DESCRIPTIVE SET-THEORETICAL PROPERTIES OF AN ABSTRACT DENSITY OPERATOR

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Abstract. Let $\mathcal{K}(\mathbb{R})$ stand for the hyperspace of all nonempty compact sets on the real line and let $d^{\pm}(x,E)$ denote the (right- or left-hand) Lebesgue density of a measurable set $E \subset \mathbb{R}$ at a point $x \in \mathbb{R}$. In [3] it was proved that

$$\{K \in \mathcal{K}(\mathbb{R}) : \forall x \in K (d^{+}(x,K) = 1 \text{ or } d^{-}(x,K) = 1)\}$$

is $\mathbf{\Pi}^1_1$–complete. In this paper we define an abstract density operator $\mathbb{D}^{\pm}$ and we generalize the above result. Some applications are included.

In descriptive set theory the following phenomenon is known – several kinds of sets with a simple description can have extremely high complexity, for example they can be $\mathbf{\Pi}^1_1$–complete. Many classical examples of such sets can be found in the Kechris monograph [5]. They appear naturally in topology, in the Banach spaces theory, the theory of real functions, and in other branches of mathematics. Descriptive properties of families of compact sets in the hyperspace of all nonempty compact sets have been considered in many papers (see [7], [8] and [6]). Recently, the notion of porosity has been studied from this point of view (see [3], [9], [12], [15]).

The motivation for this note comes from our previous paper [3] in which the operators of density and porosity are studied with the use of methods of descriptive set theory. Namely, it is proved there that the family NBP of all nowhere bilaterally porous compact sets forms a $\mathbf{\Pi}^1_1$–complete subset of the space $\mathcal{K}(\mathbb{R})$ of all nonempty compact subsets of $\mathbb{R}$ with the Vietoris topology. In [3] it is remarked that an analogous fact holds true for the operator of Lebesgue density. The both notions of porosity and Lebesgue density show the local smallness (or largeness) of a set at a point. Lebesgue density is a basic notion in classical measure theory and the theory of real functions

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(see [1]). Porosity measures the smallness of sets in normed spaces (or more generally in metric spaces); see [13], [14].

In this paper we generalize the results of [3] to several kinds of densities and porosities on the real line. To do it we define an abstract density operator $\mathbb{D}^{\pm}$ on the real line, and we prove that

$$\{K \in \mathcal{K}(\mathbb{R}) : \forall x \in K(\mathbb{D}^{+}(K,x) = 1 \text{ or } \mathbb{D}^{-}(K,x) = 1)\} \tag{1}$$

is $\mathbf{\Pi}^1_1$–complete.

The paper is organized as follows. In Section 1 we recall the preliminaries from descriptive set theory. In Section 2 we define the operator $\mathbb{D}^{\pm}$, and we discuss the meaning of each axiom defining $\mathbb{D}^{\pm}$. In Section 3 we prove the main theorem. In Section 4 we give examples of density and porosity operators which fulfil the proposed axioms, and in Section 5 we discuss the axiom (A5).

1. Preliminaries

We use standard set-theoretic notation (see [5] or [10]). We denote $\mathbb{N} = \{0, 1, ...\}$. We use the symbol $|\cdot|$ in several different meanings: the absolute value of a real number, the length of an interval, the length of a finite sequence and the cardinality of a set. This will never lead to confusion. A topological space X is Polish if it is completely metrizable and separable. From now on, let X be an uncountable Polish space. A set $A \subset X$ is analytic if it is a projection of a Borel set $B \subset X \times X$ (equivalent definitions of an analytic set can be found in [5, 14.A]). A set $C \subset X$ is coanalytic if $X \setminus C$ is analytic. The classes of analytic and coanalytic sets are denoted by $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, respectively.

By $\mathcal{K}(X)$ we denote the hyperspace of all nonempty compact subsets of X, endowed with the Vietoris topology, i.e. the topology generated by sets $\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ and $\{K \in \mathcal{K}(X) : K \subset U\}$ for any open sets U in X. The Vietoris topology is equal to the topology generated by the Hausdorff metric

$$\rho_H(K, L) = \max(\max_{x \in K} \rho(x, L), \max_{x \in L} \rho(x, K))$$

where $\rho(x, K)$ is the distance from a point x to a set K with respect to the metric ρ on X.

Let $\mathbf{\Gamma}$ be a point-class in the Borel or the projective hierarchies. We say that $A \subset X$ is $\mathbf{\Gamma}$–hard if for any zero-dimensional Polish space Y and any $B \in \mathbf{\Gamma}(Y)$ there exists a continuous function $f : Y \to X$ such that $f^{-1}(A) = B$. If additionally $A \in \mathbf{\Gamma}(X)$, then we say that A is $\mathbf{\Gamma}$-complete. If in the above definition we change the condition of the existence of a continuous function to the condition of the existence of a Borel function with the same property, then we obtain the definition of a Borel–$\mathbf{\Gamma}$–complete set. If $\mathbf{\Gamma}$ is closed under the continuous preimages, then $\mathbf{\Gamma}$–complete sets are the most complicated sets in a class $\mathbf{\Gamma}$ – they belong to $\mathbf{\Gamma}$ but they are not in any class below $\mathbf{\Gamma}$, for example a $\mathbf{\Pi}^1_1$–complete set is not in $\mathbf{\Sigma}^1_1$, and a $\mathbf{\Sigma}^0_\alpha$–complete set is not in $\mathbf{\Pi}^0_\alpha$.

