CONVERGENCE OF SERIES ON LARGE SET OF INDICES

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Abstract. We prove that if $\sum_{n=1}^{\infty} a_n = \infty$ and (a_n) is non-decreasing, then $\sum_{n\in A} a_n = \infty$ for any set $A \subset \mathbb{N}$ of positive lower density. We introduce a Cauchy - like definition of $\mathcal{I}$-convergence of series. We prove that the $\mathcal{I}$-convergence of series coincides with the convergence on large set of indexes if and only if $\mathcal{I}$ is a P-ideal. It turns out that $\mathcal{I}$-convergence of series $\sum_{n=1}^{\infty} a_n$ implies $\mathcal{I}$-convergence of (a_n) to zero. The converse implication does not hold for analytic P-ideals and it is independent of ZFC that there is $\mathcal{I}$ ideal of naturals for which $\mathcal{I}$-convergence of (a_n) to zero implies $\mathcal{I}$-convergence of series $\sum_{n=1}^{\infty} a_n = \infty$ for every sequence (a_n).

1. Introduction

The convergence of sequence x_n with respect to an ideal $\mathcal{I}$ is a natural generalization of the usual convergence and the statistical convergence. The paper by Kostyrko, Šalát, and Wilczyński [14] is a well-written introduction to this topic. Recently the large progress was done in applications of $\mathcal{I}$-convergence in analysis (see [1], [7], [9], [10], [15] and [12]).

In this note we are interested in the $\mathcal{I}$-convergence of a series $\sum_{n=1}^{\infty} a_n$. There are two approaches to that concept. The first is to consider the $\mathcal{I}$-convergence of sequence of partial sums $\sum_{n=1}^{k} a_n$ which was considered by Dindoš, Šalát and Toma in [5]. The problem with this definition of $\mathcal{I}$-convergence of a series $\sum_{n=1}^{\infty} a_n$ is that it coincides with the usual convergence if the terms a_n are nonnegative. The second approach is the following. We say that $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent if it is

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convergent on a large set of indexes, namely $\sum_{n\in A} a_n$ is convergent for some A with $\mathbb{N}\setminus A \in \mathcal{I}$. The problem with this definition is that an $\mathcal{I}$-limit of $\sum_{n=1}^{\infty} a_n$ is not well defined. Indeed, since we assume that $\mathcal{I}$ contains all singletons, then if $\sum_{n\in A} a_n$ is convergent and $\mathbb{N} \setminus A \in \mathcal{I}$, then also $\sum_{n\in A\setminus F} a_n$ is convergent and $\mathbb{N} \setminus (A \setminus F) \in \mathcal{I}$ for any finite F. Moreover, in general, the $\mathcal{I}$-convergence of a sequence does not imply the convergence on a large set of indexes. Kostyrko, Šalát, and Wilczyński in [14] proved that such an implication holds if and only if $\mathcal{I}$ is a P-ideal. We will focus on the second approach and we will show how to omit the mentioned problems and define an $\mathcal{I}$-convergence of series (see Definition 5).

Each of whose definitions of $\mathcal{I}$-convergence of series generalizes the usual notion of convergence. Therefore the most interesting question is under which conditions a divergent series is $\mathcal{I}$-convergent. First, we deal with this problem in a special case of $\mathcal{I}$-convergence, namely the statistical convergence. It was proved in [16] that if $A \subseteq \mathbb{N}$ is not of natural density zero, then

$$\sum_{n\in A} \frac{1}{n} = \infty.$$

It is a simple observation that if we change $\left(\frac{1}{n}\right)$ to any sequence (a_n) with $\sum_{n=1}^{\infty} a_n = \infty$ then $\sum_{n\in A} a_n = \infty$ need not hold even for $A \subseteq \mathbb{N}$ of density one. Indeed, take any infinite set $B \subseteq \mathbb{N}$ of density zero and define (a_n) as a characteristic function of B. One can produce a similar example with $a_n \to 0$.

