ACHIEVEMENT SETS ON THE PLANE – PERTURBATIONS OF GEOMETRIC AND MULTIGEOMETRIC SERIES

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Abstract. By $\mathrm{A}(x_n) = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : \varepsilon_n = 0, 1\}$ we denote the achievement set of the absolutely convergent series $\sum_{n=1}^{\infty} x_n$. We study the relation between the achievement set of the series on the plane and the achievement sets of its projection into two coordinates. We mainly focus on the series $\sum_{n=1}^{\infty}(x_n, y_n)$ where (x_n) is a geometric series and $y_n = x_{\sigma(n)}$ for some permutation $\sigma \in S_\infty$.

If (x_n) is a multigeometric sequence, then $\mathrm{A}(x_n, x_{\sigma(n)})$ can be one of at least seven types of sets, which are strongly related to three types of attainable achievement sets on the real line. We conjecture that if (x_n) multigeometric, then $\mathrm{A}(x_n, x_{\sigma(n)})$ can be one of eight types – none of them homeomorphic to the other one.

We prove a general fact on the Hausdorff dimension of the achievement set in Banach spaces. As a corollary we obtain that if $0 < q \leq 1/2$, $\dim_H(\mathrm{A}(q_n, q_{\sigma(n)})) = \dim_H(\mathrm{A}(x_n)) = -\log 2/\log q$ for some class of regular permutations $\sigma \in S_\infty$.

1. Introduction

Suppose that $x = (x_n)_{n=1}^{\infty} \in \ell_1$ and let

$$\mathrm{A}(x) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

denote the set of all subsums of the series $\sum_{n=1}^{\infty} x_n$, called *the achievement set* (or *a partial sumset*) of x. In 1914 Soichi Kakeya [8] initiated the study of topological properties of achievement sets presenting the following result:

Theorem 1.1 (Kakeya). *For any sequence $x \in \ell_1 \setminus c_{00}$*

1. $\mathrm{A}(x)$ *is a perfect compact set.*
2. *If $|x_n| > \sum_{i>n} |x_i|$ for almost all n, then $\mathrm{A}(x)$ is homeomorphic to the ternary Cantor set.*
3. *If $|x_n| \leq \sum_{i>n} |x_i|$ for almost all n, then $\mathrm{A}(x)$ is a finite union of closed intervals. In the case of non-increasing sequence x, the last inequality is also necessary for $\mathrm{A}(x)$ to be a finite union of intervals.*

Kakeya conjecture was that $\mathrm{A}(x)$ is either nowhere dense or a finite union of intervals. It was disproved by Weinstein and Shapiro [14] and, independently, by Ferens [5]. Guthrie and Nymann in [6] gave a simple example of sequence, namely $x = \left(\frac{5+(-1)^n}{4^n}\right)_{n=1}^{\infty}$, such that its achievement set $T = \mathrm{A}(x)$ contains an interval but it is not a finite union of intervals. In the same paper the authors formulated the following trichotomy for achievement sets, finally proved in [12]:

Theorem 1.2. *For any sequence $x \in \ell_1 \setminus c_{00}$, $\mathrm{A}(x)$ is one of the following sets:*

1. *a finite union of closed intervals;*
2. *homeomorphic to the ternary Cantor set;*
3. *homeomorphic to the set T.*

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The set T is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$, where S_n denotes the union of the 2^{n-1} open middle thirds which are removed from $[0,1]$ at the n-th step in the construction of the ternary Cantor set C. Such sets are called Cantorvals. Formally, a *Cantorval* (more precisely, an $\mathcal{M}$-Cantorval, see [9]) is a non-empty compact subset S of the real line, such that S is the closure of its interior, and both endpoints of any infinite component are accumulation points of one-point components of S. A non-empty subset C of the real plane will be called a *Cantor set* if it is compact, zero-dimensional and has no isolated points.

Note that Theorem 1.2 says that ℓ_1 can be divided into four sets: c_{00} and the sets with properties prescribed in (1), (2) and (3). Some algebraic and topological properties of these sets have been recently considered in [2].

The sequence of the form $(k_1, k_2, \dots, k_m, k_1 q, \dots, k_m q, k_1 q^2, \dots)$ is called multigeometric sequence (see [3]) and it is denoted by $(k_1, k_2, \dots, k_m; q)$. Note that Guthrie-Nymann sequence $\left(\frac{5+(-1)^n}{4^n}\right)_{n=1}^{\infty}$ is a multigeometric series of the form $(3/4, 6/4; 1/4)$. If $k_1 = \dots = k_m$, then by Kakeya Theorem $\mathrm{A}(k_1, k_2, \dots, k_m; q)$ is either a Cantor set or an interval. As in [1] we denote by Σ the set

$$\left\{ \sum_{n=1}^{m} k_n \varepsilon_n : (\varepsilon_n)_{n=1}^{m} \in \{0,1\}^m \right\}.$$

Let us write Σ as $\{\tau_1 < \dots < \tau_s\}$. Then the one-dimensional achievement set $\mathrm{A}(x)$ depends only on Σ and the ratio q. We consider the following numbers connected with Σ: $\mathrm{diam}(\Sigma) = \tau_s - \tau_1$, $\Delta(\Sigma) = \max_{i<s}(\tau_{i+1} - \tau_i)$ and $I(\Sigma) = \Delta(\Sigma)/(\Delta(\Sigma) + \mathrm{diam}(\Sigma))$. Moreover, we have $|\Sigma| = s$. It was proved in [1] that

(1) A is an interval if and only if $q \geq I(\Sigma)$.
(2) A is not a finite union of intervals if $q < I(\Sigma)$ and $\Delta \in \{\tau_2 - \tau_1, \tau_s - \tau_{s-1}\}$.
(3) A is a Cantor set of zero Lebesgue measure if $q < 1/s$.

For a metric space (X, ρ) by $\mathcal{K}(X)$ we denote the hyperspace of all non-empty compact subsets of X. There is a natural metric on $\mathcal{K}(X)$, namely the Hausdorff distance given by

$$\rho_H(K, L) = \inf\{\delta > 0 : L \subset B(K, \delta) \text{ and } K \subset B(L, \delta)\}$$

where $K, L \in \mathcal{K}(X)$ and $B(K, \delta) = \bigcup_{x \in K} B(x, \delta)$ is a δ-neighborhood of K. The iterated function system fractal (in short IFS fractal) generated by the system of affine contractions $\{f_1, \dots, f_n\}$ is the unique fixed point of the self-map $K \mapsto \bigcup_{i=1}^{n} f_i(K)$. For a positive real number s and $\delta > 0$ define $\mathcal{H}^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} (\mathrm{diam}\, A_n)^s :$ $A_1, A_2, \dots$ is a δ-cover of $F\}$ where δ-cover of F is a sequence $A_1, A_2, \dots$ of sets such that $F \subset \bigcup_{n=1}^{\infty} A_n$ and $\mathrm{diam}(A_n) \leq \delta$. The s-dimensional Hausdorff outer measure is defined as $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F)$. It is well-known that for a given Borel set F and for $0 < s < t$, if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$, and if $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$. The Hausdorff dimension $\dim_H(F)$ of a Borel set F is a critical value $s_0 \in [0, \infty]$, such that $\mathcal{H}^s(F) = \infty$ for all $s < s_0$ and $\mathcal{H}^s(F) = 0$ for all $s > s_0$.

Z. Nitecki at the end of his nice survey paper [13] on subsum sets wrote: "One might also be tempted to ask about the analogous question for null sequences in the complex plane (or more generally points in $\mathbb{R}^n$). In this context (...) the analysis of translations will be made more complicated by the need to consider directions as well as distances. Who knows where that might lead?" Following this suggestion we start investigation of multidimensional achievement sets - its topological and geometric properties.

The aim of our paper is to study the properties of the achievement sets on the plane. Let $(x_n, y_n) \in \ell_1 \times \ell_1$. By

$$\mathrm{A}(x_n, y_n) := \left\{ \sum_{n=1}^{\infty} \varepsilon_n (x_n, y_n) : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

we denote the achievement set of the series $\sum_{n=1}^{\infty} (x_n, y_n)$. The main and the most general question we are interested in, is the following:

Problem 1.3. *Let $x_n, y_n \in \ell_1$ be such that* $\mathrm{A}(x_n) = C_1$ *and* $\mathrm{A}(y_n) = C_2$. *What can be said about* $\mathrm{A}(x_n, y_n)$*?*

Achievement sets of series in $\mathbb{R}^n$ were studied by Manuel Morán in [10] and [11]. In [10] a series $\sum_{i=1}^{\infty} x_i$ is called *fractal series* if $\mathrm{A}(x_i)$ has cardinality continuum (equivalently $(x_i) \notin c_{00}$) and it has n-dimensional Lebesgue measure zero. The author has given some sufficient conditions for series $\sum_{i=1}^{\infty} x_i$ being a fractal series. Each of them implies that $\sum_{i=1}^{\infty} x_i$ is quickly convergent, which means $\|x_i\| > \sum_{k>i} \|x_k\|$ for almost every i, which is a Kakeya type condition. Morán has estimated, and in some cases precisely calculated, the Hausdorff dimension of the achievement sets.

It is easy to observe that, as in one-dimensional case, the achievement set on the plane is a compact perfect set (or finite set if elements of underlying series are eventually zero). Moreover, the set $\mathrm{A}(x_n, y_n)$ is contained in $C_1 \times C_2$ – the Cartesian product of achievement sets of (x_n) and (y_n), and $\mathrm{A}(x_n, y_n)$ is symmetric with respect to the middle point of $C_1 \times C_2$. Thus if $\mathrm{A}(x_n)$ and $\mathrm{A}(y_n)$ are Cantor sets, so is $\mathrm{A}(x_n, y_n)$.