The most standard way to prove the $\mathbf{\Gamma}$–completeness of a given set $B \in \mathbf{\Gamma}(X)$ is the following. We take a set A which is known to be $\mathbf{\Gamma}$–complete in some Polish space Y. It is usually a set with a simple combinatorial structure, convenient to deal with. Next, we find a continuous function $f : Y \to X$ with $f^{-1}(B) = A$. Then it is easy to see that B is $\mathbf{\Gamma}$-complete.

For a nonempty set A, by $A^{<\mathbb{N}}$ we denote the set of all finite sequences (together with the empty sequence $\emptyset$) of elements from A. For a sequence $s = (s(0), ..., s(n-1)) \in A^{<\mathbb{N}}$ and $a \in A$ by $s\hat{\ }a$ we denote the sequence $(s(0), ..., s(n-1), a)$.

For $s = (s(0), ..., s(n-1)) \in A^{<\mathbb{N}}$ and $0 < m \leq n$ put $|s| = n$, $s|m = (s(0), ..., s(m-1))$; additionally $|\emptyset| = 0$ and $s|0 = \emptyset$. A set $T \subset A^{<\mathbb{N}}$ is called a tree on A if the empty sequence is in T, and the following implication holds: $s \in T \Rightarrow s|k \in T$, for every $s \in A^{<\mathbb{N}}$ and every $k < |s|$. The set $\{x \in A^{\mathbb{N}} : \forall n \in \mathbb{N}\ (x(0), x(1), ..., x(n)) \in T\}$ of all infinite branches of a tree T is called a body of T and is denoted by $[T]$. A tree T is called well-founded if $[T] = \emptyset$, in other words, T is well-founded if it has no infinite branch. From now on we will consider only trees on $\mathbb{N}$. A tree can be identified with its characteristic function, hence we can identify the set of all trees Tr with a subset of $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ (this space as a countable product of discrete spaces $\{0,1\}$ is homeomorphic to the Cantor space). Since Tr is a G_δ subset of $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$, we will treat Tr as a Polish space.

Proposition 1. [5, 32.B] *A set WF of all well-founded trees forms a $\mathbf{\Pi}_1^1$–complete subset of Tr.*

Consider a subspace of Tr defined in the following way

$$\widetilde{Tr} = \{T \in Tr : \forall s \in \mathbb{N}^{<\mathbb{N}} \ \ \forall n \in \mathbb{N} \quad (s\hat{\ }n \in T \Rightarrow \forall m \in \mathbb{N} \quad s\hat{\ }m \in T)\} =$$

$$\bigcap_{s\in\mathbb{N}^{<\mathbb{N}}} \bigcap_{n\in\mathbb{N}} (\{T \in Tr : s\hat{\ }n \notin T\} \cup \bigcap_{m\in\mathbb{N}} \{T \in Tr : s\hat{\ }m \in T\}).$$

Hence $\widetilde{Tr}$ is a Polish space as a closed subset of Tr. By $\widetilde{WF}$ denote the set $WF \cap \widetilde{Tr}$.

Proposition 2. [3, Section 3] *$\widetilde{WF}$ is a $\mathbf{\Pi}_1^1$–complete subset of $\widetilde{Tr}$.*

2. Definition of $\mathbb{D}^{\pm}$

The operators of Lebesgue density and porosity are our start point to define an abstract density operator. We will define a right-hand abstract density operator $\mathbb{D}^+$ and a left-hand abstract density operator $\mathbb{D}^-$ by a list of conditions we want them to fulfil, called here the axioms. Using the symbol $\mathbb{D}^{\pm}$ in a formula we mean that something holds simultaneously for the right-hand and the left-hand abstract density operator, simultaneously.

For a given $x \in \mathbb{R}$, the operator $\mathbb{D}^{\pm}$ is defined on some family of Borel sets which will be called the family of admissible sets at x, and denoted by $\mathcal{A}^{\pm}(x)$. In the case of Lebesgue density, the family of admissible sets consists of Borel sets for which the density exists. Analogously, in the case of porosity. Usually, we will not define precisely the family of admissible sets. The number $\mathbb{D}^{\pm}(X, x) \in [0, 1]$ is called a density of a set X at a point x. Writing $\mathbb{D}^{\pm}(X, x)$ we always mean that X is admissible.

One difference between Lebesgue density and porosity is that big sets with respect to density have the density 1, but big sets with respect to porosity have the porosity 0. The following axiom is natural:

(A1) $\forall x \in \mathbb{R}(\mathbb{D}^{\pm}(\mathbb{R}, x) = 1$ and $\mathbb{D}^{\pm}(\emptyset, x) = 0)$.

The operators of density and porosity are monotonic and defined locally. These two properties of $\mathbb{D}^{\pm}$ are described by the following conditions

(A2) $\forall x \in \mathbb{R}\ \forall X, Y \in \mathcal{A}^{\pm}(x)\ (X \subset Y \Rightarrow \mathbb{D}^{\pm}(X,x) \leq \mathbb{D}^{\pm}(Y,x))$.

(A3) $\forall x \in \mathbb{R}\ \forall X \in \mathcal{A}^{+}(x) \cap \mathcal{A}^{-}(x)\ \forall \varepsilon > 0\ [\mathbb{D}^{+}(X,x) = \mathbb{D}^{+}(X \cap (x, x+\varepsilon), x)$ and $\mathbb{D}^{-}(X,x) = \mathbb{D}^{-}(X \cap (x, x-\varepsilon), x)]$.