Here we consider the following question. Can we prove a similar statement assuming that (a_n) is non-increasing? In Section 1 we show that

$$\sum_{n\in A} a_n = \infty$$

provided $\sum_{n=1}^{\infty} a_n = \infty$ and $A \subseteq \mathbb{N}$ has a positive lower density. Additionally, we give an example of a non-increasing (a_n) with $\sum_{n=1}^{\infty} a_n = \infty$, $\lim \frac{a_{n+1}}{a_n} = 1$ such that $\sum_{n\in A} a_n < \infty$ for some $A \subseteq \mathbb{N}$ with a positive upper density.

In Section 2 we introduce the notion of ideal convergence of series. Roughly speaking $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent if $\sum_{n\in A} a_n < \infty$ with $\mathbb{N}\setminus A \in \mathcal{I}$. We give some equivalent condition for $\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$.

At the end we prove that it is independent of ZFC that there is a P-ideal $\mathcal{I}$ such that $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent if and only if $a_n \to 0$ with respect to $\mathcal{I}$ for every sequence (a_n). On the other hand, for any analytic P-ideal $\mathcal{I}$, there is an $\mathcal{I}$-divergent series $\sum_{n=1}^{\infty} a_n$ such that $a_n \to 0$.

Now, recall some basic definitions. A family $\mathcal{I}$ of subsets of $\mathbb{N}$ is called an ideal if it fulfills the following conditions:

(1) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;
(2) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

We say that $\mathcal{I}$ is admissible if $\{n\} \in \mathcal{I}$ for $n \in \mathbb{N}$, and $\mathcal{I}$ is proper if $\mathbb{N} \notin \mathcal{I}$. A proper ideal $\mathcal{I}$ is called P-ideal, if for each sequence $(A_n)_{n=1}^{\infty}$ of sets from $\mathcal{I}$ there exists $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$. A proper ideal $\mathcal{I}$ has (AP) property if for any pairwise disjoint sequence $(A_n)_{n=1}^{\infty}$ of sets from $\mathcal{I}$ there exists a sequence $(B_n)_{n=1}^{\infty}$ such that $A_j \setminus B_j$ is finite set for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$. It turns out that notions of P-ideals and ideals with (AP) property coincides, see e.g. [1]. In the sequel we will need a necessary condition for non-P-ideals.

Lemma 1. *Let $\mathcal{I}$ be an admissible ideal which is not a P-ideal. Then there is a sequence (A_n) of pairwise disjoint infinite sets from $\mathcal{I}$ such that for any $A \in \mathcal{I}$ there is n such that the set $A_n \setminus A$ is infinite.*

Proof. Since $\mathcal{I}$ is not P-ideal, there is a sequence (B_n) such that $B_n \in \mathcal{I}$ and for every $A \in \mathcal{I}$ there is n such that $B_n \setminus A$ is infinite. Let $A_1 = B_1$ and $A_n = B_n \setminus \bigcup_{k=1}^{n-1} B_k$. Note that among $A_1, A_2, \dots$ there are infinitely many infinite sets. Suppose to the contrary that all but finitely many sets from $A_1, A_2, ...$ are finite. Let A be the union of all A_i which are infinite. Thus A is in $\mathcal{I}$ and $B_n \setminus A$ is finite for each n which yields a contradiction.

Let $K = \{j : A_j \text{ is infinite}\}$. For $i_0 = \min K$ let $A'_{i_0} = \bigcup_{i=0}^{i_0} A_i$. For any $i \in K \setminus \{i_0\}$ we define A'_i in the following way. If A_{i+1} is infinite then put $A'_i = A_i$, otherwise let $k = \max\{j > i : A_{i+1}, A_{i+2}, ..., A_j \text{ are finite}\}$ and put $A'_i = A_i \cup \dots \cup A_k$. Then $\{A'_i : i \in K\}$ is a family of pairwise disjoint infinite sets with $B_i = \bigcup\{A'_k : k \leq i, k \in K\}$.