If $\mathrm{A}(x_n) = C$, then $\mathrm{A}(x_n, x_n) = \sqrt{2}\,\mathrm{R}_{\frac{\pi}{4}}(C)$ where $\mathrm{R}_{\frac{\pi}{4}}$ is the anticlockwise rotation around the origin at an angle of $\frac{\pi}{4}$. On the other hand if one add zeros to the series x_n, then the one-dimensional achievement set remains unchanged. In particular

$$\mathrm{A}(x_1, 0, x_2, 0, x_3, 0, \dots) = \mathrm{A}(0, x_1, 0, x_2, 0, x_3, \dots) = \mathrm{A}(x_1, x_2, x_3, \dots) = C$$

and

$$\mathrm{A}((x_1, 0), (0, x_1), (x_2, 0), (0, x_2), (x_3, 0), (0, x_3), \dots) = C \times C.$$

This simple observation shows that to get something interesting we need to make some restrictions on the sequence (x_n, y_n). We will deal with the following more specific question.

Problem 1.4. *Let $x_n > 0$ for every $n \in \mathbb{N}$. Assume that* $\mathrm{A}(x_n) = C$. *What can be said about* $\mathrm{A}(x_n, x_{\sigma(n)})$ *where $\sigma \in S_\infty$?*

In this paper we will consider even more specific situation. Namely we will consider the case when the series $\sum_{n=1}^{\infty} x_n$ is a geometric or multigeometric series and we will restrict our attention to permutations $\sigma \in S_\infty$ which are quite regular. For $q \in (0, 1)$ the series $\sum_{n=1}^{\infty} (q^n, q^{\sigma(n)})$ will be called *perturbed geometric series*; similarly we define *perturbed multigeometric series*.

The paper is organized as follows. In Section 2 we make several general observations on the achievement sets on the real plane and we consider introductory example of perturbed geometric series to illustrate these ideas. Under some assumptions on $\sigma \in S_\infty$, the achievement set $\mathrm{A}(q^n, q^{\sigma(n)})$ is an IFS fractal, which in turn, for $0 < q \le 1/2$, fulfills the so called Moran's open set condition that allows us to give the formula for $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)}))$. In Section 3 we show that the achievement set of perturbed multigeometric series can be one of at least seven types. We pose the question if the achievement set $\mathrm{A}(x_n, x_{\sigma(n)})$ for multigeometric (x_n) can be, up to homeomorphism, one of the eight mentioned types. This would be a classification result similar that Theorem 1.2. In Section 4 we observe that the orthogonal projection of the achievement set of perturbed geometric series on the line $y = -x$ equals to the achievement set of multigeometric series. Finally in Section 5 we generalize the notion of an achievement set to the infinite algebraic sum of finite sets in Banach spaces. In this setup we estimate the Hausdorff dimension of a generalized achievement set. Under certain condition we precisely calculate the Hausdorff dimension of a generalized achievement set in $\mathbb{R}^n$. As a consequence we obtain the formula for $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)}))$ for $0 < q \le 1/2$ and some class of permutations $\sigma \in S_\infty$. The results of this section generalize that of Morán and that of Section 2.

2. General observations and an instructive example

We will focus on the particular case when the permutation σ is of the special form.

Let $0 = n_0 < n_1 < n_2 < n_3 < \dots$ and let c_i be permutation of the set $\{n_{i-1}+1, n_{i-1}+2, \dots, n_i\}$ for $i \geq 1$. Let $\sigma = c_1c_2c_3 \dots \in S_\infty$ and $\sigma_i = c_1 \dots c_i$, $i \geq 1$. Let $(x_n) \in \ell_1$.

Proposition 2.1. $\mathrm{A}(x_n, x_{\sigma_k(n)})$ *tends in the Hausdorff metric to* $\mathrm{A}(x_n, x_{\sigma(n)})$.

By

$$\mathrm{A}_k(x_n, x_{\sigma(n)}) = \left\{ \sum_{n=1}^{k} \varepsilon_n (x_n, x_{\sigma(n)}) : (\varepsilon_n) = \{0,1\}^k \right\}$$

denote the k-th approximation of the achievement set $\mathrm{A}(x_n, x_{\sigma(n)})$.

Proof. Note that the set $\bigcup_{k=1}^{\infty} \mathrm{A}_k(x_n, x_{\sigma(n)})$ is dense in $\mathrm{A}(x_n, x_{\sigma(n)})$. The sets $\mathrm{A}_k(x_n, x_{\sigma(n)})$ are finite, and consequently compact. Therefore the sequence $(\mathrm{A}_k(x_n, x_{\sigma(n)}))_{k\in\mathbb{N}}$ tends in the Hausdorff metric to $\mathrm{A}(x_n, x_{\sigma(n)})$.

Let $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \operatorname{diam} \big(\mathrm{A}((x_n, x_n), (x_{n+1}, x_{n+1}), (x_{n+2}, x_{n+2}), \dots) \big) = 0,$$

there is $N \in \mathbb{N}$ such that

$$\operatorname{diam} \big(\mathrm{A}((x_n, x_n), (x_{n+1}, x_{n+1}), (x_{n+2}, x_{n+2}), \dots) \big) < \varepsilon/2$$

for $n > N$. Note that $\mathrm{A}_{n_i}(x_n, x_{\sigma(n)}) = \mathrm{A}_{n_i}(x_n, x_{\sigma_k(n)})$, that is the n_i-th approximations of the achievement sets for sequences $(x_n, x_{\sigma(n)})$ and $(x_n, x_{\sigma_k(n)})$ are equal. Note also that

$$\mathrm{A}(x_n, x_{\sigma_i(n)}) = \mathrm{A}_{n_i}(x_n, x_{\sigma_i(n)}) + \mathrm{A}((x_{n_i+1}, x_{n_i+1}), (x_{n_i+2}, x_{n_i+2}), \dots).$$

Thus

$$\rho_H(\mathrm{A}(x_n, x_{\sigma_i(n)}), \mathrm{A}_{n_i}(x_n, x_{\sigma(n)})) < \varepsilon/2$$

if $n_i > N$. One can find large enough i such that

$$\rho_H(\mathrm{A}(x_n, x_{\sigma(n)}), \mathrm{A}_{n_i}(x_n, x_{\sigma(n)})) < \varepsilon/2.$$

Hence by the triangle inequality for the Hausdorff metric we have

$$\rho_H(\mathrm{A}(x_n, x_{\sigma(n)}), \mathrm{A}(x_n, x_{\sigma_i(n)})) < \varepsilon$$

for large enough i. □

Example 1. Let $x_n = 1/2^n$. Let $\sigma = (1,2)(3,4)(5,6)\dots$. Let $\sigma_0 = \mathrm{id}$. Then $\mathrm{A}(x_n, x_{\sigma_0(n)}) = \mathrm{A}(x_n, x_n)$ equals to the diagonal of the square $[0,1] \times [0,1]$. Let $\sigma_1 = (1,2)$. Then

$$\mathrm{A}(x_n, x_{\sigma_1(n)}) = \mathrm{A}((\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{2^3}, \frac{1}{2^3}), (\frac{1}{2^4}, \frac{1}{2^4}), \dots) = \mathrm{A}((\frac{1}{2^3}, \frac{1}{2^3}), (\frac{1}{2^4}, \frac{1}{2^4}), \dots) + \{(0,0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})\}.$$

Note that $\mathrm{A}((\frac{1}{2^3}, \frac{1}{2^3}), (\frac{1}{2^4}, \frac{1}{2^4}), \dots)$ is the diagonal of the square $[0, \frac{1}{4}] \times [0, \frac{1}{4}]$ and therefore $\mathrm{A}(x_n, x_{\sigma_1(n)})$ is the union of four pieces of the form $\mathrm{A}((\frac{1}{2^3}, \frac{1}{2^3}), (\frac{1}{2^4}, \frac{1}{2^4}), \dots)$ shifted by the vectors $(0,0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})$, see Figure 1.

Let $\sigma_2 = (1,2)(3,4)$ be the second approximation of σ. Then

$$\mathrm{A}(x_n, x_{\sigma_2(n)}) = \mathrm{A}((\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{8}, \frac{1}{16}), (\frac{1}{16}, \frac{1}{8}), (\frac{1}{2^5}, \frac{1}{2^5}), (\frac{1}{2^6}, \frac{1}{2^6}), \dots) =$$
$$\mathrm{A}((\frac{1}{2^5}, \frac{1}{2^5}), (\frac{1}{2^6}, \frac{1}{2^6}), \dots) + X$$

where $X = \{(0,0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})\} + \{(0,0), (\frac{1}{8}, \frac{1}{16}), (\frac{1}{16}, \frac{1}{8}), (\frac{3}{16}, \frac{3}{16})\}$ consists of 16 points on the plane. Therefore $\mathrm{A}(x_n, x_{\sigma_2(n)})$ is the union of 16 pieces as in Figure 2.

By Proposition 2.1 the sets, presented on Figures 1 and 2, and the next approximations, tend in the Hausdorff metric to $\mathrm{A}(x_n, x_{\sigma(n)})$ where $x_n = 1/2^n$ and $\sigma = (1,2)(3,4)\dots$. On the other hand, the set on Figure 1 equals $D_1 := T_1(D) \cup T_2(D) \cup T_3(D) \cup T_4(D)$ where D is the diagonal of $[0,1] \times [0,1]$, $T_1(x,y) = \frac{1}{4}(x,y)$, $T_2(x,y) = \frac{1}{4}(x,y) + (\frac{1}{4}, \frac{1}{2})$, $T_3(x,y) = \frac{1}{4}(x,y) + (\frac{1}{2}, \frac{1}{4})$ and $T_4(x,y) = \frac{1}{4}(x,y) + (\frac{3}{4}, \frac{3}{4})$. Then the set given on

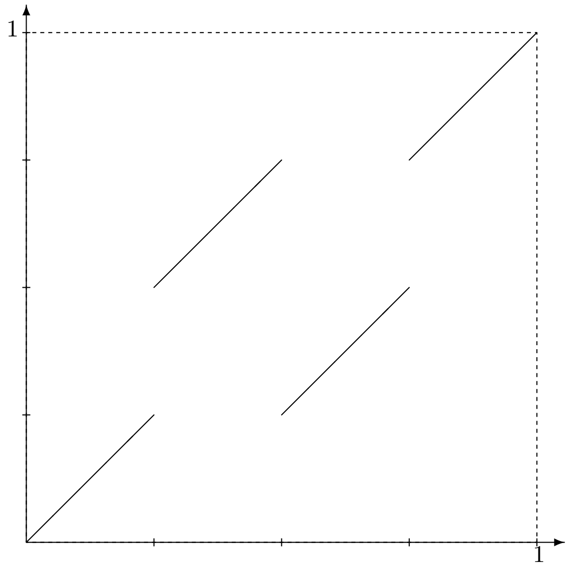

FIGURE 1. Step 1 for σ_1.

