HL}(\mathcal{F})\leq\mathcal{L}(\mathcal{F})$ for every $\mathcal{F}\subseteq V$;*
(ii) *For every successor cardinal $\kappa\leq(\dim V)^+$ there is $\mathcal{F}\subseteq V$ with $\mathcal{HL}(\mathcal{F})=\mathcal{L}(\mathcal{F})=\kappa$;*
(iii) *For every $\lambda,\kappa\leq(\dim V)^+$ such that $\lambda^+<\kappa$ there is $\mathcal{F}\subseteq V$ with $\mathcal{HL}(\mathcal{F})=\lambda^+$ and $\mathcal{L}(\mathcal{F})=\kappa$;*
(iv) *$\mathcal{HL}(\mathcal{F})$ is a successor cardinal.*

Proof. Note that the cardinal number $\mathcal{L}(\mathcal{F})$ can be defined in the similar terms as it was done for $\mathcal{HL}(\mathcal{F})$, namely $\mathcal{L}(\mathcal{F})$ is the smallest cardinal λ such that the trivial linear space $Y=\{0\}$ cannot be extended to a linear space $X\subseteq\mathcal{F}\cup\{0\}$ with $\dim V=\lambda$. Therefore $\mathcal{HL}(\mathcal{F})\leq\mathcal{L}(\mathcal{F})$.

To see (ii) take any successor cardinal $\kappa\leq|V|^+$. There is $\lambda\leq|V|$ with $\lambda^+=\kappa$. Let $\mathcal{F}$ be a linear subspace of V of dimension λ. Then $\mathcal{F}$ is λ-lineable but not λ^+-lineable. Thus $\mathcal{L}(\mathcal{F})=\kappa$. Note that any linear subspace of $\mathcal{F}$ can be extended to a λ-dimensional space $\mathcal{F}$, but cannot be extended to a κ-dimensional space. Thus $\mathcal{HL}(\mathcal{F})=\kappa$.

Let $\mathrm{Card}[\lambda,\kappa)=\{\nu : \lambda\leq\nu<\kappa \text{ and } \nu \text{ is a cardinal number}\}$. For every $\nu\in\mathrm{Card}[\lambda,\kappa)$ we find B_ν of cardinality ν such that $B=\bigcup\{B_\nu : \nu\in\mathrm{Card}[\lambda,\kappa)\}$ is a linearly independent subset of V. Let $W_\nu=[B_\nu]$. Let $\mathcal{F}=\bigcup_{\nu\in\mathrm{Card}[\lambda,\kappa)}W_\nu$. Clearly $\mathcal{F}$ is ν-lineable for any $\nu<\kappa$. Take any linear space $W\subseteq\mathcal{F}$. Since $[\mathcal{F}]=[B]$, then any element x of W is of the form $\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha_{ij}w_{ij}$ where $\alpha_{ij}\in\mathbb{K}$

and $w_{i1}, \dots, w_{in_i}$ are distinct elements of B_{ν_i}, $\lambda \le \nu_1 < \cdots < \nu_k < \kappa$. If $x \in W_\nu$ for some ν, then $x = \sum_{p=1}^m \alpha_p w_p$, $\alpha_p \in \mathbb{K}$, $w_1, \dots, w_m \in B_\nu$ are distinct. Then

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \alpha_{ij} w_{ij} = \sum_{p=1}^{m} \alpha_p w_p.$$

Now, if $\nu \notin \{n_0, \dots, \nu_k\}$, then $\alpha_{ij} = 0 = \alpha_p$ for every i, j, p. Thus $x = 0$. If $\nu = \nu_l$, then $\alpha_{ij} = 0$ for every $i \ne l$. Thus $x \in W_\nu$. This shows that W does not contain any nontrivial linear combination of $\alpha_1 x_1 + \cdots + \alpha_n x_n$ with $x_i \in W_{\nu_i}$ for distinct $\nu_1, \dots, \nu_n$. Therefore $W \subseteq W_\nu$ for some ν. Consequently $\dim W < \kappa$, which means that $\mathcal{F}$ is not κ-lineable. Hence $\mathcal{L}(\mathcal{F}) = \kappa$.