In particular, (A2) means that a superset of a set with $\mathbb{D}^{\pm}$–density 1 has $\mathbb{D}^{\pm}$-density 1 (i.e. a superset of a set with $\mathbb{D}^{\pm}$–density 1 is admissible).

The next axiom states that we can construct a so-called interval set of $\mathbb{D}^{\pm}$–density 1. A construction of an interval set of density 1 is a useful tool for dealing with different types of densities on the real line. Let sequences (a_n), (b_n) be such that $\forall n \in \mathbb{N}(b_{n+1} < a_n < b_n)$ and $x = \lim_{n\to\infty} a_n$. An interval set of $\mathbb{D}^{+}$–density 1 at a point x is a set of the form $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ with $\mathbb{D}^{+}(\bigcup_{n\in\mathbb{N}}[a_n, b_n], x) = 1$. Let (c_n), (d_n) be such that $\forall n \in \mathbb{N}(c_{n+1} > d_n > c_n)$ and $y = \lim_{n\to\infty} c_n$. An interval set of $\mathbb{D}^{-}$–density 1 at a point y is a set of the form $\bigcup_{n\in\mathbb{N}}[c_n, d_n]$ with $\mathbb{D}^{-}(\bigcup_{n\in\mathbb{N}}[c_n, d_n], y) = 1$. Now, our axiom is the following:

(A4) For every $x \in \mathbb{R}$ there exists an interval set of a $\mathbb{D}^{\pm}$–density 1 at the point x.

The next axiom is less intuitive than the previous ones. It is described by the notion of an infinite game. Consider the following game G^{+}:

Player I:	(a_0, b_0)	(a_1, b_1)	(a_2, b_2)	...
Player II:	c_0	c_1	c_2	...

The rules of G^{+} are the following: $c_i \in (a_i, b_i)$ and $[a_{i+1}, b_{i+1}] \subset (a_i, c_i)$ for each $i \in \mathbb{N}$. Let $x = \lim_{n\to\infty} c_n$. Player I wins if $\bigcup_{n\in\mathbb{N}}[b_{n+1}, c_n]$ is an admissible set with $\mathbb{D}^{+}$–density 1 at x. Otherwise, Player II wins. Now, consider the game G^{-}:

Player I:	(a_0, b_0)	(a_1, b_1)	(a_2, b_2)	...
Player II:	d_0	d_1	d_2	...

This time, the rules are the following: $d_i \in (a_i, b_i)$, $[a_{i+1}, b_{i+1}] \subset (d_i, b_i)$ for each $i \in \mathbb{N}$. Let $y = \lim_{n\to\infty} d_n$. Player I wins if $\bigcup_{n\in\mathbb{N}}[d_n, a_{n+1}]$ is an admissible set with $\mathbb{D}^{-}$–density 1 at y and otherwise, Player II wins. Now, we are ready to state the next axiom:

(A5) Player II has winning strategies in the games G^{+} and G^{-}.

The last axiom has a descriptive set-theoretical character.

(A6) The sets $\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \mathbb{D}^+(K,x) = 1\}$ and $\{(K,x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : \mathbb{D}^-(K,x) = 1\}$ are Borel.

Without (A6) we can prove only that the set in (1) is $\mathbf{\Pi}_1^1$–hard. Thanks to (A6) we are able to prove its $\mathbf{\Pi}_1^1$–completeness.

3. Main theorem

Lemma 3. *Let X be a metric space with the metric ρ. Let $F : X \to \mathcal{K}(\mathbb{R})$ be such that the set*

$$F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \cap U \neq \emptyset\})$$

is open in X, for every set U open in $\mathbb{R}$. Then F is Borel measurable.

Proof. To prove that F is Borel measurable it is enough to show that $F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \subset U\})$ is Borel for every U open in $\mathbb{R}$. It is obvious in the case $U = \mathbb{R}$ or $U = \emptyset$. Let us assume that $\emptyset \neq U \neq \mathbb{R}$ and put

$$V_n = \left\{x \in \mathbb{R} : \rho(x, \mathbb{R} \setminus U) < \frac{1}{n+1}\right\}, \quad n \in \mathbb{N}.$$

Then the set

$$F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \subset U\}) = F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \cap (\mathbb{R} \setminus U) = \emptyset\}) =$$

$$\bigcup_{n\in\mathbb{N}} F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \cap V_n = \emptyset\})$$

is of type F_σ. $\square$

Lemma 4. *Let $\{K_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ be a family of pairwise disjoint closed subintervals of $[0,1]$. Let $X \subset Tr$ (and we consider on X the topology induced from Tr). Define a function $F : X \to \mathcal{K}(\mathbb{R})$ by*

$$F(T) = cl(\bigcup_{s\in T} K_s) \quad \textit{for } T \in X.$$

Then F is Borel measurable.

Proof. Let U be any open set in $\mathbb{R}$. Let $S = \{s \in \mathbb{N}^{<\mathbb{N}} : K_s \cap U \neq \emptyset\}$. If $T \in X$ then $S \cap T \neq \emptyset \iff F(T) \cap U \neq \emptyset$. Hence

$$F^{-1}(\{K \in \mathcal{K}(\mathbb{R}) : K \cap U \neq \emptyset\}) = \bigcup_{s \in S} \{T \in X : s \in T\}$$

is open. By Lemma 3 the function F is Borel measurable. □

Now we are ready to prove a main theorem of the paper which generalizes the results of [3].

