CONVERGENCE OF SERIES ON LARGE SET OF INDICES

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Abstract. We prove that if ∞ ∑_{n=1} a_n = ∞ and (a_n) is non-decreasing, then ∑_{n ∈ A} a_n = ∞ for any set A ⊂ N of positive lower density. We introduce a Cauchy-like definition of I-convergence of series. We prove that the I-convergence of series coincides with the convergence on large set of indexes if and only if I is a P-ideal. It turns out that I-convergence of series ∑_{n=1}^N a_n implies I-convergence of (a_n) to zero. The converse implication does not hold for analytic P-ideals and it is independent of ZFC that there is I ideal of naturals for which I-convergence of (a_n) to zero implies I-convergence of series ∑_{n=1}^∞ a_n = ∞ for every sequence (a_n).

1. Introduction

The convergence of sequence x_n with respect to an ideal I is a natural generalization of the usual convergence and the statistical convergence. The paper by Kostyrko, Šalát, and Wilczyński [14] is a well-written introduction to this topic. Recently the large progress was done in applications of I-convergence in analysis (see [1], [7], [9], [10], [15] and [12]).

In this note we are interested in the I-convergence of a series ∑_{n=1}^∞ a_n. There are two approaches to that concept. The first is to consider the I-convergence of sequence of partial sums ∑_{n=1}^k a_n which was considered by Dindoš, Šalát and Toma in [5]. The problem with this definition of I-convergence of a series ∑_{n=1}^∞ a_n is that it coincides with the usual convergence if the terms a_n are nonnegative. The second approach is the following. We say that ∑_{n=1}^∞ a_n is I-convergent if it is
convergent on a large set of indexes, namely $\sum_{n \in A} a_n$ is convergent for some $A$ with $\mathbb{N} \setminus A \in \mathcal{I}$. The problem with this definition is that an $\mathcal{I}$-limit of $\sum_{n=1}^{\infty} a_n$ is not well defined. Indeed, since we assume that $\mathcal{I}$ contains all singletons, then if $\sum_{n \in A} a_n$ is convergent and $\mathbb{N} \setminus A \in \mathcal{I}$, then also $\sum_{n \in A \setminus F} a_n$ is convergent and $\mathbb{N} \setminus (A \setminus F) \in \mathcal{I}$ for any finite $F$. Moreover, in general, the $\mathcal{I}$-convergence of a sequence does not imply the convergence on a large set of indexes. Kostyrko, Šalát, and Wilczyński in [14] proved that such an implication holds if and only if $\mathcal{I}$ is a $P$-ideal. We will focus on the second approach and we will show how to omit the mentioned problems and define an $\mathcal{I}$-convergence of series (see Definition 5).

Each of whose definitions of $\mathcal{I}$-convergence of series generalizes the usual notion of convergence. Therefore the most interesting question is under which conditions a divergent series is $\mathcal{I}$-convergent. First, we deal with this problem in a special case of $\mathcal{I}$-convergence, namely the statistical convergence. It was proved in [16] that if $A \subseteq \mathbb{N}$ is not of natural density zero, then

$$\sum_{n \in A} \frac{1}{n} = \infty.$$  

It is a simple observation that if we change $(\frac{1}{n})$ to any sequence $(a_n)$ with $\sum_{n=1}^{\infty} a_n = \infty$ then $\sum_{n \in A} a_n = \infty$ need not hold even for $A \subseteq \mathbb{N}$ of density one. Indeed, take any infinite set $B \subseteq \mathbb{N}$ of density zero and define $(a_n)$ as a characteristic function of $B$. One can produce a similar example with $a_n \to 0$.

Here we consider the following question. Can we prove a similar statement assuming that $(a_n)$ is non-increasing? In Section 1 we show that

$$\sum_{n \in A} a_n = \infty$$

provided $\sum_{n=1}^{\infty} a_n = \infty$ and $A \subseteq \mathbb{N}$ has a positive lower density. Additionally, we give an example of a non-increasing $(a_n)$ with $\sum_{n=1}^{\infty} a_n = \infty$, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ such that $\sum_{n \in A} a_n < \infty$ for some $A \subseteq \mathbb{N}$ with a positive upper density.