Take any linear space $Y \subseteq \mathcal{F}$ of dimension less than λ. As before we obtain that Y is a subset of some W_ν. Therefore Y can be extended to a linear subspace of $\mathcal{F}$ of dimension λ. On the other hand $Y = W_\lambda$ cannot be extended to a linear space contained in $\mathcal{F}$ of dimension λ^+. Hence $\mathcal{HL}(\mathcal{F}) = \lambda^+$.

To prove (iv) assume that $\mathcal{HL}(\mathcal{F}) = \kappa$. Then for any $\lambda < \kappa$ and any linear space $Y \subseteq \mathcal{F} \cup \{0\}$ of dimension less than λ there is a linear space $X \supset Y$ contained in $\mathcal{F} \cup \{0\}$ of dimension λ. Suppose to the contrary that κ is a limit cardinal. There are cardinals $\tau_\nu < \kappa$, $\nu < \mathrm{cf}(\kappa) \le \kappa$ with $\bigcup_{\nu < \mathrm{cf}(\kappa)} \tau_\nu = \kappa$. Since $\mathcal{HL}(\mathcal{F}) = \kappa$, then for any linear space $Y \subseteq \mathcal{F}$ of dimension less than κ we can inductively define an increasing chain $\{Y_\nu : \dim Y < \nu < \mathrm{cf}(\kappa)\}$ of linear spaces with $\dim Y_\nu = \tau_\nu$ and $Y_\nu \subseteq \mathcal{F} \cup \{0\}$. Then $Y' = \bigcup Y_\nu$ is a linear space of dimension κ such that $Y \subseteq Y' \subseteq \mathcal{F}$. Hence $\mathcal{HL}(\mathcal{F}) \ge \kappa^+$ which is a contradiction. □

Theorem 2.5. *Assume that $3 \le \kappa \le \mu$, $\mathbb{K}$ is a field of cardinality μ and V is a linear space over $\mathbb{K}$ with $\dim V = 2^\mu$. Then there is a star-like family $\mathcal{F} \subseteq V$ with $\mathcal{A}(\mathcal{F}) = \kappa$ which is not 2-lineable.*

Proof. Let $\{G_\xi : \xi < 2^\mu\}$ be an enumeration of all subsets of V of cardinality less than κ. Let $\mathcal{I} \subseteq V$ be a linearly independent set of cardinality κ. Inductively for any $\xi < 2^\mu$ we construct $\varphi_\xi \in V$ and $X_\xi \subseteq V$ such that

(a) $\varphi_\xi + f \notin Y_\xi := [\mathcal{I} \cup \bigcup_{\beta < \xi} X_\beta]$ for any $f \in G_\xi$;
(b) $X_\xi = \bigcup_{f \in G_\xi} [\varphi_\xi + f]$.

Suppose that we have already constructed φ_ξ and X_ξ for every $\xi < \alpha$. Let $Y_\alpha = [\mathcal{I} \cup \bigcup_{\xi < \alpha} X_\xi]$. Since $\dim [X_\xi] < \kappa$, then $\dim Y_\alpha \le |\alpha|\kappa + \kappa$. Thus $|Y_\alpha| < 2^\mu$ and we can choose $\varphi_\alpha \notin Y_\alpha - G_\alpha$ (equivalently $\varphi_\alpha + f \notin Y_\alpha$ for every $f \in G_\alpha$). Define $X_\alpha = \bigcup_{f \in G_\alpha} [\varphi_\alpha + f]$.

Observe that $X_\xi \cap X_{\xi'} = \{0\}$ and $Y_\xi \subseteq Y_{\xi'}$ for $\xi < \xi'$. Moreover $Y_\xi \cap X_{\xi'} = \{0\}$ and $X_\xi \cap [\mathcal{I}] = \{0\}$ for $\xi \le \xi'$. Define $\mathcal{F} = \bigcup_{\xi < 2^\mu} X_\xi$. Take any $G \subseteq V$ with $|G| < \kappa$. There is ξ with $G = G_\xi$. Then $\varphi_\xi + G_\xi \subseteq X_\xi \subseteq \mathcal{F}$. Therefore $\mathcal{A}(\mathcal{F}) \ge \kappa$.