Theorem 5. *Assume that $\mathbb{D}^{\pm}$ is an abstract density operator fulfilling (A1)–(A5). Then the set*

$$\mathcal{X} = \{K \in \mathcal{K}(\mathbb{R}) : \forall x \in K (\mathbb{D}^{+}(K, x) = 1 \text{ or } \mathbb{D}^{-}(K, x) = 1)\}$$

is $\mathbf{\Pi}^1_1$–hard. If additionally (A6) holds for $\mathbb{D}^{\pm}$, then $\mathcal{X}$ is $\mathbf{\Pi}^1_1$–complete.

Proof. If $\mathbb{D}^{\pm}$ fulfills the axiom (A6), then it is standard to prove that $\mathcal{X}$ is coanalytic. We need only to show that the axioms (A1)–(A5) imply the $\mathbf{\Pi}^1_1$–hardness of $\mathcal{X}$.

We define, by induction with respect to the length of $s \in \mathbb{N}^{<\mathbb{N}}$, closed subintervals K_s of $[0, 2]$ and real numbers c^s, d^s, p^s_m, r^s_m, for every $m \in \mathbb{N}$, such that the following conditions hold:

(i) $d^s < p^s_m < r^s_m < c^s$ and $r^s_0 - d^s < 1/|s|$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ and $m \in \mathbb{N}$;

(ii) $K_{s\hat{\ }m} = [p^s_m, r^s_m]$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ and $m \in \mathbb{N}$;

(iii) $\bigcup_{m \in \mathbb{N}} [p^s_m, r^s_m]$ is an interval set with $\mathbb{D}^+$–density 1 at d^s, and $\forall m \in \mathbb{N}(r^s_{m+1} < p^s_m < r^s_m)$ for every $s \in \mathbb{N}^{<\mathbb{N}}$;

(iv) $(d^{s\hat{\ }m}, c^{s\hat{\ }m}) \subset (r^s_{m+1}, p^s_m)$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ and $m \in \mathbb{N}$;

(v) for $\alpha \in \mathbb{N}^{\mathbb{N}}$ let x_α be such that $\{x_\alpha\} = \bigcap_{n \in \mathbb{N}} (d^{\alpha|n}, c^{\alpha|n})$, then the set $cl(\bigcup_{s \in \mathbb{N}^{<\mathbb{N}}} K_s)$ has not $\mathbb{D}^{\pm}$–density 1 at x_α.

Let $K_\emptyset = [1, 2]$. By (A5), Player II has winning strategies in the games G^+ and G^-, say σ^+ and σ^-. Let $c^\emptyset \in (0, 1)$ be such that $((0, 1), c^\emptyset) \in \sigma^+$, i.e. let $c^\emptyset$ be an answer of II for $(0, 1)$ according to the strategy σ^+. Next let $d^\emptyset \in (0, c^\emptyset)$ be such that $((0, c^\emptyset), d^\emptyset) \in \sigma^-$. By (A4) there exists an interval set with $\mathbb{D}^+$–density 1 at $d^\emptyset$. By (A3) we may assume that this interval set is contained in $(d^{\emptyset}, \min\{c^{\emptyset}, d^{\emptyset} + 1/2\})$; suppose that it is of the form $\bigcup_{n\in\mathbb{N}}[p_n^{\emptyset}, r_n^{\emptyset}]$ with $\forall n \in \mathbb{N}(r_{n+1}^{\emptyset} < p_n^{\emptyset} < r_n^{\emptyset})$. Put $K_{(n)} = [p_n^{\emptyset}, r_n^{\emptyset}]$.

Let $k \in \mathbb{N}$. Assume that we have already defined intervals $K_{s\hat{\ }m}$ and numbers c^s, d^s, p_m^s, r_m^s which fulfil (i)–(iv) for every $|s| \le k$ and $m \in \mathbb{N}$. Let $s \in \mathbb{N}^{<\mathbb{N}}$ be such that $|s| = k$ and let $l \in \mathbb{N}$. Let $c^{s\hat{\ }l}$ be an answer of Player II to the k-th move (r_{l+1}^s, p_l^s) of Player I in G^+ according to σ^+; more precisely let $c^{s\hat{\ }l}$ be such that

$$((0,1); c^{\emptyset}; (r_{s(0)+1}^{\emptyset}, p_{s(0)}^{\emptyset}); c^{s|1}; (r_{s(1)+1}^{s|1}, p_{s(1)}^{s|1}); c^{s|1}; ...; c^s; (r_{l+1}^s, p_l^s); c^{s\hat{\ }l}) \in \sigma^+.$$

Next let $d^{s\hat{\ }l}$ be an answer of Player II to the k-th move $(r_{l+1}^s, c^{s\hat{\ }l})$ of Player I in G^- according to σ^-; precisely let $d^{s\hat{\ }l}$ be such that

$$((0, c^{\emptyset}); d^{\emptyset}; (r_{s(0)+1}^{\emptyset}, c^{s|1}); d^{s|1}; (r_{s(1)+1}^{s|1}, c^{s|1}); d^{s|1}; ...; d^s; (r_{l+1}^s, c^{s\hat{\ }l}); d^{s\hat{\ }l}) \in \sigma^-.$$

By (A4) there exists an interval set with $\mathbb{D}^+$–density 1 at $d^{s\hat{\ }l}$. By (A3) we may assume that it is contained in $(d^{s\hat{\ }l}, \min\{c^{s\hat{\ }l}, d^{s\hat{\ }l} + 1/(k+1)\})$; suppose that it is of the form $\bigcup_{n\in\mathbb{N}}[p_n^{s\hat{\ }l}, r_n^{s\hat{\ }l}]$ with $\forall n \in \mathbb{N}(r_{n+1}^{s\hat{\ }l} < p_n^{s\hat{\ }l} < r_n^{s\hat{\ }l})$. Put $K_{(s\hat{\ }l)\hat{\ }n} = [p_n^{s\hat{\ }l}, r_n^{s\hat{\ }l}]$.