Suppose that there is $C \in \mathcal{I}$ such that the set $A'_i \setminus C$ is finite for each $i \in K$. Then the set $B_i \setminus C = \bigcup\{A'_k : k \leq i, k \in K\} \setminus C = \bigcup\{A'_k \setminus C : k \leq i, k \in K\}$ is finite for $i \in K$. If $i \notin K$ then either B_i is finite or there is $j \in K$ with $j < i$ and $B_i \setminus B_j$ is finite and in the both cases $B_i \setminus C$ is finite. This yields a contradiction. $\square$

A function $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is called a submeasure if $\varphi(A) \le \varphi(A \cup B) \le \varphi(A) + \varphi(B)$ for any $A, B \in \mathcal{P}(\mathbb{N})$. A submeasure φ is called lower semicontinuous if $\lim_{n\to\infty} \varphi(A \cap n) = \varphi(A)$. By $\mathrm{Exh}(\varphi)$ denote the set of all $A \subset \mathbb{N}$ with $\lim_{n\to\infty} \varphi(A \setminus n) = 0$. The celebrated Solecki's characterization states that an ideal $\mathcal{I}$ is an analytic P-ideal if and only if it is of the form $\mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ on $\mathbb{N}$.

Let $A \subseteq \mathbb{N}$. By

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, ..., n\}|}{n},$$

we denote the upper density of A where $|A|$ stands for the cardinality of A. In a similar way we define the lower density $\underline{d}(A)$ of A. If $\bar{d}(A) = \underline{d}(A)$, then this common value we denote by $d(A)$ and we call it the density of A. It is well known that the family $\mathcal{I}_d$ of all subsets A of $\mathbb{N}$ with $d(A) = 0$ is an analytic P-ideal.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers. Let $\mathcal{I}_{(a_n)} = \left\{ A \subseteq \mathbb{N} : \sum_{n\in A} a_n < \infty \right\}$. Then $\mathcal{I}_{(a_n)}$ is called a summable ideal. If $\sum_{n=1}^{\infty} a_n = \infty$, then $\mathcal{I}_{(a_n)}$ is a proper P-ideal.

2. Divergent monotone series diverges on large sets of indexes

Theorem 2. *Let $(a_n)_{n=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Assume that $A \subseteq \mathbb{N}$ has a positive lower density. Then $\sum_{n\in A} a_n = \infty$.*

Proof. Since A has a positive lower density, there exists $m \in \mathbb{N}$ such that $\underline{d}(A) > \frac{1}{m}$. By the definition of lower density there is n_0 such that

$$\frac{|A \cap \{1, ..., n\}|}{n} > \frac{1}{m} \tag{1}$$

for every $n \ge n_0$. In particular, for $n = mn_0 \ge n_0$, the set $A \cap \{1, 2, \dots, mn_0\}$ contains at least n_0 elements. Moreover, for $n = 2mn_0$, the set $A \cap \{1, 2, \dots, 2mn_0\}$ contains at least $2n_0$ elements. Thus the following inequalities hold

$$\sum_{k\in A\cap\{1,\dots,mn_0\}} a_k \ge \sum_{k=(m-1)n_0+1}^{mn_0} a_k$$

and

$$\sum_{k\in A\cap\{1,\dots,2mn_0\}} a_k \ge \sum_{k=(m-1)n_0+1}^{mn_0} a_k + \sum_{k=(2m-1)n_0+1}^{2mn_0} a_k.$$

Now let $i \geq 2$. By the same argument as above we obtain that

$$\sum_{k \in A \cap \{1,...,imn_0\}} a_k \geq \sum_{k=(m-1)n_0+1}^{mn_0} a_k + ... + \sum_{k=(im-1)n_0+1}^{imn_0} a_k. \tag{2}$$