Now, we will show that for any $\varphi \in V$ there is $i \in \mathcal{I}$ with $\varphi + i \notin \mathcal{F}$. Suppose to the contrary that it is not the case, that is there is $\varphi \in V$ such that for any $i \in \mathcal{I}$ we have $\varphi + i \in \mathcal{F}$. Then there are distinct $i, i' \in \mathcal{I}$ with $\varphi + i, \varphi + i' \in \mathcal{F}$. Suppose first that $\varphi + i \in X_\xi$ and $\varphi + i' \in X_{\xi'}$ with $\xi < \xi'$. Then

$$X_{\xi'} \ni \varphi + i' = \varphi + i + (i' - i) \in [X_\xi \cup \mathcal{I}] \subseteq Y_\xi.$$

Thus $\varphi + i' = 0$. Therefore $\varphi \in [\mathcal{I}]$ and consequently $\varphi + \mathcal{I} \subseteq [\mathcal{I}]$. Since $i \ne i'$, then $X_\xi \ni \varphi + i = \varphi + i' + (i - i') = i - i' \in [\mathcal{I}]$ which contradicts the fact that $X_\xi \cap [\mathcal{I}] = \{0\}$. Hence there is ξ such that $\varphi + i \in X_\xi$ for every $i \in \mathcal{I}$. Since $|G_\xi| < \kappa$ and $|\mathcal{I}| = \kappa$, there are two distinct $i, i' \in \mathcal{I}$ such that

$\varphi+i=a(\varphi_\xi+f)$ and $\varphi+i'=a'(\varphi_\xi+f)$ for some $a,a'\in\mathbb{K}$ and $f\in G_\xi$. Thus $i-i'=(a-a')(\varphi_\xi+f)$ and therefore $\varphi_\xi+f\in[\mathcal{I}]$ which is a contradiction.

Finally we obtain that $\varphi+\mathcal{I}\nsubseteq\mathcal{F}$ for every $\varphi\in V$, which means that $\mathcal{A}(\mathcal{F})\leq\kappa$. Hence $\mathcal{A}(\mathcal{F})=\kappa$.

Let $U=[h,h']$ for two linearly independent elements $h,h'\in\mathcal{F}$. Then $h\in X_\xi$ and $h'\in X_{\xi'}$ for some ξ and ξ'. If $\xi<\xi'$, then $h\in Y_{\xi'}$ and $h'\notin Y_{\xi'}$. Let $f\in U\setminus([h]\cup[h'])$. Then $f=ah+a'h'$ for some $a,a'\in\mathbb{K}\setminus\{0\}$. If $f\in Y_{\xi'}$, then $h'=(f-ah)/a'\in Y_{\xi'}$ which leads to contradiction. Thus $f\notin Y_{\xi'}$ which means that $U\cap Y_{\xi'}=[h]$. Since two-dimensional space U cannot be covered by less than μ many sets of the form $[g]$, then $U\nsubseteq Y_{\xi'}\cup X_{\xi'}$. However $U\subseteq[Y_{\xi'}\cup X_{\xi'}]$. Therefore $U\cap X_\alpha=\{0\}$ for every $\alpha>\xi'$. Hence $U\nsubseteq\mathcal{F}$.

If $h,h'\in X_\xi$, then $U\cap\bigcup_{\beta\neq\xi}X_\beta=\{0\}$ and $U\nsubseteq X_\xi$. That implies that $\mathcal{F}$ does not contains two-dimensional vector space.

Finally, note that $\mathcal{F}$ is star-like. □

The next result, which is a modification of Theorem 2.5, shows that if $\mathcal{A}(\mathcal{F})\leq|\mathbb{K}|$, then $\mathcal{L}(\mathcal{F})$ can be any cardinal not greater than $(2^\mu)^+$.