In this way we have defined intervals $K_{s\hat{\ }m}$ and numbers c^s, d^s, p_m^s, r_m^s fulfilling the conditions (i)–(iv) for every $s \in \mathbb{N}^{<\mathbb{N}}$ and $m \in \mathbb{N}$. We show that the condition (v) holds. Let x_α be such that $\{x_\alpha\} = \bigcap_{n\in\mathbb{N}}(d^{\alpha|n}, c^{\alpha|n})$ (by (A1) this intersection is a singleton). Since σ^+ is a winning strategy in G^+, we obtain that $X^+ = \bigcup_{n\in\mathbb{N}}[p_{\alpha(n)}^{\alpha|n}, c^{\alpha|(n+1)}]$ is not a set with $\mathbb{D}^+$–density 1 at x_α. Note that for each $n \in \mathbb{N}$: if $m \le \alpha(n)$, then $K_{(\alpha|n)\hat{\ }m} = [p_m^{\alpha|n}, r_m^{\alpha|n}] \subset X^+$ and if $m > \alpha(n)$, then $r_m^{\alpha|n} \le r_{\alpha(n)+1}^{\alpha|n} < d^{\alpha|(n+1)} < x_\alpha$. Since $cl(X^+) = X^+ \cup \{x_\alpha\}$, then $(x_\alpha, \infty) \cap cl(\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} K_s) \subset X^+$, and by (A2) and (A3) we obtain that $cl(\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} K_s)$ has not $\mathbb{D}^+$–density 1 at x_α. Analogously one can show that $cl(\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} K_s)$ has not $\mathbb{D}^-$–density 1 at x_α.

Consider the function $T \mapsto cl(\bigcup_{s\in T} K_s)$ which maps $\widetilde{Tr}$ to $\mathcal{K}(\mathbb{R})$. By Lemma 4 this is a Borel function. By Proposition 2, to prove the Theorem it is enough to show that for every $T \in \widetilde{Tr}$ the following equivalence holds: $T \in \widetilde{WF} \iff cl(\bigcup_{s\in T} K_s) \in \mathcal{X}$ (recall that the notions of $\mathbf{\Pi}_1^1$–completeness and Borel–$\mathbf{\Pi}_1^1$–completeness are equivalent – see [4]).

Assume that $T \in \widetilde{WF}$ and $x \in cl(\bigcup_{s\in T} K_s)$. We consider two cases:

1) If $x \in K_s$ for some $s \in T$, then since K_s is an interval, then by (A1)–(A3) we have that $\mathbb{D}^-(cl(\bigcup_{t\in T} K_t), x) = 1$ or $\mathbb{D}^+(cl(\bigcup_{t\in T} K_t), x) = 1$.
2) If $x \notin K_s$ for every $s \in T$, then $x = d^s$ for some $s \in T$. Indeed, note that d^s is a limit point of each sequence (z_n) with $z_n \in K_{s\hat{}k_n}$ for every n, where (k_n) is an increasing sequence of natural numbers. By the construction if $d^s \in cl(\bigcup_{t\in T} K_t)$, then $\bigcup_{n\in\mathbb{N}} K_{s\hat{}n} \subset cl(\bigcup_{t\in T} K_t)$ (here we use the fact that $T \in \widetilde{Tr}$). Suppose now that $x \neq d^s$ for every $s \in T$. Then there exist sequences $(s_n) \in T$ and (w_n) such that $w_n \to x$ $w_n \in K_{s_n}$ for every $n \in \mathbb{N}$. Since $x \neq d^{\emptyset}$, then $(s_n(0))_{n\in\mathbb{N}}$ is bounded. Then there is $k_0 \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : s_n(0) = k_0\}$ is infinite. Proceeding inductively we define a sequence $\alpha = (k_0, k_1, k_2, ...)$ with $\alpha|n \in T$ for $n \in \mathbb{N}$. A contradiction. Then let s be such that $x = d^s$. Note that by (ii) and (iii), $\{x\} \cup \bigcup_{n\in\mathbb{N}} K_{s\hat{}n}$ is an interval set with $\mathbb{D}^+$–density 1 at x. Since $\{x\}\cup\bigcup_{n\in\mathbb{N}} K_{s\hat{}n} \subset cl(\bigcup_{t\in T} K_t)$, then by (A2) we obtain $\mathbb{D}^+(cl(\bigcup_{t\in T} K_t), x) = 1$. Then $cl(\bigcup_{t\in T} K_t) \in \mathcal{X}$.

Assume now that $T \notin \widetilde{WF}$. Then the body $[T]$ of T is nonempty. Let $\alpha \in [T]$ and let x_α be the unique point of $\bigcap_{n\in\mathbb{N}}(d^{\alpha|n}, c^{\alpha|n})$. Note that $\{p^{\alpha|n}_{\alpha(n)}\}_{n\in\mathbb{N}}$ is a sequence of elements of $\bigcup_{s\in T} K_s$ tending to x_α. Then $x_\alpha \in cl(\bigcup_{s\in T} K_s)$. By (v) and $cl(\bigcup_{s\in T} K_s) \subset cl(\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} K_s)$ we obtain that $cl(\bigcup_{s\in T} K_s) \notin \mathcal{X}$. $\square$

4. Applications

4.1. **Lebesgue density.** Let μ be Lebesgue measure on $\mathbb{R}$. For a measurable set $E \subset \mathbb{R}$ and a point $x \in \mathbb{R}$, by $d^+(x, E)$ we denote the right-hand Lebesgue density of the set E at x, i.e. the number $d^+(x, E) = \lim_{h\to 0^+} \frac{\mu([0,h]\cap E)}{h}$, provided this limit exists. Analogously we define $d^-(x, E)$.