Let $B_p = \sum_{i=1}^{\infty} \sum_{k=(im-p)n_0+1}^{(im-p+1)n_0} a_k$ for $p = 1, ..., m$. By (2) we have $\sum_{k \in A} a_k \geq B_1$. Since (a_k) is non-increasing, then $B_1 \leq B_2 \leq ... \leq B_m$. If $B_p^{'} = \sum_{i=2}^{\infty} \sum_{k=(im-p)n_0+1}^{(im-p+1)n_0} a_k$, then $B_2^{'} \leq B_3^{'} \leq ... \leq B_m^{'} \leq B_1$ and $B_p^{'} < \infty$ iff $B_p < \infty$. Suppose that B_1 is finite. Then each $B_p^{'}$ is also finite, and therefore every B_p is finite. But this means that $B_1 + B_2 + ... + B_m = \sum_{k=1}^{\infty} a_k$ is finite and we reach a contradiction. Thus B_1 is infinite which implies that $\sum_{k \in A} a_k$ is infinite. □

We cannot strengthen Theorem 2 assuming only that the set A has positive upper density. Even if the assumption that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ is added.

Proposition 3. *There exists a non-increasing sequence $(a_n)_{n=1}^{\infty}$ of positive reals such that $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, and there is $A \subseteq \mathbb{N}$ with $\overline{d}(A) = 1, \underline{d}(A) = 0$ and $\sum_{n \in A} a_n < \infty$.*

Proof. Consider a sequence (a_n) of the form

$$1, \underbrace{\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n_1^2}}, \underbrace{\frac{1}{(n_1+1)^2}, \frac{1}{(n_1+2)^2}, \dots, \frac{1}{n_2^2}}, \underbrace{\frac{1}{n_2^2+1}, \frac{1}{n_2^2+2}, \dots, \frac{1}{n_3^2}},$$

$$\underbrace{\frac{1}{(n_3+1)^2}, \frac{1}{(n_3+2)^2}, \dots, \frac{1}{n_4^2}}, \dots$$

The sequence (a_n) is a mixture of elements of the harmonic series $\sum_{n=1}^{\infty} 1/n$ and the 2-series $\sum_{n=1}^{\infty} 1/n^2$. Clearly (a_n) is decreasing and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ for any choice $n_1 < n_2 < \dots$. Let A consist of those indexes where elements of the 2-series are used in the definition of (a_n). We can choose integers n_k such that

$$\frac{1}{n_{2j}^2+1} + \frac{1}{n_{2j}^2+2} + \dots + \frac{1}{n_{2j+1}^2} > 1$$

for every j, and such that $\overline{d}(A) = 1$. Clearly $\sum_{k\in A} a_k < \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$. By Theorem 2 the set A does not contain a subset of positive density, and therefore $\underline{d}(A) = 0$. $\square$

Corollary 4. *Let $\alpha \in (0,1]$. Then $\sum_{n\in A} \frac{1}{n^\alpha} = \infty$ for $A \notin \mathcal{I}_{\left(\frac{1}{n}\right)}$. In particular if $\bar{d}(A) > 0$ then $\sum_{n\in A} \frac{1}{n^\alpha} = \infty$.*

It is well known that $\mathcal{I}_{\left(\frac{1}{n}\right)} \subseteq \mathcal{I}_d$. It is not true in general that $\mathcal{I}_{(a_n)} \subseteq \mathcal{I}_d$ even if one assumes that $(a_n)_{n=1}^{\infty}$ is non-decreasing. This follows from Proposition 3.

Remark. An anonymous referee pointed out that Theorem 2 was actually proved by Šalát in [17] using a substantially different method.