Theorem 2.6. *Assume that $3\leq\kappa\leq\mu$, $\mathbb{K}$ is a field of cardinality μ and V is a linear space over $\mathbb{K}$ with $\dim V=2^\mu$. Let $1<\lambda\leq(2^\mu)^+$. There is star-like family $\mathcal{F}\subseteq V$ such that $\kappa\leq\mathcal{A}(\mathcal{F})\leq\kappa+1$ and $\mathcal{L}(\mathcal{F})=\lambda$.*

Proof. Let $\{G_\xi:\xi<2^\mu\}$ be an enumeration of all subsets of V of cardinality less than κ. Write V as a direct sum $V_1\oplus V_2$ of two vector spaces V_1 and V_2 with $\dim V_1=\dim V_2=2^\mu$. Let $\mathrm{Card}(\lambda)=\{\nu<\lambda:\nu$ is a cardinal number$\}$. As in the proof of Proposition 2.4(iii) we can find vector spaces $W_\nu\subseteq V_1$, $\nu\in\mathrm{Card}(\lambda)$ such that $\dim W_\nu=\nu$ and the union of bases of all W_ν's forms a linearly independent set. Put $Z=\bigcup_{\nu\in\mathrm{Card}(\lambda)}W_\nu\subseteq V_1$ and note that $\mathcal{L}(Z)=\lambda$. Let $\mathcal{I}\subseteq V_2$ be a linearly independent set of cardinality $\kappa+1$. Inductively for any $\xi<2^\mu$ we construct $\varphi_\xi\in V$ and $X_\xi\subseteq V$ such that φ_ξ and X_ξ satisfy the formulas (a) and (b) from the proof of Theorem 2.5. Define $\mathcal{F}=Z\cup\bigcup_{\xi<2^\mu}X_\xi$. Then $\mathcal{A}(\mathcal{F})\geq\kappa$.

Now, we will show that for any $\varphi\in V$ there is $i\in\mathcal{I}$ with $\varphi+i\notin\mathcal{F}$. Suppose to the contrary that it is not the case, that is there is $\varphi\in V$ such that for any $i\in\mathcal{I}$ we have $\varphi+i\in\mathcal{F}$. Then there are $i,i'\in\mathcal{I}$ with $\varphi+i,\varphi+i'\in\mathcal{F}$. Suppose first that $\varphi+i\in Z$ and $\varphi+i'\in Z$. Then $i-i'\in V_1$. Since $[\mathcal{I}]\cap V_1=\{0\}$, then $i=i'$. Thus there is at most one element $i\in\mathcal{I}$ with $\varphi+i\in Z$. Now there are at least $\kappa\geq3$ elements $i\in\mathcal{I}$ such that $\varphi+i\in\bigcup_{\xi<2^\mu}X_\xi$. Using the argument as in the proof of Theorem 2.5 we reach a contradiction and we obtain that $\varphi+\mathcal{I}\nsubseteq\mathcal{F}$ for every $\varphi\in V$, which means that $\mathcal{A}(\mathcal{F})\leq\kappa+1$. Hence $\kappa\leq\mathcal{A}(\mathcal{F})\leq\kappa+1$.

Since $Z\subseteq\mathcal{F}$ and $\mathcal{L}(Z)=\lambda$, then $\mathcal{L}(\mathcal{F})\geq\lambda$. From the proof of Theorem 2.5 we obtain that $\bigcup_{\xi<2^\mu}X_\xi$ does not contain 2-dimensional vector space. To show that $\mathcal{L}(\mathcal{F})=\lambda$ it suffices to show that each 2-dimensional space W contained in $\mathcal{F}$ must be a subset of Z.

Let W be a 2-dimensional space which is not contained in Z. In the proof of Theorem 2.5 we have shown that the cardinality of the family of one-dimensional subspaces of W is less than μ. Moreover, by the construction of Z, W has at most two one-dimensional subspaces contained in Z. Consequently W is not a subset of $\mathcal{F}$. □

Note that $\mathcal{HL}(\mathcal{F})=2$ for $\mathcal{F}$ constructed in the proofs of Theorem 2.5 and Theorem 2.6.