Clearly the conditions (A1)–(A3) hold for Lebesgue density. To prove (A4) it is enough to establish the existence of an interval set with d^+–density 1 at 0, since $d^{\pm}(E, x) = d^{\pm}(E-x, 0)$ and $d^-(E, 0) = d^+(-E, 0)$ where $E-x = \{y-x : y \in E\}$ and $-E = \{-y : y \in E\}$. Let $a_n = \frac{1}{10^n}$, $b_n = \frac{10^{n-1}-1}{10^{2(n-1)}}$ for $n \geq 1$. Then for $t \in (\frac{1}{10^{n+1}}, \frac{1}{10^n})$

we have

$$\mu(\bigcup_{k=1}^{\infty}[a_k,b_k]\cap(0,t))=\mu(\bigcup_{k=n+2}^{\infty}[a_k,b_k]\cup[a_{n+1},\min(t,b_{n+1})])>$$

$$t-2|a_{n+1}-b_{n+2}|=t+\frac{2}{10^{2n+2}}>t(1-10^{n+1}\frac{2}{10^{2n+2}})=t(1-\frac{2}{10^{n+1}}).$$

Hence $\bigcup_{n\in\mathbb{N}}[a_n,b_n]$ is an interval set with d^+–density 1.

Now we describe a strategy for Player II in G^+: after the n-th move (a_n,b_n) of Player I, let Player's II answer be $c_n=1/2(a_n+b_n)$. Let $x=\lim_{n\to\infty}a_n$. Then $x\in(a_n,c_n)$ for every $n\in\mathbb{N}$. Hence $\mu[(\bigcup_{k=1}^{\infty}[b_{k+1},c_k])\cap(x,b_n)]\leq c_n-x<b_n-c_n$ and

$$\frac{\mu[(\bigcup_{k=1}^{\infty}[b_{k+1},c_k])\cap(x,b_n)]}{b_n-x}\leq\frac{c_n-x}{b_n-x}=\frac{c_n-x}{b_n-c_n+c_n-x}<\frac{1}{2}.$$

This shows that this is a winning strategy for Player II. A winning strategy for II in G^- can be defined in a similar way. Finally this shows that $d^{\pm}$ satisfies (A5).

Note that

$$\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:d^+(K,x)=1\}=$$

$$\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:\forall\varepsilon>0\exists\delta>0\forall t\in(0,\delta)\frac{\mu(K\cap(x,x+t))}{t}\geq1-\varepsilon\}=$$

$$\bigcap_{n\in\mathbb{N}}\bigcup_{\delta\in\mathbb{Q}_+}\bigcap_{t\in(0,\delta)\cap\mathbb{Q}_+}\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:\frac{\mu(K\cap(x,x+t))}{t}\geq1-\frac{1}{n}\}.$$

To prove the Borelness of the above set, it is enough to show that the set

$$\mathcal{T}=\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:\mu(K\cap(x,x+t))<t(1-\frac{1}{n})\}$$

is open. Let us fix $(K_0,x_0)\in\mathcal{T}$. Since $\mu(K_0\cap(x_0,x_0+t))<t(1-\frac{1}{n})$, then there exists an open set U with $\mu(U\cap(x_0,x_0+t))<t(1-\frac{1}{n})$. Let $\varepsilon>0$ be such that $\mu(U\cap(x_0,x_0+t))+\varepsilon<t(1-\frac{1}{n})$. Then $\mu(U\cap(x,x+t))<t(1-\frac{1}{n})$ for each $x\in(x_0-\varepsilon,x_0+\varepsilon)$. Then the following set $\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:K\subset U,\ x\in(x_0-\varepsilon,x_0+\varepsilon)\}$ is open. Moreover it contains (K_0,x_0) and it is contained in $\mathcal{T}$. This shows that $\mathcal{T}$ is open. Hence (A6) holds for $d^{\pm}$.

4.2. **Porosity.** Let $E \subset \mathbb{R}$, $x \in \mathbb{R}$ and $R > 0$. By $\lambda^+(x, R, E)$ we denote $\sup\{b - a : (a, b) \subset (x, x + R) \setminus E\}$ (if there is no interval (a, b) with $(a, b) \subset (x, x + R) \setminus E$ then we put $\lambda^+(x, R, E) = 0$). The right-hand porosity of the set E at the point x is defined as

$$p^+(E, x) = \limsup_{R\to 0^+} \frac{\lambda^+(x, R, E)}{R}.$$

Analogously we define the left-hand porosity of the set E at the point x and we denote it by $p^-(E, x)$. We say that E is porous (strongly porous) from the right at x if $p^+(E, x) > 0$ ($p^+(E, x) = 1$, respectively).