In Section 2 we introduce the notion of ideal convergence of series. Roughly speaking $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent if $\sum_{n \in A} a_n < \infty$ with $\mathbb{N} \setminus A \in \mathcal{I}$. We give some equivalent condition for $\mathcal{I}$-convergence of $\sum_{n=1}^{\infty} a_n$.  

At the end we prove that it is independent of ZFC that there is a $P$-ideal $I$ such that $\sum_{n=1}^{\infty} a_n$ is $I$-convergent if and only if $a_n \to 0$ with respect to $I$ for every sequence $(a_n)$. On the other hand, for any analytic $P$-ideal $I$, there is an $I$-divergent series $\sum_{n=1}^{\infty} a_n$ such that $a_n \to 0$.

Now, recall some basic definitions. A family $I$ of subsets of $\mathbb{N}$ is called an ideal if it fulfills the following conditions:

1. if $A \in I$ and $B \subseteq A$, then $B \in I$;
2. if $A, B \in I$, then $A \cup B \in I$.

We say that $I$ is admissible if $\{n\} \in I$ for $n \in \mathbb{N}$, and $I$ is proper if $\mathbb{N} \notin I$. A proper ideal $I$ is called $P$-ideal, if for each sequence $(A_n)_{n=1}^{\infty}$ of sets from $I$ there exists $A_{\infty} \in I$ such that $A_n \setminus A_{\infty}$ is finite for all $n \in \mathbb{N}$. A proper ideal $I$ has (AP) property if for any pairwise disjoint sequence $(A_n)_{n=1}^{\infty}$ of sets from $I$ there exists a sequence $(B_n)_{n=1}^{\infty}$ such that $A_j \setminus B_j$ is finite set for all $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} B_n \in I$. It turns out that notions of $P$-ideals and ideals with (AP) property coincides, see e.g. [1]. In the sequel we will need a necessary condition for non-$P$-ideals.

**Lemma 1.** Let $I$ be an admissible ideal which is not a $P$-ideal. Then there is a sequence $(A_n)$ of pairwise disjoint infinite sets from $I$ such that for any $A \in I$ there is $n$ such that the set $A_n \setminus A$ is infinite.

**Proof.** Since $I$ is not $P$-ideal, there is a sequence $(B_n)$ such that $B_n \in I$ and for every $A \in I$ there is $n$ such that $B_n \setminus A$ is infinite. Let $A_1 = B_1$ and $A_n = B_n \setminus \bigcup_{k=1}^{n-1} B_k$. Note that among $A_1, A_2, \ldots$ there are infinitely many infinite sets. Suppose to the contrary that all but finitely many sets from $A_1, A_2, \ldots$ are finite. Let $A$ be the union of all $A_i$ which are infinite. Thus $A$ is in $I$ and $B_n \setminus A$ is finite for each $n$ which yields a contradiction.

Let $K = \{ j : A_j \text{ is infinite} \}$. For $i_0 = \min K$ let $A'_0 = \bigcup_{i=0}^{i_0} A_i$. For any $i \in K \setminus \{i_0\}$ we define $A'_i$ in the following way. If $A_{i+1}$ is infinite then put $A'_i = A_i$, otherwise let $k = \max\{ j > i : A_{i+1}, A_{i+2}, \ldots, A_j \text{ are finite} \}$ and put $A'_i = A_i \cup \ldots \cup A_k$. Then $\{A'_i : i \in K\}$ is a family of pairwise disjoint infinite sets with $B_i = \bigcup\{A'_k : k \leq i, k \in K\}$.

Suppose that there is $C \in I$ such that the set $A'_i \setminus C$ is finite for each $i \in K$. Then the set $B_i \setminus C = \bigcup\{A'_k : k \leq i, k \in K\} \setminus C = \bigcup\{A'_k \setminus C : k \leq i, k \in K\}$ is finite for $i \in K$. If $i \notin K$ then either $B_i$ is finite or there is $j \in K$ with $j < i$ and $B_i \setminus B_j$ is finite and in the both cases $B_i \setminus C$ is finite. This yields a contradiction. \qed
A function $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is called a submeasure if $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for any $A, B \in \mathcal{P}(\mathbb{N})$. A submeasure $\varphi$ is called lower semicontinuous if $\lim_{n \to \infty} \varphi(A \cap n) = \varphi(A)$. By $\text{Exh}(\varphi)$ denote the set of all $A \subset \mathbb{N}$ with $\lim_{n \to \infty} \varphi(A \setminus n) = 0$. The celebrated Solecki’s characterization states that an ideal $\mathcal{I}$ is an analytic $\mathcal{P}$-ideal if and only if it is of the form $\text{Exh}(\varphi)$ for some lower semicontinuous submeasure $\varphi$ on $\mathbb{N}$.