3. lineability of residual star-like subsets of Banach spaces

Let us present the following example. Let $\mathcal{J}$ be an ideal of subsets of some set X which does not contain X. By $\operatorname{add}(\mathcal{J})$ we denote the cardinal number defined as $\min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{J} \text{ and } \bigcup \mathcal{G} \notin \mathcal{J}\}$ where $|\mathcal{G}|$ stands for cardinality of $\mathcal{G}$. Let $\mathcal{N}$ and $\mathcal{M}$ stands for σ-ideal of null and meager subsets of the real line, respectively. Then $\omega_1 \leq \operatorname{add}(\mathcal{N}) \leq \operatorname{add}(\mathcal{M}) \leq \mathfrak{c}$. Moreover, if X is an uncountable complete separable metric space and $\mathcal{M}_X$ is an ideal of meager subsets of X, then $\operatorname{add}(\mathcal{M}) = \operatorname{add}(\mathcal{M}_X)$.

Let $V = \mathbb{R}$ be a linear space over $\mathbb{K} = \mathbb{Q}$. Let $\mathcal{J}$ be a translation invariant proper σ-ideal of subsets of $\mathbb{R}$ and let $\mathcal{F}$ be a $\mathcal{J}$-residual subset of $\mathbb{R}$, i.e. $\mathcal{F}^c \in \mathcal{J}$. It turns out that $\mathcal{A}(\mathcal{F}) \geq \operatorname{add}\mathcal{J}$. To see it fix $F \subseteq \mathbb{R}$ with $|F| < \operatorname{add}\mathcal{J}$. For any $f \in F$ consider a set

$$T_f = \{t \in \mathbb{R} : t + f \notin \mathcal{F}\} = \{t \in \mathbb{R} : \exists g \in \mathcal{F}^c (t = g - f)\} \subseteq \mathcal{F}^c - f.$$

Thus $T_f \in \mathcal{J}$. Since $|F| < \operatorname{add}\mathcal{J}$, then also $\bigcup_{f\in F} T_f \in \mathcal{J}$. Therefore there is $t \in \mathbb{R}$ such that $t + f \in \mathcal{F}$ for any $f \in F$.

If we additionally assume that $\mathcal{F}$ is star-like, then by Theorem 2.2 we obtain that $\mathcal{HL}(\mathcal{F}) > \operatorname{add}\mathcal{J}$. In particular if A is positive Lebesgue measure (non-meager with Baire property), then $\mathbb{Q}A = \{qa : q \in \mathbb{Q}, a \in A\}$ is ($\mathbb{Q}$-)star-like of full measure (residual) and therefore $\mathcal{HL}(\mathcal{F}) > \operatorname{add}(\mathcal{N})$ ($> \operatorname{add}(\mathcal{M})$).

Using a similar reasoning one can prove the following.

Theorem 3.1. *Assume that X is a separable Banach space. Let $\mathcal{F} \subseteq X$ be residual and star-like, and let $\mathbb{K} \subseteq \mathbb{R}$ be a field of cardinality less than* $\operatorname{add}(\mathcal{M})$. *Consider X as a linear space over $\mathbb{K}$. Then* $\mathcal{HL}_{\mathbb{K}}(\mathcal{F}) > \operatorname{add}(\mathcal{M})$. *In particular $\mathcal{F}$ contains an uncountably dimensional vector space over $\mathbb{K}$.*

Let $\hat{C}[0,1]$ stand for the family of functions from $C[0,1]$ which attain the maximum only at one point. Then $\hat{C}[0,1]$ is star-like and residual but not 2-lineable, see [15] and [28] for details. This shows that Theorem 3.1 would be false for $\mathbb{K} = \mathbb{R}$. On the other hand, Theorem 3.1 shows that for any field $\mathbb{K} \subseteq \mathbb{R}$ of cardinality less than $\operatorname{add}(\mathcal{M})$ there is uncountable family $\mathcal{F} \subseteq \hat{C}[0,1]$ such that any nontrivial linear combination of elements from $\mathcal{F}$ with coefficients from $\mathbb{K}$ attains the maximum only at one point.

4. Homogeneous lineability and lineability numbers of some subsets of $\mathbb{R}^{\mathbb{R}}$

In this section we will apply Theorem 2.2 to obtain homogeneous lineability of families of functions from $\mathbb{R}^{\mathbb{R}}$. We will consider those families for which the additivity has been already computed.