3. $\mathcal{I}$-convergence of series

Dindoš, Šalát and Toma introduced in [5] the statistical convergence of series in the following way. A series $\sum_{n=1}^{\infty} a_n$ is statistically convergent to some L provided the sequence $s_n = \sum_{k=1}^{n} a_k$ of partial sums converges statistically to L. In a similar way, one can define a convergence of a series with respect to $\mathcal{I}$, namely as the $\mathcal{I}$-convergence of partial sums. Our approach is different. Since we cannot define an $\mathcal{I}$-sum of a series, we define $\mathcal{I}$-convergence of series by the Cauchy condition. Let us mention that Červeňanský, Šalát and Toma proved in [4] that in general these two definitions of $\mathcal{I}$-convergence of a series do not coincide and any of them do not imply the other.

Definition 5. *Let $\mathcal{I}$ be an admissible ideal. We say that a series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent if it satisfies the $\mathcal{I}$–Cauchy condition, that is if for every $\varepsilon > 0$ there are $n_\varepsilon \in \mathbb{N}$ and $A_\varepsilon \in \mathcal{I}$ such that*

$$\left| \sum_{m\in\{l,\ldots,k\}\setminus A_\varepsilon} a_m \right| < \varepsilon$$

for any $k > l > n_\varepsilon$.

Definition 6. *Let $\mathcal{I}$ be an admissible ideal. We say that a series $\sum_{n=1}^{\infty} a_n$ satisfies the (*)-$\mathcal{I}$-Cauchy condition if there exists set $A \in \mathcal{I}$ such that $\sum_{n\in\mathbb{N}\setminus A} a_n$ satisfies the Cauchy condition. We say that a series $\sum_{n=1}^{\infty} a_n$ is (*)-$\mathcal{I}$-convergent if there* *is $A \in \mathcal{I}$ such that $\sum_{n\in\mathbb{N}\setminus A} a_n$ converges. Clearly the series $\sum_{n=1}^{\infty} a_n$ satisfies the (*)-$\mathcal{I}$-Cauchy condition if and only if $\sum_{n\in\mathbb{N}\setminus A} a_n$ is (*)-$\mathcal{I}$-convergent.*

Now, we will show how these two definitions of Cauchy conditions are related each to other. The following is a counterpart of [1, Proposition 3].

Lemma 7. *Let $\mathcal{I}$ be an admissible ideal. If $\sum_{n=1}^{\infty} a_n$ satisfies the (*)-$\mathcal{I}$-Cauchy condition, then it satisfies the $\mathcal{I}$-Cauchy condition.*

Proof. Since $\sum_{n=1}^{\infty} a_n$ satisfies the (*)–$\mathcal{I}$–Cauchy condition, there is $A \in \mathcal{I}$ such that $\sum_{n\in\mathbb{N}\setminus A} a_n$ satisfies the Cauchy condition. Let $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N}\setminus A$ such that $\left|\sum_{m\in\{l,...,k\}\setminus A} a_m\right| < \varepsilon$ for any $k > l > n_\varepsilon$. Put $A_\varepsilon = A \cup \{1, ..., n_\varepsilon\}$. Then $A_\varepsilon \in \mathcal{I}$ and $\left|\sum_{m\in\{l,...,k\}\setminus A_\varepsilon} a_m\right| < \varepsilon$ for any $k > l > n_\varepsilon$. □

It turns out that the reverse implication is true if and only if $\mathcal{I}$ is a P-ideal. This is a counterpart of [14, Theorem 3.2].

Theorem 8. *Let $\mathcal{I}$ be an admissible ideal. Then the following are equivalent:*

(1) *$\mathcal{I}$ is a P-ideal,*

(2) *$\sum_{n=1}^{\infty} a_n$ satisfies the $\mathcal{I}$-Cauchy condition if and only if satisfies the (*)-$\mathcal{I}$-Cauchy condition.*