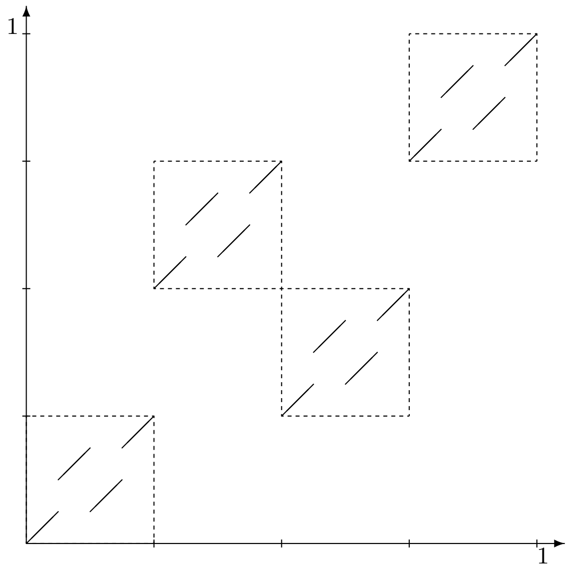

FIGURE 2. Step 2 for σ_2.

Figure 2 equals $D_2 := T_1(D_1) \cup T_2(D_1) \cup T_3(D_1) \cup T_4(D_1)$. This suggests that $\mathrm{A}(x_n, x_{\sigma(n)})$ is the IFS fractal generated by four contractions T_1, T_2, T_3, T_4.

Let us consider now the permutations more regular than these from Proposition 2.1 – the permutations c_i will be cycles with the same length.

Let $m \in \mathbb{N}$ and let $c \in S_m$. Divide $\mathbb{N}$ into consecutive segments $I_1, I_2, \ldots$ of length m, that is $I_i = \{m(i-1)+1, m(i-1)+2, \ldots, mi\}$. Let c_i be a permutation on I_i given by $c_i(m(i-1)+k) = c(k)$ for $k = 1, \ldots, m$. Define $\sigma_c = c_1 c_2 c_3 \ldots$. We say that σ_c is a regular permutation generated by c. Let $q \in (0,1)$. For $\varepsilon \in \{0,1\}^m$ put

$$T_\varepsilon(x,y) = q^m(x,y) + \sum_{n=1}^{m} \varepsilon_n \left(q^n, q^{c(n)}\right).$$

Proposition 2.2. *The achievement set* $\mathrm{A}(q^n, q^{\sigma_c(n)})$ *is equal to the IFS fractal generated by the system of affine contractions* $\{T_\varepsilon : \varepsilon \in \{0,1\}^m\}$.

Proof. At first note that

$$\mathrm{A}((q^n, q^{\sigma_c(n)})_{n=m+1}^\infty) = q^m\,\mathrm{A}((q^n, q^{\sigma_c(n)})_{n=1}^\infty).$$

Therefore

$$\mathrm{A}((q^n, q^{\sigma_c(n)})_{n=1}^\infty) = \left\{\sum_{n=1}^m \varepsilon_n \left(q^n, q^{c(n)}\right) : \varepsilon \in \{0,1\}^m\right\} + \mathrm{A}\left((q^n, q^{\sigma_c(n)})_{n=m+1}^\infty\right) =$$

$$= \bigcup_{\varepsilon\in\{0,1\}^m} \left\{\sum_{n=1}^m \varepsilon_n \left(q^n, q^{c(n)}\right)\right\} + q^m\,\mathrm{A}\left((q^n, q^{\sigma_c(n)})_{n=1}^\infty\right) =$$

$$= \bigcup_{\varepsilon\in\{0,1\}^m} T_\varepsilon\left(\mathrm{A}\left((q^n, q^{\sigma_c(n)})_{n=1}^\infty\right)\right).$$

Thus the achievement set $\mathrm{A}(q^n, q^{\sigma_c(n)})$ is a fixed point of the set function $X \mapsto \bigcup_{\varepsilon\in\{0,1\}^m} T_\varepsilon(X)$, and consequently it is the IFS fractal generated by the system of affine contractions $\{T_\varepsilon : \varepsilon \in \{0,1\}^m\}$. □

After [4] we say that the iterated function system $(f_1, f_2, \ldots, f_n)$ satisfies Moran's open set condition if and only if there exists a nonempty open set U, for which we have $f_i[U]\cap f_j[U] = \emptyset$ for $i \neq j$ and $U \supset f_i[U]$ for all i. Such an open set U will be called a Moran open set for the iterated function system. Assume that $f_1, \ldots, f_n$ are affine contractions with ratios $r_1, \ldots, r_n$, respectively. Let s be the unique solution of the equation $\sum_{i=1}^n r_i^s = 1$ and assume that $(f_1, f_2, \ldots, f_n)$ satisfies Moran's open set condition. If K is the fixed point of $X \mapsto \bigcup f_i(X)$, then the Hausdorff dimension $\dim_H(K)$ of K equals s. For details see [4, Section 6.5, p. 190–199].

Theorem 2.3. *Let* $0 < q \leq 1/2$. *Then the system of affine contractions* $\{T_\varepsilon : \varepsilon \in \{0,1\}^m\}$ *fulfills the open set condition. In particular* $\dim_H(\mathrm{A}(q^n, q^{\sigma_c(n)})) = \dim_H(\mathrm{A}(q^n)) = -\log 2/\log q$.

Proof. Note that $\mathrm{A}(q^n)$ is (f_1, f_2)-invariant where $f_1(x) = qx$ and $f_2(x) = q + qx$. Moreover, $f(0,1) \subset (0,1)$ and $f_1(0,1) \cap f_2(0,1) = \emptyset$. Thus (f_1, f_2) satisfies Moran's open set condition. If $q^s + q^s = 1$, then $s = \frac{\log 2}{\log(1/q)}$ and consequently $\dim_H(\mathrm{A}(x_n)) = \frac{\log 2}{\log(1/q)}$.

Let $d_{\max}((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Since $\mathrm{A}(q^n, q^{\sigma_c(n)}) \subset [0,1]\times[0,1]$, then

$$\mathrm{diam}_{\max}(\mathrm{A}\left((q^n, q^{\sigma_c(n)})_{n=m+1}^\infty\right)) = \mathrm{diam}_{\max}(q^m\,\mathrm{A}\left((q^n, q^{\sigma_c(n)})_{n=1}^\infty\right)) \leq q^m.$$

Moreover, the last inequality is strict if $q < 1/2$. Note that

$$d_{\max}\left(\sum_{n=1}^m \varepsilon(n)\left(q^n, q^{c(n)}\right), \sum_{n=1}^m \varepsilon'(n)\left(q^n, q^{c(n)}\right)\right) \geq q^m$$

if $\varepsilon \neq \varepsilon'$. Therefore, the system $\{T_\varepsilon : \varepsilon \in \{0,1\}^m\}$ satisfies Moran's open set condition with the Moran's set equal to $(0,1)\times(0,1)$. See also that the unique solution of the equation $2^m(q^m)^s = 1$ equals $s = \frac{\log 2}{\log(1/q)}$. □

Example 1 continued. By Theorem 2.3 the achievement set has the Hausdorff dimension 1. Moreover, it is homeomorphic to the Cantor set.

3. Eight types of achievement sets

As it was mentioned in the Introduction, the achievement set of multigeometric series is one of three types – finite union of intervals, Cantor set or $\mathcal{M}$-Cantorval. In this section we will prove that the achievement set of perturbed multigeometric series can be, up to linear isomorphism, of one of the following forms – Cantor set (Proposition 3.1(i)), product of a Cantor set and an interval (Example 3), product of two intervals (Proposition 3.1(ii)), product of two $\mathcal{M}$-Cantorvals (Example 4), product of an interval and an $\mathcal{M}$-Cantorval (Example 5). We truly believe that, using our method, one can prove that such achievement set can be, up to linear isomorphism, a product of an $\mathcal{M}$-Cantorval and a Cantor set. Note that if perturbation is trivial, namely σ is the identity, then $\mathrm{A}(x_n, x_n)$ has the same type as $\mathrm{A}(x_n)$. Hence, perturbed multigeometric series can be linearly isomorphic to a finite union of intervals or an $\mathcal{M}$-Cantorval. We also conjecture that the achievement set $\mathrm{A}(x_n, x_\sigma)$ for $(x_n) \in \ell_1 \setminus c_{00}$ can be, up to homeomorphism, one of eight mentioned types.

Let us start from the following observation.