Let $f \in \mathbb{R}^{\mathbb{R}}$. We will say that

(1) $f \in \mathrm{D}(\mathbb{R})$ (f is Darboux) if f maps connected sets onto connected sets.
(2) $f \in \mathrm{ES}(\mathbb{R})$ (f is everywhere surjective) if $f(U) = \mathbb{R}$ for every nonempty open set U;
(3) $f \in \mathrm{SES}(\mathbb{R})$ (f is strongly everywhere surjective) if f takes each real value $\mathfrak{c}$ many times in each interval;
(4) $f \in \mathrm{PES}(\mathbb{R})$ (f is perfectly everywhere surjective) if $f(P) = \mathbb{R}$ for every perfect set P;
(5) $f \in \mathrm{J}(\mathbb{R})$ (f is Jones function) if the graph of f intersects every closed subset of $\mathbb{R}^2$ with uncountable projection on the x-axis.
(6) $f \in \mathrm{AC}(\mathbb{R})$ (f is almost continuous, in the sense of J. Stallings) if every open set containing the graph of f contains also the graph of some continuous function.

(7) If $h : X \to \mathbb{R}$, where X is a topological space, $h \in \mathrm{Conn}(X)$ (h is a connectivity function) if the graph of $h|_C$ is connected for every connected set $C \subseteq X$.

(8) $f \in \mathrm{Ext}(\mathbb{R})$ (f is extendable) if there is a connectivity function $g : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $f(x) = g(x, 0)$ for every $x \in \mathbb{R}$.

(9) $f \in \mathrm{PR}(\mathbb{R})$ (f is a perfect road function) if for every $x \in \mathbb{R}$ there is a perfect set $P \subseteq \mathbb{R}$ such that x is a bilateral limit point of P and $f|_P$ is continuous at x.

(10) $f \in \mathrm{PC}(\mathbb{R})$ (f is peripherally continuous) if for every $x \in \mathbb{R}$ and pair of open sets $U, V \subseteq \mathbb{R}$ such that $x \in U$ and $f(x) \in V$ there is an open neighborhood W of x with $W \subseteq U$ and $f(\mathrm{bd}(W)) \subseteq V$.

(11) $f \in \mathrm{SZ}(\mathbb{R})$ (f is a Sierpiński–Zygmund function) if f is not continuous on any subset of the real line of cardinality $\mathfrak{c}$.

We start from recalling two cardinal numbers:

$$e_{\mathfrak{c}} = \min\{|F| : F \subseteq \mathbb{R}^{\mathbb{R}}, \forall \varphi \in \mathbb{R}^{\mathbb{R}} \exists f \in F(\mathrm{card}(f \cap \varphi) < \mathfrak{c})\},$$

$$d_{\mathfrak{c}} = \min\{|F| : F \subseteq \mathbb{R}^{\mathbb{R}}, \forall \varphi \in \mathbb{R}^{\mathbb{R}} \exists f \in F(\mathrm{card}(f \cap \varphi) = \mathfrak{c})\}.$$

It was proved in [17] that $\mathcal{A}(\mathrm{D}(\mathbb{R})) = \mathcal{A}(\mathrm{AC}(\mathbb{R})) = e_{\mathfrak{c}}$. Therefore $\mathcal{HL}(\mathrm{D}(\mathbb{R})), \mathcal{HL}(\mathrm{AC}(\mathbb{R})) \geq e_{\mathfrak{c}}^+$. More recently in [21] it was proved that $\mathcal{A}(\mathrm{J}(\mathbb{R})) = e_{\mathfrak{c}}$. Thus $\mathcal{HL}(\mathrm{J}(\mathbb{R})) \geq e_{\mathfrak{c}}^+$. On the other hand, by the result of [20], $\mathcal{L}(\mathrm{J}(\mathbb{R})) = (2^{\mathfrak{c}})^+$. Since $\mathrm{J}(\mathbb{R}) \subseteq \mathrm{PES}(\mathbb{R}) \subseteq \mathrm{SES}(\mathbb{R}) \subseteq \mathrm{ES}(\mathbb{R}) \subseteq \mathrm{D}(\mathbb{R})$, then additivity number for the classes $\mathrm{PES}(\mathbb{R}), \mathrm{SES}(\mathbb{R}), \mathrm{ES}(\mathbb{R})$ is $e_{\mathfrak{c}}$ while their lineability number is largest possible. Since in some model of ZFC we have $e_{\mathfrak{c}} < 2^{\mathfrak{c}}$, our method does not give optimal solution for lineability number in this cases.