We define two abstract density operators: $\mathbb{D}_1^\pm(E, x) = 1 - p^\pm(E, x)$ and

$$\mathbb{D}_2^\pm(E, x) = \begin{cases} 1, & \text{if } p^\pm(E, x) < 1; \\ 0, & \text{if } p^\pm(E, x) = 1. \end{cases}$$

$\mathbb{D}_1^\pm(E, x) = 1$ means that E is not porous at x, and $\mathbb{D}_2^\pm(E, x) = 1$ means that E is not strongly porous at x. The conditions (A1)–(A3) are immediate. Since a strong porous set is porous, it is enough to verify (A4) for $\mathbb{D}_1$ and (A5) for $\mathbb{D}_2$.

Similarly as in the case of Lebesgue density, it is enough to define an interval set at 0. We claim that $\bigcup_{n\in\mathbb{N}}[\frac{1}{2n+2}, \frac{1}{2n+1}]$ has the $\mathbb{D}_1^+$–density 1 at 0. Note that

$$p^+\left(\bigcup_{n\in\mathbb{N}}\left[\frac{1}{2n+2}, \frac{1}{2n+1}\right], 0\right) = \limsup_{n\to\infty} \frac{\frac{1}{2n+2} - \frac{1}{2n+3}}{\frac{1}{2n+2}} = \lim_{n\to\infty} \frac{1}{2n+3} = 0.$$

Thus we show (A4) for $\mathbb{D}_1^\pm$.

We describe a winning strategy for Player II in G^+ with respect to the operator $\mathbb{D}_2^+$: if (a_n, b_n) is the n-th move of Player I, then Player II plays $c_n = \frac{n+1}{n+2}a_n + \frac{1}{n+2}b_n$. Let $x = \lim_{n\to\infty} a_n$. Then $x \in (a_n, c_n)$ for each $n \in \mathbb{N}$. Note that

$$\lambda^+(x, b_n - x, \bigcup_{k=1}^{\infty}[b_{k+1}, c_k]) = b_n - c_n = \frac{n+1}{n+2}(b_n - a_n).$$

Thus

$$\frac{\lambda^+(x, b_n - x, \bigcup_{k=1}^{\infty}[b_{k+1}, c_k])}{b_n - x} \geq \frac{\frac{n+1}{n+2}(b_n - a_n)}{b_n - a_n} = \frac{n+1}{n+2}.$$

Hence $p^+(\bigcup_{k=1}^{\infty}[b_{k+1}, c_k], x) = 1$ which proves (A5) for $\mathbb{D}_2^\pm$.

In [3, Section 3] it is proved that

$$A = \{(K, x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : p^+(K, x) = 0\}$$

is Borel. Consequently, we have (A6) for $\mathbb{D}_1^{\pm}$. The proof that (A6) holds for $\mathbb{D}_2^{\pm}$ is analogous.

4.3. **$\mathcal{J}$–density points.** Suppose that $\mathcal{J}$ is a nontrivial ideal of subsets of the real line, i.e. $\mathcal{J}$ does not contain $\mathbb{R}$ and contains all singletons. We say that a is a $\mathcal{J}$–density point of a set $A \subset \mathbb{R}$ if for every increasing sequence $(n_m)_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $(n_{m_p})_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}(A-a)\cap(-1,1)} = \chi_{(-1,1)}, \ \mathcal{J} - \text{a.e.}$$

There are also one-sided versions of the definition of $\mathcal{J}$–density points. Namely, we say that a is a right-hand $\mathcal{J}$–density point of a set $A \subset \mathbb{R}$ if for every increasing sequence $(n_m)_{m\in\mathbb{N}}$ of natural numbers there exists its subsequence $(n_{m_p})_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}(A-a)\cap[0,1)} = \chi_{[0,1)}, \quad \mathcal{J}\text{– a.e.}$$

Analogously we may define a left-hand $\mathcal{J}$–density points. These definitions are due to Wilczyński [11], see also [2].

Now, we define an abstract density operator by

$$\mathbb{D}^+(E, x) = \begin{cases} 1, & \text{if } x \text{ is a right-hand } \mathcal{J}\text{–density point;} \\ 0, & \text{otherwise.} \end{cases}$$

Analogously we define $\mathbb{D}^-(E, x)$.

Since $\mathcal{J}$ is nontrivial, we have (A1). The conditions (A2) and (A3) can be easily derived directly from the definition of $\mathcal{J}$–density points. The existence of an interval sets with $\mathbb{D}^{\pm}$–density 1 follows by Lemma [2, 2.1.4]. So, we have (A4). We shall define a winning strategy for Player II in G^+. Suppose that (a_0, b_0) is the first move of Player I. The answer of Player II is $c_0 = a_0 + (b_0 - a_0)/2$. Additionally put $k_0 = 1$. Let $(a_1, b_1) \subset (a_0, c_0)$ be the second move of Player I. Let

$$k_1 = \min\{k \in \mathbb{N} : k\frac{b_1 - a_1}{2} \geq b_0 - a_1\}.$$

The answer of Player II is $c_1 = a_1 + (b_1 - a_1)/2k_1$. If (a_2, b_2) is the next move of Player I, then let

$$k_2 = \min\{k \in \mathbb{N} : k\frac{b_2 - a_2}{2} \geq b_0 - a_2\}$$

and II plays $c_2 = a_2 + (b_2 - a_2)/2k_2$, etc.

Let $x = \lim_{n\to\infty} a_n$. Consider the interval set $E = \bigcup_{n=0}^{\infty}[b_{n+1}, c_n]$. We claim that

$$\forall y \in (x, b_0) \ \ \{n \in \mathbb{N} : y - x \in k_n(E - x)\} \text{ is finite,}$$

in particular, x is not a right $\mathcal{J}$–density point of E, and the condition (A5) holds.