Example 2. Let $q = \frac{1}{\sqrt{2}}$. Then

$$\mathrm{A}((q^n, q^n)_{n=3}^{\infty}) = \{(x, x) : 0 \le x \le \frac{q^3}{1-q}\} = \{(x, x) : 0 \le x \le \frac{\sqrt{2}+1}{2}\}$$

is the diagonal of the square $[0, \frac{\sqrt{2}+1}{2}] \times [0, \frac{\sqrt{2}+1}{2}]$. Note that $q + q^2 = \frac{\sqrt{2}+1}{2}$. Let $\sigma_1 = (1, 2)$. Then

$$\mathrm{A}(q^n, q^{\sigma_1(n)}) = \mathrm{A}((q^n, q^n)_{n=3}^{\infty})) + \{(0, 0), (\frac{1}{\sqrt{2}}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{\sqrt{2}}), (\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2})\}.$$

Therefore $\mathrm{A}(q^n, q^{\sigma_1(n)})$ contains the diagonal of the square $[0, \sqrt{2}+1] \times [0, \sqrt{2}+1]$ which is, in turn, equal to the achievement set $\mathrm{A}(q^n, q^n)$.

By the simple inductive argument and Proposition 2.1 the achievement set $\mathrm{A}(q^n, q^{\sigma(n)})$, where $\sigma = (1, 2)(3, 4)\ldots$, contains the diagonal of the square $[0, \sqrt{2}+1] \times [0, \sqrt{2}+1]$. Thus $\mathrm{A}(q^n, q^{\sigma(n)})$ **is not** homeomorphic to the Cantor set. Clearly the same argument holds true for any $\sqrt{2}/2 \le q < 1$.

As we have proved, the achievement set for geometric series is homeomorphic to the Cantor set if $0 < q \le 1/2$ and it contains homeomorph of the unit interval if $1/\sqrt{2} \le q < 1$. Now we prove more general fact which shed the light on these phenomena.

Proposition 3.1. *Let $q \in (0, 1)$ and let $\sigma = (1, 2)(3, 4)\ldots$. Then*
(1) $\mathrm{A}(q^n, q^{\sigma(n)})$ is the rhombus with vertices $(0, 0); (\frac{q}{1-q^2}, \frac{q^2}{1-q^2}); (\frac{q^2}{1-q^2}, \frac{q}{1-q^2})$ and $(\frac{q}{1-q}, \frac{q}{1-q})$ if and only if $q \ge \frac{\sqrt{2}}{2}$. In particular $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) = 2$ in this case.
(2) $\mathrm{A}(q^n, q^{\sigma(n)})$ is homeomorphic to a Cantor set if and only if $q < \frac{\sqrt{2}}{2}$. Moreover, the rhombus with vertices $(0, 0); (\frac{q}{1-q^2}, \frac{q^2}{1-q^2}); (\frac{q^2}{1-q^2}, \frac{q}{1-q^2})$ and $(\frac{q}{1-q}, \frac{q}{1-q})$ is a Moran's open set for $\{T_\varepsilon : \varepsilon \in \{0, 1\}^2\}$, and consequently $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) = -\frac{\log 2}{\log q}$.

Proof. Let $(x, y) \in \mathrm{A}(q^n, q^{\sigma(n)})$. There is a sequence $(\varepsilon_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \varepsilon_n q^n$ and $y = \sum_{n=1}^{\infty} \varepsilon_n q^{\sigma(n)}$. Let

$$x_1 = \sum_{n=1}^{\infty} \varepsilon_{2n} q^{2n},\ x_2 = \sum_{n=1}^{\infty} \varepsilon_{2n-1} q^{2n-1},\ y_1 = \sum_{n=1}^{\infty} \varepsilon_{2n} q^{2n-1},\ y_2 = \sum_{n=1}^{\infty} \varepsilon_{2n-1} q^{2n}.$$

Then $x = x_1 + x_2$ and $y = y_1 + y_2$. Note that the point (x_1, y_1) lies on the line $y = x/q$ and the point (x_2, y_2) lies on the line $y = qx$. It shows that any point of $\mathrm{A}(q^n, q^{\sigma(n)})$ can be represented as algebraic sum $A_{\text{top}} + A_{\text{bot}}$ of two sets $A_{\text{top}} := \{(x, x/q) : x \in \mathrm{A}(q^{2n})\}$ and $A_{\text{bot}} := \{(x/q, x) : x \in \mathrm{A}(q^{2n})\}$. Note that $A_{\text{top}} + A_{\text{bot}}$ is homeomorphic to $\mathrm{A}(q^{2n}) \times \mathrm{A}(q^{2n})$ via the linear isomorphism $(x, y) \mapsto (x + qy, qx + y)$. If $q < \frac{\sqrt{2}}{2}$ then $\mathrm{A}(q^{2n})$ is a Cantor set, and so is $A_{\text{top}} + A_{\text{bot}}$. The second part of the assertion (2) can be easily checked. If $q \ge \frac{\sqrt{2}}{2}$

then $A(q^{2n})$ is an interval, and so $A_{\rm top}+A_{\rm bot}$ is the rhombus with edges $(0,0);(\frac{q}{1-q^2},\frac{q^2}{1-q^2});(\frac{q^2}{1-q^2},\frac{q}{1-q^2})$ and $(\frac{q}{1-q},\frac{q}{1-q})$. □

Example 3. Let $\sigma=(1,2,3)(4,5,6)\dots$. Let $(x,y)\in A(q^n,q^{\sigma(n)})$. There is a sequence $(\varepsilon_n)_{n=1}^\infty\in\{0,1\}^{\mathbb N}$ such that $x=\sum_{n=1}^\infty\varepsilon_nq^n$ and $y=\sum_{n=1}^\infty\varepsilon_nq^{\sigma(n)}$. Let

$$y_1=\varepsilon_1q+\varepsilon_4q^4+\dots,\ x_1=\varepsilon_1q^3+\varepsilon_4q^6+\dots$$

$$y_2=(\varepsilon_2q^2+\varepsilon_3q^3)+(\varepsilon_5q^5+\varepsilon_6q^6)+\dots,\ x_2=(\varepsilon_2q+\varepsilon_3q^2)+(\varepsilon_5q^4+\varepsilon_6q^5)+\dots$$

Then $x=x_1+x_2$ and $y=y_1+y_2$. Note that the point (x_1,y_1) lies on the line $y=x/q^2$ and the point (x_2,y_2) lies on the line $y=qx$. Then $A(q^n,q^{\sigma(n)})=A_{\rm bot}+A_{\rm top}$ where $A_{\rm top}:=\{(x,x/q^2):x\in A(q^{3n})\}$ and $A_{\rm bot}:=\{(x,qx):x\in A(q,q^2;q^3)\}$. Note that $A(q^{3n})$ is a Cantor set with the Hausdorff dimension $-\log 2/(3\log q)$ if and only if $q<1/\sqrt[3]{2}$; otherwise it is an interval.

Now, consider the achievement set $A(q,q^2;q^3)$ of a multigeometric series $(q,q^2;q^3)$. Note that $\Delta(\Sigma)=q^2$ and $\mathrm{diam}(\Sigma)=q+q^2$. Then $A(q,q^2;q^3)$ is an interval if $q^3\geq\frac{\Delta(\Sigma)}{\Delta(\Sigma)+\mathrm{diam}(\Sigma)}=\frac{q}{2q+1}$, that is if $q\geq q^*$ where q^* is a positive solution of the equation $2q^3+q^2-1=0$. Note that $q^*<1/\sqrt{2}<1/\sqrt[3]{2}$. Therefore the achievement set $A(q^n,q^{\sigma(n)})$ is linearly isomorphic to the product of an interval and a Cantor set for $q\in[q^*,1/\sqrt[3]{2})$. Thus $\dim_H(A(q^n,q^{\sigma(n)}))=1-\log 2/(3\log q)$ for $q\in[q^*,1/\sqrt[3]{2})$. Moreover, if $q<1/\sqrt[3]{4}$, then $A(q,q^2;q^3)$ is a Cantor set.

Unfortunately, the topological properties of $A(q,q^2;q^3)$ are not known for $1/\sqrt[3]{4}\leq q<q^*$. The methods used by the authors in [1] do not work since the elements of $\Sigma=\{0,q^2,q,q+q^2\}$ and the self-similarity ratio q^3 depend on the same parameter q.

Remark. If $x_n=q^n$ is a geometric sequence with $0<q\leq 1/2$ and $\sigma\in S_\infty$ is regular (see Corollary 5.4 for details) we have $\dim_H(A(x_n))=\dim_H(A(x_n,x_{\sigma(n)}))$. This is not longer true for $q>1/2$ and regular permutations σ. Note that $\dim_H(A(x_n))=1$ for $q>1/2$. On the other hand if $\sigma=(1,2)(3,4)\dots$, then $\dim_H(A(x_n,x_{\sigma(n)}))=-\log 2/\log q$ for $q\leq 1/\sqrt{2}$ and $\dim_H(A(x_n,x_{\sigma(n)}))=2$ otherwise. If $\sigma=(1,2,3)(4,5,6)\dots$, then $\dim_H(A(x_n,x_{\sigma(n)}))=1-\log 2/(3\log q)$ for $q\in[q^*,1/\sqrt[3]{2})$, and $\dim_H(A(x_n,x_{\sigma(n)}))=2$ for $q\geq 1/\sqrt[3]{2}$.