Proof. Let $\mathcal{I}$ be a P-ideal and assume that $\sum_{n=1}^{\infty} a_n$ satisfies the $\mathcal{I}$–Cauchy condition. Then for every $j \in \mathbb{N}$ there exist $A_j \in \mathcal{I}$ and q such that for any $k > l > q$ we have $\left|\sum_{m\in\{l,...,k\}\setminus A_j} a_m\right| < \frac{1}{j}$. Since $\mathcal{I}$ is a P-ideal, there exists $A_\infty \in \mathcal{I}$ such that $A_j\setminus A_\infty$ is finite for all $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ and let $p \in \mathbb{N}$ be such that $A_j\setminus A_\infty \subset \{1, ..., p\}$. Thus for any $k > l > p$ if $k, l \notin A_j$, then $k, l \notin A_\infty$ and therefore $\left|\sum_{m\in\{l,...k\}\setminus A_\infty} a_m\right| < \frac{1}{j}$.

Assume now that $\mathcal{I}$ is not a P-ideal. Then by Lemma 1 there is a sequence $A_1, A_2, ...$ of pairwise disjoint infinite sets in $\mathcal{I}$ such that for any $A \in \mathcal{I}$ there is n such that $A_n \setminus A$ is infinite. Let $\{k_1^n < k_2^n < \dots\}$ be an increasing enumeration of A_n. Define $a_{k_i^n} = (-1)^i/2^n$ and $a_m = 0$ if $m \notin \bigcup_{n=1}^{\infty} A_n$.

Let $\varepsilon > 0$. There is n with $1/2^n < \varepsilon$. Take $m < k$ and consider

$$t := \sum_{m=l, m \notin A_1 \cup \dots \cup A_n}^{k} a_m.$$

By the construction of series $\sum_{m=1}^{\infty} a_m$, we have

$$|t| < \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots = \frac{1}{2^n} < \varepsilon.$$

Hence $\sum_{m=1}^{\infty} a_m$ fulfills the $\mathcal{I}$–Cauchy condition.

Let $A \in \mathcal{I}$. Let n_0 be such that $A_{n_0} \setminus A$ is infinite. Then the set $\{s_m : m \notin A\}$ contains infinitely many elements of the form $(-1)^i/2^{n_0}$. Therefore $\sum_{m \in \mathbb{N} \setminus A} a_m$ does not converge, and thus $\sum_{m \in \mathbb{N} \setminus A} a_m$ does not fulfill the Cauchy condition. Hence $\sum_{m=1}^{\infty} a_m$ does not fulfill the (*)–$\mathcal{I}$–Cauchy condition. □

4. When $\mathcal{I}$-$\lim_n a_n = 0$ implies $\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$

In this section we will prove two facts. The first fact states that for a large class of ideals, namely analytic P-ideals $\mathcal{I}$, there is an $\mathcal{I}$ divergent series $\sum_{n=1}^{\infty} a_n$ such that $\mathcal{I}$-$\lim_{n \to \infty} a_n = 0$. The second fact states that there is a maximal P-ideal $\mathcal{I}$ such that $\mathcal{I}$-$\lim_n a_n = 0$ implies the $\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$. But first let us note the following basic fact.

Proposition 9. *Assume that $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent. Then (a_n) is $\mathcal{I}$-convergent to zero.*

Proof. Since $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent, then

$$\forall \varepsilon > 0 \; \exists A_\varepsilon \in \mathcal{I} \; \exists n_\varepsilon \; \forall k > l > n_\varepsilon \; \Big| \sum_{m \in \{l, \dots, k\} \setminus A_\varepsilon} a_m \Big| < \varepsilon.$$

Thus $|a_m| \le \varepsilon$ for every $m > n_\varepsilon$, $m \notin A_\varepsilon$. Hence (a_n) is $\mathcal{I}$-convergent to zero. □

Theorem 10. *For any analytic P-ideal $\mathcal{I}$ there exists an $\mathcal{I}$-divergent series $\sum_{n=1}^{\infty} a_n$ such that $(a_n)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent to zero.*