Let $y \in (x, b_0)$. Note that $k_0(E - y) + y = E$, so $E \cap (c_0, b_0) = \emptyset$. Let $n \in \mathbb{N}$. Assume that $y \in (a_n, c_n) = (a_n, \frac{b_n - a_n}{2k_n} + a_n)$. We shall prove that $(k_n(E - y) + y) \cap (b_n, b_0) = \emptyset$. Then

$$k_n(b_n - y) + y \geq k_n(b_n - \frac{b_n - a_n}{2k_n} - a_n) + a_n = 2k_n\frac{b_n - a_n}{2} - \frac{b_n - a_n}{2} + a_n \geq$$

$$2b_0 - \frac{b_n - a_n}{2} - a_n = b_0 + [(b_0 - a_n) - \frac{b_n - a_n}{2}] \geq b_0.$$

Moreover

$$k_n(c_n - y) + y \leq k_n(\frac{b_n - a_n}{2k_n} + a_n - a_n) + \frac{b_n - a_n}{2k_n} + a_n = a_n + (1 + \frac{1}{k_n})\frac{b_n - a_n}{2} \leq b_n.$$

Finally, conditions (A1)–(A5) hold for every nontrivial ideal $\mathcal{J}$ of subsets of the real line. The assumption that an ideal is nontrivial is so general that it would be hard to expect that also (A6) holds in this case. However, we show that condition (A6) is fulfilled in a very important case when we consider the ideal $\mathcal{M}$ of meager sets on $\mathbb{R}$.

First note that $\{(K, x) \in \mathcal{K}(\mathbb{R}) \times \mathbb{R} : (K - x) \cap U \neq \emptyset\}$ is open in $\mathcal{K}(\mathbb{R}) \times \mathbb{R}$ for every U open. For $B \subset \mathbb{R}$ with the Baire property, by $\tilde{B}$ we denote the unique regular open set with $(B \setminus \tilde{B}) \cup (\tilde{B} \setminus B) \in \mathcal{M}$. Note that for any $K \in \mathcal{K}(\mathbb{R})$ we have $\widetilde{(\mathbb{R} \setminus K)} = \mathbb{R} \setminus cl(\mathbb{R} \setminus cl(\mathbb{R} \setminus K))$. Let I be an open interval. Thus

$$I \cap \widetilde{(\mathbb{R} \setminus K)} = \emptyset \iff I \cap (\mathbb{R} \setminus cl(\mathbb{R} \setminus cl(\mathbb{R} \setminus K))) = \emptyset \iff I \subset cl(\mathbb{R} \setminus cl(\mathbb{R} \setminus K)) \iff$$

$$\mathbb{R} \setminus cl(\mathbb{R} \setminus K) \text{ is dense in } I \iff cl(\mathbb{R} \setminus K) \text{ is nowhere dense in } I \iff$$

$$\mathbb{R} \setminus K \text{ is nowhere dense in } I \iff K \text{ is dense in } I \iff I \subset K.$$

Using Theorem [2, 2.2.2] we obtain that

$$\mathbb{D}^+(K,x)=1 \iff \forall (a,b)\subset(0,1), a,b\in\mathbb{Q}\exists\varepsilon\in\mathbb{Q}_+\exists n_0\in\mathbb{N}\forall n\geq n_0$$

$$\exists (c,d)\subset(a,b), c,d\in\mathbb{Q}\ (|d-c|>\varepsilon \text{ and } (c,d)\cap n(\widetilde{(\mathbb{R}\setminus K)}-x)=\emptyset).$$

Thus $\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:\mathbb{D}^+(K,x)=1\}$ is Borel. Analogously we show that $\{(K,x)\in\mathcal{K}(\mathbb{R})\times\mathbb{R}:\mathbb{D}^-(K,x)=1\}$ is also Borel. Hence (A6) holds.

5. Discussion on (A5)

Now we give an example of an operator $\mathbb{D}$ which fulfills (A1)–(A4) but does not fulfil (A5).

Example 6. Here all Borel sets are admissible. Let $\mathbb{D}^+:\mathcal{B}(\mathbb{R})\times\mathbb{R}\to[0,1]$ be given by

$$\mathbb{D}^+(X,x)=\begin{cases}1, & \text{if } X\cap(x,x+\varepsilon) \text{ is uncountable for every } \varepsilon>0;\\ 0, & \text{if } X\cap(x,x+\varepsilon) \text{ is countable for some } \varepsilon>0.\end{cases}$$

Analogously we define $\mathbb{D}^-$. Note that every interval set has $\mathbb{D}^\pm$–density 1. Then (A5) does not hold. It is clear that (A1)–(A4) are fulfilled for $\mathbb{D}$. Note also that

$$\{K\in\mathcal{K}(\mathbb{R}):\forall x\in K(\mathbb{D}^+(K,x)=1 \text{ or } \mathbb{D}^-(K,x)=1)\}$$

is the family of all perfect compact subsets of the real line. Hence it is a G_δ set. This shows that axioms (A1)–(A4) are not sufficient to prove the $\mathbf{\Pi}^1_1$-hardness of the above set and some additional axiom is needed for this purpose. □

In our consideration, it is important that Lebesgue density and porosity are defined with the use of limits. One can establish (A5) for several kinds of densities and porosities on the real line until in their definitions the limit or the upper limit are used. If we consider lower density, or in the definition of porosity we change lim sup to lim inf, then (A5) simply does not hold. Player II can always make so small holes in an interval that the lower limit of an interval set will be equal to 1.

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