Now, using the same method one can prove the following:

Proposition 3.2. *Let $q\in(0,1)$ and let $\sigma=(1,\dots,k)(k+1,\dots,2k)\dots$. Then*
(1) $A(q^n,q^{\sigma(n)})$ is contained in the parallelogram with vertices

$$(0,0),(\frac{q^k}{1-q^k},\frac{q}{1-q^k});(\frac{q}{1-q}-\frac{q^k}{1-q^k},\frac{q^2}{1-q}-\frac{q^{k+1}}{1-q^k})\ \text{and}\ (\frac{q}{1-q},\frac{q}{1-q})$$

(2) $A(q^n,q^{\sigma(n)})=A_{\rm bot}+A_{\rm top}$, where

$$A_{\rm top}=\{\sum_{n=1}^\infty\varepsilon_{nk}(q^{nk},q^{nk-k+1}):\varepsilon_n=0,1\}\ \text{and}\ A_{\rm bot}=\{\sum_{n=0}^\infty\sum_{i=1}^{k-1}\varepsilon_{nk+i}(q^{nk+i},q^{nk+i+1}):\varepsilon_n=0,1\}.$$

(3) $A_{\rm bot}$ lays on the line $y=qx$, $A_{\rm top}$ lays on the line $y=x/q^{k-1}$, and these two lines contain two edges of the parallelogram;
(4) $A(q^n,q^{\sigma(n)})$ is linearly isomorphic to the product $A(q,q^2,\dots,q^{k-1};q^k)\times A(q^k;q^k)$ of achievement sets of a multigeometric series $(q,q^2,\dots,q^{k-1};q^k)$ and a geometric series $(q^k;q^k)$ via $(x,y)\mapsto(x+qy,q^{k-1}x+y)$;
(5) the set $A_{\rm bot}$ is a Cantor set if $q<\sqrt[k]{1/2^{k-1}}$; in this case the parallelogram is a Moran's open set for $\{T_\varepsilon:\varepsilon\in\{0,1\}^k\}$, and consequently $\dim_H(A(q^n,q^{\sigma(n)}))=-\frac{\log 2}{\log q}$.
(6) the set $A_{\rm bot}$ is an interval if $q\geq q^*$ where q^* is the smallest positive solution of the equation $2q^k+q^{k-1}+\dots+q^2=1$;
(7) the set $A_{\rm top}$ is a Cantor set if and only if $q<\sqrt[k]{1/2}$ if and only if $A_{\rm top}$ is not an interval.

Example 4. Let $q = 1/2$ and let $\sigma = (1,3)(2,4)(5,7)(6,8)\dots$. Let $(v_n) = (3q, 2q; q)$ be a multigeometric series. Note that the series $(3,2;1/4)$ is a linear transformation of Guthrie-Nymann sequence, and therefore the achievement set $\mathrm{A}(3,2;1/4)$ is an $\mathcal{M}$-Cantorval. Let $(x,y) \in \mathrm{A}(v_n, v_{\sigma(n)})$. There is a sequence $(\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \varepsilon_n v_n$ and $y = \sum_{n=1}^{\infty} \varepsilon_n v_{\sigma(n)}$. Let

$$x_1 = \varepsilon_1 3q + \varepsilon_2 2q + \varepsilon_5 3q^3 + \varepsilon_6 2q^3 + \dots,\ x_2 = \varepsilon_3 3q^2 + \varepsilon_4 2q^2 + \varepsilon_7 3q^4 + \varepsilon_8 2q^4 + \dots,$$

$$y_1 = \varepsilon_1 3q^2 + \varepsilon_2 2q^2 + \varepsilon_5 3q^4 + \varepsilon_6 2q^4 + \dots,\ y_2 = \varepsilon_3 3q + \varepsilon_4 2q + \varepsilon_7 3q^3 + \varepsilon_8 2q^3 + \dots.$$

Then $x = x_1 + x_2$ and $y = y_1 + y_2$. Note that the point (x_1, y_1) lies on the line $y = qx$, while the point (x_2, y_2) lies on the line $y = x/q$. It shows that any point of $\mathrm{A}(v_n, v_{\sigma(n)})$ can be represented as the algebraic sum $A_{\mathrm{bot}} + A_{\mathrm{top}}$ of two sets $A_{\mathrm{bot}} := \{(x/q, x) : x \in \mathrm{A}(3q^2, 2q^2; q^2)\}$ and $A_{\mathrm{top}} := \{(x, x/q) : x \in \mathrm{A}(3q^2, 2q^2; q^2)\}$. Then the achievement set $\mathrm{A}(v_n, v_{\sigma(n)})$ is linearly isomorphic to the product $\mathrm{A}(3,2;1/4) \times \mathrm{A}(3,2;1/4)$ of two $\mathcal{M}$-Cantorvals.

Example 5. Let $q = 1/\sqrt[3]{4}$ and let $\sigma = (1,3,5)(2,4,6)(7,9,11)(8,10,12)\dots$. Let $(v_n) = (3q, 2q; q)$ be a multigeometric series. Let $(x,y) \in \mathrm{A}(v_n, v_{\sigma(n)})$. There is a sequence $(\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \varepsilon_n v_n$ and $y = \sum_{n=1}^{\infty} \varepsilon_n v_{\sigma(n)}$. Let

$$x_1 = (\varepsilon_1 3q + \varepsilon_2 2q) + (\varepsilon_7 3q^4 + \varepsilon_8 2q^4) + \dots,\ y_1 = (\varepsilon_1 3q^3 + \varepsilon_2 2q^3) + (\varepsilon_7 3q^6 + \varepsilon_8 2q^6) + \dots,$$

$$x_2 = (\varepsilon_3 3q^2 + \varepsilon_4 2q^2 + \varepsilon_5 3q^3 + \varepsilon_6 2q^3) + (\varepsilon_9 3q^5 + \varepsilon_{10} 2q^5 + \varepsilon_{11} 3q^6 + \varepsilon_{12} 2q^6) + \dots,$$
$$y_2 = (\varepsilon_3 3q + \varepsilon_4 2q + \varepsilon_5 3q^2 + \varepsilon_6 2q^2) + (\varepsilon_9 3q^4 + \varepsilon_{10} 2q^4 + \varepsilon_{11} 3q^5 + \varepsilon_{12} 2q^5) + \dots$$

Then $x = x_1 + x_2$ and $y = y_1 + y_2$. Note that the point (x_1, y_1) lies on the line $y = q^2 x$, while the point (x_2, y_2) lies on the line $y = x/q$. It shows that any point of $\mathrm{A}(v_n, v_{\sigma(n)})$ can be represented as the algebraic sum $A_{\mathrm{bot}} + A_{\mathrm{top}}$ of two sets $A_{\mathrm{bot}} := \{(x/q^2, x) : x \in \mathrm{A}(3q^3, 2q^3; q^3)\}$ and $A_{\mathrm{top}} := \{(x, x/q) : x \in \mathrm{A}(3q^2, 2q^2, 3q^3, 2q^3; q^3)\}$. The set A_{bot} is an $\mathcal{M}$-Cantorval for given q.

Now, consider the multigeometric series $(3q^2, 2q^2, 3q^3, 2q^3; q^3)$. Note that $\Delta(\Sigma) = 2q^3$ and $\mathrm{diam}(\Sigma) = 5q^2 + 5q^3$. It is simple to show that $q^3 \geq \frac{\Delta(\Sigma)}{\Delta(\Sigma)+\mathrm{diam}(\Sigma)} = \frac{2q}{5+7q}$. Thus $\mathrm{A}(3q^2, 2q^2, 3q^3, 2q^3; q^3)$ is an interval. Therefore the achievement set $\mathrm{A}(v_n, v_{\sigma(n)})$ is linearly isomorphic to the product of an $\mathcal{M}$-Cantorval and an interval.

4. The orthogonal projection of the achievement sets

For better understanding of the plane achievement set structure we will study its projection onto the line $\tilde{D} = \{(x, -x) : x \in \mathbb{R}\}$. Let $\pi : \mathbb{R}^2 \to \tilde{D}$, given by $\pi(x,y) = (\frac{x-y}{2}, \frac{y-x}{2})$, be the projection onto $\tilde{D}$. It is well-known that $\mathrm{A}(q^n) - \mathrm{A}(q^n)$ is homeomorphic to the Cantor set if $0 < q < 1/3$, and otherwise $\mathrm{A}(q^n) - \mathrm{A}(q^n)$ is an interval, see [3].

Proposition 4.1. *Let $x_n = q^n$ and $\sigma = (1,2)(3,4)(5,6)\dots$. Then*

$$\pi(\mathrm{A}(x_n, x_{\sigma(n)})) = \{(x, -x) : x \in \frac{1-q}{2q}\mathrm{A}(q^{2k}) - \frac{1-q}{2q}\mathrm{A}(q^{2k})\}.$$

In particular, $\pi(\mathrm{A}(x_n, x_{\sigma(n)}))$ is homeomorphic to the Cantor set iff $q < 1/\sqrt{3}$; otherwise it is an interval.

Proof. We have

$$\pi(\mathrm{A}(x_n, x_{\sigma(n)})) = \{\sum_{n=1}^{\infty} \varepsilon_n (\frac{x_n - x_{\sigma(n)}}{2}, \frac{x_{\sigma(n)} - x_n}{2}) : \varepsilon \in \{0,1\}^{\mathbb{N}}\}.$$

If $n = 2k-1$, then $x_n - x_{\sigma(n)} = q^{2k-1} - q^{2k}$, and if $n = 2k$, then $x_n - x_{\sigma(n)} = q^{2k} - q^{2k-1}$. Then the projection of $\pi(\mathrm{A}(x_n, x_{\sigma(n)}))$ on the first coordinate equals

$$\{\sum_{k=1}^{\infty} \varepsilon_{2k-1} \frac{q^{2k-1} - q^{2k}}{2} + \sum_{k=1}^{\infty} \varepsilon_{2k} \frac{q^{2k} - q^{2k-1}}{2} : \varepsilon_n \in \{0,1\}^{\mathbb{N}}\} =$$

$$= \{\frac{1-q}{2q}\sum_{k=1}^{\infty}\varepsilon_{2k-1}q^{2k} - \frac{1-q}{2q}\sum_{k=1}^{\infty}\varepsilon_{2k}q^{2k} : \varepsilon_n \in \{0,1\}^{\mathbb{N}}\} =$$
$$= \frac{1-q}{2q}\,\mathrm{A}(q^{2k}) - \frac{1-q}{2q}\,\mathrm{A}(q^{2k}).$$