It was proved in [19] that $\mathcal{A}(\mathrm{Ext}(\mathbb{R})) = \mathcal{A}(\mathrm{PR}(\mathbb{R})) = \mathfrak{c}^+$. Thus $\mathcal{HL}(\mathrm{Ext}(\mathbb{R})), \mathcal{HL}(\mathrm{Ext}(\mathbb{R})) \geq \mathfrak{c}^{++}$. These two classes were not considered in the context of lineability.

In [18] it was proved that $\mathcal{A}(\mathrm{SZ}(\mathbb{R})) = d_{\mathfrak{c}}$. Thus $\mathcal{HL}(\mathrm{SZ}(\mathbb{R})) \geq d_{\mathfrak{c}}^+$. In [7] it was shown that $\mathrm{SZ}(\mathbb{R})$ is κ-lineable if there exists a family of cardinality κ consisting of almost disjoint subsets of $\mathfrak{c}$. On the other hand in [24] the authors proved that if there is no family of cardinality κ consisting of almost disjoint subsets of $\mathfrak{c}$, then $\mathrm{SZ}(\mathbb{R})$ is not κ-lineable. In [18] the authors proved that it is consistent with ZFC+CH and $\mathcal{A}(\mathrm{SZ}(\mathbb{R})) = \mathfrak{c}^+ < 2^{\mathfrak{c}}$. However, if CH holds, then there is a family of cardinality $2^{\mathfrak{c}}$ consisting of almost disjoint subsets of $\mathfrak{c}$ and therefore $\mathrm{SZ}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable. Consequently, as in previous examples, consistently $\mathcal{A}(\mathrm{SZ}(\mathbb{R}))^+ < \mathcal{L}(\mathrm{SZ}(\mathbb{R}))$.

It was proved in [19] that $\mathcal{A}(\mathrm{PC}(\mathbb{R})) = 2^{\mathfrak{c}}$. Therefore $\mathcal{HL}(\mathrm{PC}(\mathbb{R})) = (2^{\mathfrak{c}})^+$ is the largest possible. Note that Darboux functions are peripherally continuous. Since the set of all functions which everywhere discontinuous and Darboux are strongly $2^{\mathfrak{c}}$-algebrable, see [8], then so is $\mathrm{PC}(\mathbb{R})$, which is much stronger property that $2^{\mathfrak{c}}$-lineability.

If V is a vector space over $\mathbb{K}$ with $|\mathbb{K}| = \mu \geq \omega$, $\mathcal{F}$ is star-like, $\mathcal{F} \subseteq V$ and $\mathcal{A}(\mathcal{F}) = \kappa > \mu$, then $\mathcal{F} \cup \{0\} = \bigcup \mathcal{B}$, where every member B of $\mathcal{B}$ is a linear space of dimension $\dim B \geq \kappa$. It follows from the fact that for any $f \in \mathcal{F} \setminus \{0\}$ and any maximal vector space B contained in $\mathcal{F}$ such that $[f] \subseteq B$ by Theorem 2.2 we have $\dim B \geq \kappa$.

Theorem 4.1. *Let $\mathcal{F} \subseteq V$ be such that $\mathcal{A}(\mathcal{F}) = \kappa > |\mathbb{K}|$. Assume that there is a vector space $X \subseteq V$ such that $X \cap \mathcal{F} \subseteq \{0\}$ with $\dim X = \tau \leq \kappa$. Then $\mathcal{B}$ contains at least τ many pairwise distinct elements.*

Proof. Assume first that $\dim X = \kappa$ Let $X = \bigcup_{\xi<\kappa} X_\xi$ be such that $X_\xi \subseteq X_{\xi'}$ provided $\xi < \xi'$ and X_ξ is a linear space with $\dim X_\xi = |\xi|$ for every $\xi < \kappa$. Since $|X_\xi| < \mathcal{A}(\mathcal{F})$, there is $\varphi_\xi \in \mathcal{F}$ with $\varphi_\xi + X_\xi \subseteq \mathcal{F}$. Take any two distinct elements $x, y \in X$. There is ξ such that $x, y \in X_\xi$ and $\varphi_\xi + x, \varphi_\xi + y \in \mathcal{F}$. Thus $x - y \in X$. Hence x, y are not in the same B from $\mathcal{B}$. Consequently $|\mathcal{B}| \geq \kappa$. If $\dim X < \kappa$, the proof is similar and a bit simpler. □