Proof. Let φ be a submeasure witnessing that $\mathcal{I}$ is an analytic P-ideal. Let $M' = \lim_{n\to\infty} \varphi(\mathbb{N}\backslash n) > 0$. Since $\mathcal{I}$ does not contain $\mathbb{N}$, then M' is a positive real number or $M' = \infty$. By M denote $M'/2$ if M' is finite or 1 if $M' = \infty$. Let $0 = n_0 < n_1 < n_2 < n_3 < ...$ be such that $\varphi(n_{k+1}\backslash n_k) \geq M$. Let $A_k = \{n \in \mathbb{N} : n_{k-1} \leq n < n_k\}$ and for every index n from A_k define $a_n = 1/k$. It can be easily seen that $\bigcup_{k=0}^{\infty} A_k = \mathbb{N}$ and the sequence $(a_n)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent to zero.

Let $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A \in \mathcal{I}$. Therefore $\lim_{n\to\infty} \varphi((\mathbb{N}\backslash A) \setminus n) = 0$. Now, we want to show that there exists $l \in \mathbb{N}$ such that for all $n > l$ we have $A_n \cap A \neq \emptyset$. Suppose to the contrary that for any $l \in \mathbb{N}$ there exists $k > l$ that $A_k \cap A = \emptyset$. This means that $\lim_{n\to\infty} \varphi((\mathbb{N}\backslash A) \setminus n) \geq M$ which is a contradiction. Hence series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-divergent since it contains a subseries of the form $\sum_{n=k}^{\infty} 1/n$. □

Now, we will prove that Theorem 10 is not true for all P-ideals. To do that we will need the following set-theoretic statement proved by Bartoszewicz, Głąb and Wachowicz in [2]. We refer the reader to [2] for the notation used in this section.

Theorem 11. *Assume that $\mathfrak{p} = \mathfrak{c}$. Let $\tau < \mathfrak{p}$. Suppose that $\mathcal{B}_1$, $\mathcal{B}_2$ are two properties of sequences $x \in \mathbb{R}^{\mathbb{N}}$ such that:*

- (a) *for all $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in [\mathbb{N}]^{\mathbb{N}}$, if $x_{\restriction K}$ has $\mathcal{B}_1$, then there is $L \in [\mathbb{N}]^{\mathbb{N}}$, $L \subset K$, such that $x_{\restriction L}$ has $\mathcal{B}_2$;*
- (b) *$\mathcal{B}_1$ is closed under taking subsequences, i.e. for all $x \in \mathbb{R}^{\mathbb{N}}$, $L, K \in [\mathbb{N}]^{\mathbb{N}}$, if $L \subset K$ and $x_{\restriction K}$ has $\mathcal{B}_1$, then $x_{\restriction L}$ has $\mathcal{B}_1$.*

If a filter $\mathcal{F}$ is τ-generated, then $\mathcal{F}$ can be extended to a filter $\mathcal{F}'$ such that for any $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in \mathcal{F}'$, if $x_{\restriction K}$ has $\mathcal{B}_1$, then there is $L \in \mathcal{F}'$, $L \subset K$, such that $x_{\restriction L}$ has $\mathcal{B}_2$.

Theorem 12. *Assume that $\mathfrak{p} = \mathfrak{c}$. There exists a P-ideal $\mathcal{I}$ such that if $(a_n)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent to zero then the series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent.*

Proof. We say that a sequence (a_n) has the property $\mathcal{B}_1$ if (a_n) is bounded, and we say that a sequence (a_n) has the property $\mathcal{B}_2$ if (a_n) is convergent and $\sum_{n=1}^{\infty}(a_n - \lim_k a_k)$ is finite. Clearly conditions (a) and (b) of Theorem 11 are fulfilled. Let $\mathcal{F}$ be a Frechet filter, i.e. a filter which consists of cofinite subsets of $\mathbb{N}$. Then by Theorem 11 there is a filter $\mathcal{F}' \supset \mathcal{F}$ such that if (a_n) is bounded on a set $K \in \mathcal{F}'$, then $\sum_{n=1}^{\infty}(a_n - \lim_k a_k)$ converges on a set $L \in \mathcal{F}'$. By $\mathcal{I}'$ denote the dual ideal to $\mathcal{F}'$. In particular we obtain that $l^{\infty}(\mathcal{I}') = c^*(\mathcal{I}')$ and therefore by [14, Theorem 3.2] and [6, Proposition 3] $\mathcal{I}'$ is a maximal P-ideal.