The set $\frac{1-q}{2q}\,\mathrm{A}(q^{2k}) - \frac{1-q}{2q}\,\mathrm{A}(q^{2k})$ has the same topological properties as $\mathrm{A}(q^{2k}) + \mathrm{A}(q^{2k})$. Indeed, it is equal to $\frac{1-q}{2q}\,\mathrm{A}(q^2, -q^2, q^4, -q^4, \dots)$, so by the well-known properties of achievement sets in the real line, it is the translation of the set $\frac{1-q}{2q}\,\mathrm{A}(q^2, q^2, q^4, q^4, \dots)$, and consequently, it is the affine image of $\mathrm{A}(q^2, q^2, q^4, q^4, \dots)$. Such set is an interval for $q^2 \geq 1/3$ and it is homeomorphic to the Cantor set if $q^2 < 1/3$. Hence, for $q < \sqrt{3}/3$ the projection of $\mathrm{A}(q^n, q^{\sigma(n)})$ is a Cantor set and for $q \geq \sqrt{3}/3$ it is an interval. □

In the same way as Proposition 4.1, one can prove the following:

Theorem 4.2. *Let σ_c be a regular permutation generated by a cycle c of the length m. Then the orthogonal projection of $\mathrm{A}(q^n, q^{\sigma(n)})$ on $\tilde{D}$ is the affine image of a set*

$$\mathrm{A}(q^m, q\cdot q^m, q^2\cdot q^m, \dots, q^{m-2}\cdot q^m, (1+q+\dots+q^{m-2})q^m,$$
$$(q^m)^2, q\cdot(q^m)^2, q^2\cdot(q^m)^2, \dots, q^{m-2}\cdot(q^m)^2, (1+q+\dots+q^{m-2})(q^m)^2, \dots).$$

We are interested in the problem how topological properties of this set depend on q. The sequence x, which achievement set is the orthogonal projection from Theorem 4.2, can be written as $x = (1, q, q^2, \dots, q^{m-2}, 1 + q + \dots + q^{m-2}; q^m)$. We have

$$\Sigma = \{1, q, q^2, \dots, q^{m-2}, 1+q+\dots+q^{m-2}, q+q^2, \dots, 2+2q+\dots+2q^{m-2}\}.$$

Thus $|\Sigma| \leq 2^m - 1$. Therefore for $q^m < \frac{1}{2^m-1}$ the achievement set $\mathrm{A}(x)$ is a Cantor set with measure zero. If $q > \frac{1}{2}$, then $\Delta(\Sigma) = q^{m-2}$ and the greatest gap is the first one in Σ. By (1), $\mathrm{A}(x)$ is an interval if and only if

$$q^m \geq \frac{q^{m-2}}{2+2q+\dots+2q^{m-3}+3q^{m-2}}$$

what means that q is not less than the positive solution q_m of the equation $3q^m + 2q^{m-1} + \dots + 2q^2 = 1$. Observe that for $m = 2$ we have $q_2 = 1/\sqrt{2}$ and we obtain the dichotomy from Proposition 4.1: $\mathrm{A}(x)$ is either a Cantor set or an interval. For $m > 2$ we have $q_m > \sqrt[m]{\frac{1}{2^m-1}}$.

Unfortunately, as in the case of Example 3 and Proposition 3.2, we do know almost nothing on the topological properties of the set $\mathrm{A}(x)$ for $\sqrt[m]{\frac{1}{2^m-1}} \leq q < q_m$. We only know that it is not a finite union of intervals. At the end let us observe that if $|\Sigma| < 2^m - 1$, the set $\mathrm{A}(x)$ is an interval.

5. Achievement sets in Banach spaces

Let X be a Banach space. Let $X_1, X_2, \dots$ be finite subsets of X such that a series $\sum_{n=1}^{\infty} y_n$ is absolutely convergent for any choice $y_n \in X_n$. By $\mathrm{A}((X_n)_{n=1}^{\infty})$ we denote the set $\{\sum_{n=1}^{\infty} y_n : y_n \in X_n\}$. Put $Y = \prod_{i\geq 1} X_i$. If one consider X_i with the discrete topology, then Y, considered with the product topology, is homeomorphic to the Cantor set. Let $F : Y \to \mathrm{A}((X_n)_{n=1}^{\infty})$ be given by $F((y_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} y_n$.

Lemma 5.1. *The function F is continuous. In particular $\mathrm{A}((X_n)_{n=1}^{\infty})$ is compact.*

Proof. Let $\delta > 0$. Since a series $\sum_{n=1}^{\infty} y_n$ is absolutely convergent for any choice of $y_n \in X_n$, the series of non-negative numbers $\sum_{n=1}^{\infty} d(0, X_n)$ is convergent where $d(0, X_n) = \max\{\|y\| : y \in X_n\}$ is a distance from X_n to zero. Find N such that $\sum_{n=N+1}^{\infty} d(0, X_n) < \delta/2$. Let U_N be a basic open neighborhood of $(y_n)_{n=1}^{\infty}$ of the form $U_N = \{(x_n)_{n=1}^{\infty} \in Y : x_n = y_n \text{ for } n \leq N\}$. Then

$$\|\sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} y_n\| \leq \|\sum_{n=N+1}^{\infty} x_n\| - \|\sum_{n=N+1}^{\infty} y_n\| \leq 2\sum_{n=N+1}^{\infty} d(0, X_n) < \delta$$

for every $(x_n)_{n=1}^\infty \in U_N$. Thus $F(U_N) \subset B(\sum_{n=1}^\infty y_n, \delta)$, which means that F is continuous. □

Let $\mathcal{R}_i = \{\sum_{n=i+1}^\infty y_n : y_n \in X_n\} = \mathrm{A}((X_n)_{n=i+1}^\infty)$. By Lemma 5.1, the set $\mathcal{R}_i$ is compact. Therefore it is bounded in X and the following number is well-defined

$$R_i := \inf\{r : \mathcal{R}_i \subset B(x, r) \text{ for some } x \in X\}.$$

Let $t_i = \min\{\|x - y\| : x \neq y, x, y \in X_1 + \cdots + X_i\}$. The following is the main result of this section.

Theorem 5.2. *(i) If $t_i > 2R_i$ for every $i \in \mathbb{N}$, then $\mathrm{A}((X_n)_{n=1}^\infty)$ is a Cantor set.*
(ii) $\dim_H(\mathrm{A}((X_n)_{n=1}^\infty)) \leq \liminf_{i\to\infty}(-\frac{\log n_i}{\log R_i})$, where $n_i = |X_1 + X_2 + \cdots + X_i|$;
(iii) Assume that $X = \mathbb{R}^m$ and $t_i > 2R_i$ for every $i \in \mathbb{N}$. If $\inf \frac{R_{i+1}}{R_i} > 0$, then $\dim_H(\mathrm{A}((X_n)_{n=1}^\infty)) = \liminf_{i\to\infty}(-\frac{\log n_i}{\log R_i})$.

Proof. (i) For any $i \in \mathbb{N}$ find R_i' with $R_i < R_i' < t_i/2$ and $v_i \in X$ such that $\mathcal{R}_i \subset B(v_i, R_i')$. Note that $\mathrm{A}((X_n)_{n=1}^\infty) \subset \bigcup\{\overline{B}(x + v_i, R_i') : x \in X_1 + \cdots + X_i\}$ for every $i \in \mathbb{N}$. To see it, fix $y_n \in X_n$. Then $\sum_{n=1}^\infty y_n = x + \sum_{n=i+1}^\infty y_n$ where $x = \sum_{n=1}^i y_n$. Moreover $x \in X_1 + \cdots + X_i$ and $\sum_{n=i+1}^\infty y_n \in \mathcal{R}_i \subset B(v_i, R_i')$. Thus $\sum_{n=1}^\infty y_n \in \overline{B}(x + v_i, R_i')$. Therefore

$$\mathrm{A}((X_n)_{n=1}^\infty) \subset \bigcap_{i\geq 1} \bigcup\{\overline{B}(x + v_i, R_i') : x \in X_1 + \cdots + X_i\}.$$

Now we will show that the function F is one-to-one. Take two distinct points $(y_n), (y_n') \in Y$. Let $i = \min\{n : y_n \neq y_n'\}$. Then $F(y_n) \in \overline{B}(y_1 + \cdots + y_i + v_i, R_i')$, $F(y_n') \in \overline{B}(y_1' + \cdots + y_i' + v_i, R_i')$ and $t_i \leq \|(y_1 + \cdots + y_i) - (y_1' + \cdots + y_i')\|$. Since $R_i' < t_i/2$, then $\overline{B}(y_1 + \cdots + y_i + v_i, R_i') \cap \overline{B}(y_1' + \cdots + y_i' + v_i, R_i') = \emptyset$. Thus F as a continuous bijection is a homeomorphism. Therefore $\mathrm{A}((X_n)_{n=1}^\infty)$ is a Cantor set.

(ii) Let $s > \liminf_{i\to\infty}(-\frac{\log n_i}{\log R_i})$. We need to show that $\mathcal{H}^s\big(\mathrm{A}((X_n)_{n=1}^\infty)\big) < \infty$. Let $0 < \delta < 1$. Find $i \geq 1$ such that $-\frac{\log n_i}{\log R_i} < s$ and $R_i < \delta$. Next, find t such that $-\frac{\log n_i}{\log R_i} < t < s$ and $n_i^{-1/t} < \delta$. Then $R_i < n_i^{-1/t}$. Put $R_i' = n_i^{-1/t}$. Then $\{\overline{B}(x + v_i, R_i') : x \in X_1 + X_2 + \cdots + X_i\}$ is a δ-covering of $\mathrm{A}((X_n)_{n=1}^\infty)$. Then for this covering we have

$$\sum_{x\in X_1+X_2+\cdots+X_i} \operatorname{diam}(\overline{B}(x + v_i, R_i'))^s = n_i(2R_i')^s < 2^s n_i (R_i')^t = 2^s.$$

Therefore $\mathcal{H}^s_\delta\big(\mathrm{A}((X_n)_{n=1}^\infty)\big) < 2^s$ for every $\delta > 0$. Hence $\mathcal{H}^s\big(\mathrm{A}((X_n)_{n=1}^\infty)\big) \leq 2^s < \infty$.