Let $A \subseteq \mathbb{N}$. By $\bar{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$, we denote the upper density of $A$ where $|A|$ stands for the cardinality of $A$. In a similar way we define the lower density $\underline{d}(A)$ of $A$. If $\bar{d}(A) = \underline{d}(A)$, then this common value we denote by $d(A)$ and we call it the density of $A$. It is well known that the family $\mathcal{I}_d$ of all subsets $A$ of $\mathbb{N}$ with $d(A) = 0$ is an analytic $\mathcal{P}$-ideal.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers. Let $\mathcal{I}_{(a_n)} = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} a_n < \infty \right\}$. Then $\mathcal{I}_{(a_n)}$ is called a summable ideal. If $\sum_{n=1}^{\infty} a_n = \infty$, then $\mathcal{I}_{(a_n)}$ is a proper $\mathcal{P}$-ideal.

2. Divergent monotone series diverges on large sets of indexes

**Theorem 2.** Let $(a_n)_{n=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $\lim_{n \to \infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Assume that $A \subseteq \mathbb{N}$ has a positive lower density. Then $\sum_{n \in A} a_n = \infty$.

**Proof.** Since $A$ has a positive lower density, there exists $m \in \mathbb{N}$ such that $d(A) > \frac{1}{m}$. By the definition of lower density there is $n_0$ such that

$$\frac{|A \cap \{1, \ldots, n\}|}{n} > \frac{1}{m}$$

for every $n \geq n_0$. In particular, for $n = mn_0 \geq n_0$, the set $A \cap \{1, 2, \ldots, mn_0\}$ contains at least $n_0$ elements. Moreover, for $n = 2mn_0$, the set $A \cap \{1, 2, \ldots, 2mn_0\}$ contains at least $2n_0$ elements. Thus the following inequalities hold

$$\sum_{k \in A \cap \{1, \ldots, mn_0\}} a_k \geq \sum_{k=(m-1)n_0+1}^{mn_0} a_k$$

and

$$\sum_{k \in A \cap \{1, \ldots, 2mn_0\}} a_k \geq \sum_{k=(m-1)n_0+1}^{mn_0} a_k + \sum_{k=(2m-1)n_0+1}^{2mn_0} a_k.$$
Now let $i \geq 2$. By the same argument as above we obtain that
\[
\sum_{k \in A \cap \{1, \ldots, imn_0\}} a_k \geq \sum_{k = (im-1)n_0+1}^{mn_0} a_k + \sum_{k = (im-1)n_0+1}^{imn_0} a_k.
\] (2)

Let $B_p = \sum_{i=1}^{\infty} \left(\sum_{k = (im-p+1)n_0+1}^{imn_0} a_k\right)$ for $p = 1, \ldots, m$. By (2) we have $\sum_{k \in A} a_k \geq B_1$. Since $(a_k)$ is non-increasing, then $B_1 \leq B_2 \leq \ldots \leq B_m$ and $B_p < \infty$ iff $B_p < \infty$. Suppose that $B_1$ is finite. Then each $B_p$ is also finite, and therefore every $B_p$ is finite. But this means that $B_1 + B_2 + \ldots + B_m = \sum_{k=1}^{\infty} a_k$ is finite and we reach a contradiction. Thus $B_1$ is infinite which implies that $\sum_{k \in A} a_k$ is infinite.

We cannot strengthen Theorem 2 assuming only that the set $A$ has positive upper density. Even if the assumption that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ is added.

**Proposition 3.** There exists a non-increasing sequence $(a_n)_{n=1}^{\infty}$ of positive reals such that $\lim_{n \to \infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$, and there is $A \subseteq \mathbb{N}$ with $\overline{d}(A) = 1, \underline{d}(A) = 0$ and $\sum_{n \in A} a_n < \infty$.