The fact that $\mathcal{F} \cup \{0\}$ can be represented as a union of at least τ linear spaces, each of dimension at least κ we denote by saying that $\mathcal{F}$ has property $B(\kappa, \tau)$. Surprisingly families of strange function defined by non-linear properties can be written as unions of large linear spaces.

Corollary 4.2. (1) $\mathrm{D}(\mathbb{R})$, $\mathrm{ES}(\mathbb{R})$, $\mathrm{SES}(\mathbb{R})$, $\mathrm{PES}(\mathbb{R})$ *and* $\mathrm{J}(\mathbb{R})$ *have* $B(e_\mathfrak{c}, e_\mathfrak{c})$.
(2) $\mathrm{AC}(\mathbb{R})$ *has* $B(e_\mathfrak{c}, \mathfrak{c})$.
(3) $\mathrm{PC}(\mathbb{R})$ *has* $B(2^\mathfrak{c}, \mathfrak{c})$.
(4) $\mathrm{PR}(\mathbb{R})$ *have* $B(\mathfrak{c}^+, \mathfrak{c}^+)$.
(5) $\mathrm{SZ}(\mathbb{R})$ *has* $B(d_\mathfrak{c}, d_\mathfrak{c})$.

Proof. We need only to show that for each of the given families of functions there is a large linear space disjoint from it. This is in fact the same as saying that complements of these families are τ-lineable for an appropriate τ. Note that $X_1 = \{f \in \mathbb{R}^\mathbb{R} : f(x) = 0 \text{ for every } x \leq 0\}$ is disjoint from $\mathrm{ES}(\mathbb{R})$, $\mathrm{SES}(\mathbb{R})$, $\mathrm{PES}(\mathbb{R})$, $\mathrm{J}(\mathbb{R})$ and $\mathrm{SZ}(\mathbb{R})$. Let X_2 be a space of dimension $2^\mathfrak{c}$ such that $X_2 \setminus \{0\}$ consists of nowhere continuous functions with a finite range. Such a space X_2 was constructed in [23]. Clearly, X_2 is disjoint from $\mathrm{D}(\mathbb{R})$. Let X_3 be a linear space of dimension $\mathfrak{c}$ such that $X_3 \setminus \{0\}$ consists of functions which has dense set of jump discontinuities. Such a space X_3 was constructed in [27]. Then $X_3 \cap \mathrm{AC}(\mathbb{R}) = X_3 \cap \mathrm{PC}(\mathbb{R}) = \{0\}$. Let X_4 be a linear space of dimension $2^\mathfrak{c}$ such that $X_4 \setminus \{0\} \subseteq \mathrm{PES}(\mathbb{R})$. Such a space X_4 was constructed in [22]. Then $X_4 \cap \mathrm{PR}(\mathbb{R}) = \{0\}$. □

We end the paper with the list of open questions:

1. Is it true that $\mathcal{A}(\mathcal{F})^+ \geq \mathcal{HL}(\mathcal{F})$ for any family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$?
2. What is are homogeneous lineability numbers of the following families $\mathrm{D}(\mathbb{R})$, $\mathrm{AC}(\mathbb{R})$, $\mathrm{PES}(\mathbb{R})$, $\mathrm{J}(\mathbb{R})$, $\mathrm{Ext}(\mathbb{R})$, $\mathrm{PR}(\mathbb{R})$ and $\mathrm{SZ}(\mathbb{R})$?
3. Are the families $\mathrm{Ext}(\mathbb{R})$ and $\mathrm{PR}(\mathbb{R})$ $2^\mathfrak{c}$-lineable in ZFC?
4. Are the complements of the families $\mathrm{Ext}(\mathbb{R})$, $\mathrm{PC}(\mathbb{R})$ and $\mathrm{AC}(\mathbb{R})$ $2^\mathfrak{c}$-lineable?

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