(iii) Let $s < \liminf_{i\to\infty} -\frac{\log n_i}{\log R_i}$. There is i_0 such that $-\frac{\log n_i}{\log R_i} > s$ for every $i \geq i_0$. Therefore for every $i \geq i_0$ there is R_i' such that $R_i < R_i' < \min\{t_i/2, 2R_i\}$ and $-\frac{\log n_i}{\log R_i'} > s$. For $i < i_0$ take any R_i' with $R_i < R_i' < t_i$. Since $\inf \frac{R_{i+1}}{R_i} > 0$, then $\inf \frac{R_{i+1}'}{R_i'} \geq \inf \frac{R_{i+1}}{2R_i} > 0$. Since F is one-to-one, $n_i = |X_1 + \cdots + X_i| = |X_1| \cdots |X_i|$.

For $i \geq 1$ let μ_i be a uniform probability distribution on a finite set X_i, that is $\mu_i(\{t\}) = 1/|X_i|$ for every $t \in X_i$. Let $\mu = \prod_{i\geq 1} \mu_i$ be a probability product measure on $Y = \prod_{i\geq 1} X_i$. Let λ be a probability measure on $\mathrm{A}((X_n)_{n=1}^\infty)$ defined as $\lambda(E) = \mu(F^{-1}(E))$. Since F is one-to-one, for $x \in \mathrm{A}((X_n)_{n=1}^\infty)$, $x = \sum_{i=1}^\infty x_i$, $x_i \in X_i$ we have

$$\lambda(\overline{B}(v_i + \sum_{n=1}^i x_n, R_i')) = \mu(F^{-1}(\overline{B}(v_i + \sum_{n=1}^i x_n, R_i'))) = \mu(\{x_1\} \times \ldots \{x_i\} \times \prod_{k>i} X_k) = \frac{1}{|X_1| \cdot |X_2| \cdots |X_i|}.$$

We will need the following.

Claim. *There is $M > 0$ such that for any $\rho > 0$ there is $i \in \mathbb{N}$ such that $\rho > R_i'$ and*

$$|\{x \in X_1 + X_2 + \cdots + X_i : B \cap \overline{B}(x + v_i, R_i') \neq \emptyset\}| \leq M$$

for every ball B with radius ρ.

Proof of the Claim. Let B be a ball with radius ρ and let i be the smallest natural number such that $R_i' < \rho$. Let B' be a ball, concentric with B with radius 3ρ. The number M is not greater than the number M' of balls

$\overline{B}(x+v_i,R_i')$, $x\in X_1+\dots+X_i$, contained in B' which, in turn, fulfills the inequality $M'(R_i')^m<3^m\rho^m$ (since $X=\mathbb{R}^m$, the sum of volumes of $\overline{B}(x+v_i,R_i')$ cannot exceed the volume of B'). Put $\tau:=\inf\frac{R_{i+1}'}{R_i'}>0$. Then

$$M<\left(\frac{3\rho}{R_i'}\right)^m<\left(\frac{3R_{i-1}'}{R_i'}\right)^m\leq\left(\frac{3}{\tau}\right)^m.$$

In both cases M does not depend on i.

Let $0<\delta<\min\{R_{i_0}',1\}$ and $\varepsilon>0$. Take a δ-covering U_p of $\mathrm{A}((X_n)_{n=1}^\infty)$ such that

$$\sum_{p=1}^\infty\operatorname{diam}(U_p)^s<\mathcal{H}_\delta^s(\mathrm{A}((X_n)_{n=1}^\infty))+\varepsilon.$$

We can cover $\mathrm{A}((X_n)_{n=1}^\infty)$ with balls B_p with $U_p\subset B_p$ and $\operatorname{diam}(B_p)<2\operatorname{diam}(U_p)$. Then

$$\sum_{p=1}^\infty\operatorname{diam}(U_p)^s\geq2^{-s}\sum_{p=1}^\infty\operatorname{diam}(B_p)^s.$$

Using the Claim for $\rho=\operatorname{diam}(B_p)/2$ we find i such that $\lambda(B_p)\leq\sum\lambda(\overline{B}(x+v_i,R_i'))$, where the sum is over all balls $\overline{B}(x+v_i,R_i')$ such that $B_p\cap\overline{B}(x+v_i,R_i')\neq\emptyset$, $x\in X_1+\dots+X_i$ and $\operatorname{diam}(B_p)>2R_i'$. Moreover

$$\lambda(B_p)\leq\frac{M}{|X_1|\cdot|X_2|\cdots|X_i|}=\frac{M}{n_i}<M(R_i')^s<M\left(\frac{\operatorname{diam}(B_p)}{2}\right)^s.$$

Therefore

$$1\leq\sum_{p=1}^\infty\lambda(B_p)<\frac{M}{2^s}\sum_{p=1}^\infty(\operatorname{diam}(B_p))^s\leq M\sum_{p=1}^\infty\operatorname{diam}(U_p)^s<M(\mathcal{H}_\delta^s(\mathrm{A}(X_n)_{n=1}^\infty)+\varepsilon).$$

This shows that $\mathcal{H}^s(\mathrm{A}(X_n)_{n=1}^\infty)>0$, and consequently by (ii) we have $\dim_H(\mathrm{A}(X_n)_{n=1}^\infty)=\liminf_{i\to\infty}(-\frac{\log n_i}{\log R_i})$. □

Let us remark that one can easily generalize Theorem 5.2 for the achievement set in a Polish Abelian group considered with an invariant metric. Then the condition $X=\mathbb{R}^m$ in part (iii) of Theorem 5.2 can be change to the following condition for groups with an invariant metric: there is a constant $C>0$ such that for every $R>0$ any ball of radius R contains at most C many pairwise disjoint balls of radius $R/2$. This condition is fulfilled for $\mathbb{R}^m$ and in $\mathbb{Z}_2^\mathbb{N}$ considered with an invariant metric $d((x_n),(y_n))=2^{-\min\{n:x_n\neq y_n\}}$. It fails in infinitely dimensional Banach spaces and in the group $\prod_{n=2}^\infty\mathbb{Z}_n$ endowed with an invariant metric $d((x_n),(y_n))=2^{-\min\{n:x_n\neq y_n\}}$.

Now we present several applications of Theorem 5.2.

Let $\sum_{n=1}^\infty x_n$ be an absolutely convergent series in a Banach space X. For an increasing sequence $1=k_0<k_1<k_2<\dots$ of natural numbers we define

$$X_i=\{\sum_{n=k_{i-1}}^{k_i-1}\varepsilon_nx_n:\varepsilon_n=0,1\},$$

$$t_i=\min\{\|x-y\|:x\neq y\text{ and }x,y\in X_1+X_2+\dots+X_i\}$$

and

$$r_i=\max\{\|\sum_{n=k_i}^\infty\varepsilon_nx_n\|:\varepsilon_n=-1,0,1\}=\operatorname{diam}(\mathcal{R}_i).$$

Corollary 5.3. *(i) If $t_i>r_i$ for every $i\in\mathbb{N}$, then $\mathrm{A}(x_n)$ is a Cantor set;*
(ii) $\dim_H(\mathrm{A}(x_n))\leq\liminf_{i\to\infty}(-\frac{\log n_i}{\log r_i})$, where $n_i=|X_1+X_2+\dots+X_i|$;
(iii) Assume that $X=\mathbb{R}^m$ and $t_i>r_i$ for every $i\in\mathbb{N}$. If $\inf\frac{r_{i+1}}{r_i}>0$, then $\dim_H(\mathrm{A}(x_n))=\liminf_{i\to\infty}(-\frac{\log n_i}{\log r_i})$.

Proof. It is enough to show that $r_i = 2R_i$. Clearly $r_i \leq 2R_i$. Note that

$$r_i = \max\{\| \sum_{n=k_i}^{\infty} \varepsilon_n x_n \| : \varepsilon_n = -1, 1\}.$$

Indeed, let $y = \sum_{n=k_i}^{\infty} \varepsilon_n x_n$ and $\varepsilon_n = 0$ for some $n \geq k_i$. Then $\|y\| \leq \|y + x_n\|$ or $\|y\| \leq \|y - x_n\|$. This means that we enlarge the value of $\| \sum_{n=k_i}^{\infty} \varepsilon_n x_n \|$ changing each $\varepsilon_n = 0$ to ± 1.

Let $\overline{x} = \frac{1}{2} \sum_{n=k_i}^{\infty} x_n$. Then for every $\varepsilon_n \in \{-1, 1\}^{\mathbb{N}}$ we have

$$R_i \leq \max_{(\varepsilon_n)} \|\overline{x} - \sum_{n=k_i}^{\infty} \varepsilon_n x_n \| = \max_{(\varepsilon_n)} \frac{1}{2} \| \sum_{n=k_i}^{\infty} \varepsilon_n x_n \| = r_i/2.$$

Therefore $r_i = 2R_i$. □

Note that if $\sum_{n=1}^{\infty} x_n$ is quickly convergent, that is if $\|x_n\| > \sum_{i>n} \|x_i\|$, then for $k_i = i$ we have $t_i > r_i$. Therefore the achievement set of a quickly convergent series is a Cantor set. This shows that our result generalizes two first parts of Kakeya Theorem.

Let us go back to the achievement set of perturbed geometric series on the real plane. Let $\sigma = c_1 c_2 c_3 \ldots$ and let $|c_i|$ stand for the length (the cardinality of domain) of c_i.

Corollary 5.4. *(i) Let $0 < q < \sqrt{2}/2$. Then $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) \leq -\log 2/\log q$. In particular the achievement set $\mathrm{A}(q^n, q^{\sigma(n)})$ has Lebesque measure zero.*
(ii) Let $0 < q \leq 1/2$. Assume that there is M such that $|c_i| \leq M$ for every $i \geq 1$. Then $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) = -\log 2/\log q$.
(iii) Let $q = 1/2$. Then $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) = 1$.