Let (a_n) be $\mathcal{I}'$-convergent to zero. Since $\mathcal{I}'$ is P-ideal there is $L \in \mathcal{F}'$ such that $\lim_{n\in L} a_n = 0$. By the $\mathcal{I}'$-boundedness of (a_n) there is $K \in \mathcal{F}'$ such that $\sum_{k\in K} a_n$ is finite, which means that $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}'$-convergent. □

The anonymous referee suggested that the notion of rapid filter is crucial for the property of ideals studied in this section. A filter $\mathcal{F}$ on $\mathbb{N}$ is called a rapid filter, if for any sequence (ε_n) such that $\varepsilon_n \to 0$, there exists $X \in \mathcal{F}$ such that $\sum_{n\in X} \varepsilon_n < \infty$. (There are several equivalent definitions of rapid filters, see e.g. [3, Lemma 4.6.2].)

Proposition 13. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The (*)-$\mathcal{I}$-convergence of (a_n) to zero implies the (*)-$\mathcal{I}$-convergence of series $\sum_{n=1}^{\infty} a_n$ for every sequence (a_n) of real numbers if and only if the filter $\mathcal{F}$ dual to $\mathcal{I}$ is a rapid filter.*

Proof. Assume that $\mathcal{F}$ is not a rapid filter. Then there is a sequence (ε_n) tending to zero such that $\sum_{n\in X} \varepsilon_n = \infty$ for every $X \in \mathcal{F}$. Note that (ε_n) is (*)-$\mathcal{I}$-convergent to zero while $\sum_{n=1}^{\infty} a_n$ is not (*)-$\mathcal{I}$-convergent.

Assume now that $\mathcal{F}$ is a rapid filter. Let (a_n) be a sequence of real numbers which is (*)-$\mathcal{I}$-convergent. Thus there is a set $A \in \mathcal{F}$ such that $\lim_{n\in A} a_n = 0$. Put $a'_n = a_n$ if $n \in A$ and $a'_n = 0$ otherwise. Then (a'_n) tends to zero. Since $\mathcal{F}$ is a rapid filter, there is $B \in \mathcal{F}$ such that $\sum_{n\in B} a'_n < \infty$. Note that $A \cap B \in \mathcal{F}$ and $\sum_{n\in B\cap A} a_n = \sum_{n\in B\cap A} a'_n < \infty$. Therefore $\sum_{n=1}^{\infty} a_n$ is (*)-$\mathcal{I}$-convergent. □

In the light of Proposition 13 what we proved in Theorem 12 is that under the assumption $\mathfrak{p} = \mathfrak{c}$ there is a rapid filter. However this is a known fact (see e.g. [11]). Theorem 10 can be read as follows – there are no analytic rapid P-ideals. On the other hand, by the result of Judah and Shelah [13], there is a model of ZFC in which there are no rapid filters. Therefore we have the following.

Corollary 14. *It is independent of ZFC that there exists an ideal $\mathcal{I}$ on $\mathbb{N}$ such that the (*)-$\mathcal{I}$-convergence of (a_n) to zero implies the (*)-$\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$ for every sequence (a_n) of real numbers. In particular, it is independent of ZFC that there exists a P-ideal on $\mathbb{N}$ such that the $\mathcal{I}$-convergence of (a_n) to zero implies the $\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$ for every sequence (a_n) of real numbers.*

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