Note that for $q \geq \sqrt{2}/2$ the estimation $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) \leq -\log 2/\log q$ is valid but trivial. This follows from the fact that $\dim_H(B) \leq 2 < -\log 2/\log q$ for any compact set $B \subset \mathbb{R}^2$.

Proof. (i) We will use the notation from Corollary 5.3. Let $k_0 = 1$ and $k_{n+1} = k_n + |c_n|$. Then $X_i = \{\sum_{n=k_{i-1}}^{k_i - 1} \varepsilon_n (q^n, q^{\sigma(n)}) : \varepsilon_n = 0, 1\}$. Since $\sigma(n) \geq n$ for every $n \geq k_i$, $\sigma(\{n, n+1, \ldots\}) = \{n, n+1, \ldots\}$ and

$$r_i = \mathrm{diam}(\mathcal{R}_i) = \mathrm{diam}\{\sum_{n \geq k_i} \varepsilon_n (q^n, q^{\sigma(n)}) : \varepsilon_n = 0, 1\} = \| \sum_{n \geq k_i} (q^n, q^{\sigma(n)}) \| = \|(\sum_{n \geq k_i} q^n, \sum_{n \geq k_i} q^{\sigma(n)})\| =$$

$$\|(\frac{q^{k_i}}{1-q}, \frac{q^{k_i}}{1-q})\| = \frac{q^{k_i}}{1-q} \|(1,1)\| = \frac{\sqrt{2} q^{k_i}}{1-q}.$$

Note that $n_i = |X_1 + \cdots + X_i| \leq 2^{|c_1| + \ldots |c_i|} = 2^{k_i - 1}$. By Corollary 5.3 we obtain

$$\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) \leq \liminf_{i \to \infty} -\frac{\log n_i}{\log r_i} \leq \liminf_{i \to \infty} -\frac{(k_i - 1)\log 2}{\log \frac{\sqrt{2}}{1-q} + k_i \log q} = -\frac{\log 2}{\log q}.$$

(ii) Note that

$$\frac{r_{i+1}}{r_i} = \frac{q^{k_{i+1}}}{q^{k_i}} = \frac{q^{k_i + |c_i|}}{q^{k_i}} = q^{|c_i|} \geq q^M > 0.$$

Let $m(k) = |c_1| + \cdots + |c_k|$. Note that $\mathrm{A}((q^n, q^{\sigma(n)})_{n \geq m(k)+1})$ is contained in the square $S_k = [0, \frac{q^{m(k)+1}}{1-q}]^2$. Then $\mathrm{A}(q^n, q^{\sigma(n)})$ is covered with $2^{m(k)}$ squares of the form

$$C_\varepsilon = (\sum_{n=1}^{m(k)} \varepsilon_n q^n, \sum_{n=1}^{m(k)} \varepsilon_n q^{\sigma(n)}) + S_k$$

where $\varepsilon = (\varepsilon_n) \in \{0,1\}^{m(k)}$. Note that for each k the family $\{C_\varepsilon : \varepsilon \in \{0,1\}^{m(k)}\}$ of squares is pairwise disjoint if $q < 1/2$. Then $r_i \leq \frac{\sqrt{2} q^{m(k)+1}}{1-q} < \sqrt{2} q^{m(k)} \leq t_i$. Therefore by Corollary 5.3 we have $\dim_H(\mathrm{A}(q^n, q^{\sigma(n)})) = -\log 2/\log q$.

(iii) By (i) we already know that $\dim_H(\mathrm{A}(1/2^n, 1/2^{\sigma(n)})) \leq 1$. Let $\delta > 0$, $s > 0$. Note that for each k the family $\{C_\varepsilon : \varepsilon \in \{0,1\}^{m(k)}\}$ of squares is pairwise non-overlapping if $q = 1/2$. Let $\{U_i\}$ be δ-covering of $\mathrm{A}(1/2^n, 1/2^{\sigma(n)})$ such that

$$\sum_{i=1}^{\infty} \mathrm{diam}(U_i) < \mathcal{H}^1(\mathrm{A}(1/2^n, 1/2^{\sigma(n)})) + s.$$

By the proof of Theorem 5.2 the measure $\mathcal{H}^1(\mathrm{A}(1/2^n, 1/2^{\sigma(n)}))$ is finite, and therefore $\sum_{i=1}^{\infty} \mathrm{diam}(U_i) < \infty$. We can cover $\mathrm{A}(1/2^n, 1/2^{\sigma(n)})$ with open balls B_i with $\mathrm{diam}(B_i) < 2\,\mathrm{diam}(U_i)$. Since $\mathrm{A}(1/2^n, 1/2^{\sigma(n)})$ is compact, there is t such that $B_1, \ldots, B_t$ cover $\mathrm{A}(1/2^n, 1/2^{\sigma(n)})$ and $\sum_{i=t+1}^{\infty} \mathrm{diam}(B_i) < s$. Let $\delta' = \min\{\mathrm{diam}(B_i) : i \leq t\}$. As in (i) we find k such that $\sqrt{2}/2^{m(k)} < \delta'$. Let $J_1, \ldots, J_n$ be subsets of $\{0,1\}^{m(k)}$ defined as $J_i = \{\varepsilon : C_\varepsilon \cap B_i \neq \emptyset\}$. Since the family $\{C_\varepsilon : \varepsilon \in \{0,1\}^{m(k)}\}$ is pairwise non-overlapping, $\bigcup_{i<t} J_i = \{0,1\}^{m(k)}$. Let B_i' be an open ball concentric with B_i with $\mathrm{diam}(B_i') = 3\,\mathrm{diam}(B_i)$. Then $C_\varepsilon \subset B_i'$ for every $\varepsilon \in J_i$. If $\varepsilon, \varepsilon' \in \{0,1\}^{m(k)}$ are distinct, then

$$\left| \sum_{n=1}^{m(k)} \varepsilon(n) 1/2^n - \sum_{n=1}^{m(k)} \varepsilon'(n) 1/2^n \right| \geq 1/2^{m(k)}.$$

Therefore the set $\{\pi_1(C_\varepsilon) : \varepsilon \in \{0,1\}^{m(k)}\}$ of projection on the first coordinate of sets C_ε consists of non-overlapping intervals on the line. Similarly $\{\pi_2(C_\varepsilon) : \varepsilon \in \{0,1\}^{m(k)}\}$ consists of non-overlapping intervals on the line. Then

$$|J_i|\,\mathrm{diam}(C_\varepsilon) \leq \mathrm{diam}(B_i') = 3\,\mathrm{diam}(B_i),$$

and consequently

$$\sum_{\varepsilon} \mathrm{diam}(C_\varepsilon) \leq \sum_{i=1}^{t} \sum_{\varepsilon \in J_i} \mathrm{diam}(C_\varepsilon) \leq \sum_{i=1}^{t} \sum_{\varepsilon \in J_i} |J_i|\,\mathrm{diam}(C_\varepsilon) \leq$$

$$\leq \sum_{i=1}^{t} 3\,\mathrm{diam}(B_i) \leq \sum_{i=1}^{\infty} 3\,\mathrm{diam}(B_i) \leq \sum_{i=1}^{\infty} 6\,\mathrm{diam}(U_i) < 6(\mathcal{H}^1(\mathrm{A}(1/2^n, 1/2^{\sigma(n)})) + s).$$

Therefore $\mathcal{H}^1(\mathrm{A}(1/2^n, 1/2^{\sigma(n)})) > 0$. Consequently $\dim_H(\mathrm{A}(1/2^n, 1/2^{\sigma(n)})) = 1$. □

Proposition 5.5. *The set of all permutations σ of the form $c_1 c_2 c_3 \ldots$ is residual in S_∞.*

Proof. Note that the set

$$X_n := \big\{\sigma \in S_\infty : \exists k \geq n\ \sigma(\{1, \ldots, k\}) = \{1, \ldots, k\}\big\}$$

is open and dense in S_∞. Next, see that

$$\bigcap_{n=1}^{\infty} X_n = \big\{\sigma \in S_\infty : \forall n \exists k \geq n\ \sigma(\{1, \ldots, k\}) = \{1, \ldots, k\}\big\} =$$

$$= \big\{\sigma \in S_\infty : \exists k_1 < k_2 < \ldots \forall i\ \sigma(\{k_i + 1, \ldots, k_{i+1}\}) = \{k_i + 1, \ldots, k_{i+1}\}\big\}$$

which is in turn the set of all permutations σ of the form $c_1 c_2 c_3 \ldots$. □

Corollary 5.6. *The assertion of Corollary 5.4(iii) holds for almost every, in the sense of Baire category, permutation $\sigma \in S_\infty$.*

Problem 5.7. *Let $x_n = q^n$, $0 < q \leq 1/2$. Let $\sigma \in S_\infty$ be an arbitrary permutation. Is it true that $\dim_H(\mathrm{A}(x_n, x_{\sigma(n)})) = \frac{\log 2}{\log(1/q)}$?*

If X is an infinitely dimensional Banach space, the achievement set $\mathrm{A}(x_n)$, as a compact set, is small in many senses (nowhere dense, porous, Haar null etc.) in X. The next example shows that it can be big in other meaning, for example it can be homeomorphic to the Hilbert cube and in particular of infinite dimension.

Example 6. Let $E_1, E_2, \dots$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets. Let $E_i = \{m_{i1} < m_{i2} < m_{i3} < \dots\}$. Let $x_n \in c_0$ be defined as follows: $x_n = \frac{1}{2^{i+k}} \cdot e_i$ if $n = m_{ik}$ where $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ otherwise. Then $\sum_{n=1}^{\infty} \|x_n\| = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^{i+k}} = 1$. Then the achievement set A equals $[0,1] \times [0,1/2] \times [0,1/4] \times \dots$, and consequently it is homeomorphic to the Hilbert cube $[0,1]^{\mathbb{N}}$.

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