**Proof.** Consider a sequence $(a_n)$ of the form
\[
\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \ldots, \frac{1}{n_j^2}, \frac{1}{(n_1+1)^2}, \frac{1}{(n_1+2)^2}, \ldots, \frac{1}{n_j^2}, \frac{1}{n_1^2 + 1}, \frac{1}{n_2^2 + 1}, \ldots, \frac{1}{n_j^2 + 1}, \ldots
\]

The sequence $(a_n)$ is a mixture of elements of the harmonic series $\sum_{n=1}^{\infty} 1/n$ and the 2-series $\sum_{n=1}^{\infty} 1/n^2$. Clearly $(a_n)$ is decreasing and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ for any choice $n_1 < n_2 < \ldots$. Let $A$ consist of those indexes where elements of the 2-series are used in the definition of $(a_n)$. We can choose integers $n_k$ such that
\[
\frac{1}{n_{2j}^2 + 1} + \frac{1}{n_{2j}^2 + 2} + \cdots + \frac{1}{n_{2j+1}^2} > 1
\]
for every \( j \), and such that \( d(A) = 1 \). Clearly \( \sum_{k \in A} a_k < \infty \) and \( \sum_{n=1}^{\infty} a_n = \infty \). By Theorem 2 the set \( A \) does not contain a subset of positive density, and therefore \( d(A) = 0 \). □

**Corollary 4.** Let \( \alpha \in (0, 1] \). Then \( \sum_{n \in A} \frac{1}{n^\alpha} = \infty \) for \( A \notin \mathcal{I}_{\alpha} \). In particular if \( d(A) > 0 \) then \( \sum_{n \in A} \frac{1}{n^\alpha} = \infty \).

It is well known that \( \mathcal{I}_{\alpha} \subseteq \mathcal{I}_d \). It is not true in general that \( \mathcal{I}_{(a_n)} \subseteq \mathcal{I}_d \) even if one assumes that \( (a_n)_{n=1}^{\infty} \) is non-decreasing. This follows from Proposition 3.

**Remark.** An anonymous referee pointed out that Theorem 2 was actually proved by Šalát in [17] using a substantially different method.

### 3. \( \mathcal{I} \)-convergence of series

Dindoš, Šalát and Toma introduced in [5] the statistical convergence of series in the following way. A series \( \sum_{n=1}^{\infty} a_n \) is statistically convergent to some \( L \) provided the sequence \( s_n = \sum_{k=1}^{n} a_k \) of partial sums converges statistically to \( L \). In a similar way, one can define a convergence of a series with respect to \( \mathcal{I} \), namely as the \( \mathcal{I} \)-convergence of partial sums. Our approach is different. Since we cannot define an \( \mathcal{I} \)-sum of a series, we define \( \mathcal{I} \)-convergence of series by the Cauchy condition. Let us mention that Červeňanský, Šalát and Toma proved in [4] that in general these two definitions of \( \mathcal{I} \)-convergence of a series do not coincide and any of them do not imply the other.

**Definition 5.** Let \( \mathcal{I} \) be an admissible ideal. We say that a series \( \sum_{n=1}^{\infty} a_n \) is \( \mathcal{I} \)-convergent if it satisfies the \( \mathcal{I} \)-Cauchy condition, that is if for every \( \varepsilon > 0 \) there are \( n_\varepsilon \in \mathbb{N} \) and \( A_\varepsilon \in \mathcal{I} \) such that

\[
\left| \sum_{m \in \{l, \ldots, k\} \setminus A_\varepsilon} a_m \right| < \varepsilon
\]

for any \( k > l > n_\varepsilon \).

**Definition 6.** Let \( \mathcal{I} \) be an admissible ideal. We say that a series \( \sum_{n=1}^{\infty} a_n \) satisfies the (*)-\( \mathcal{I} \)-Cauchy condition if there exists set \( A \in \mathcal{I} \) such that \( \sum_{n \in \mathbb{N} \setminus A} a_n \) satisfies the Cauchy condition. We say that a series \( \sum_{n=1}^{\infty} a_n \) is (*)-\( \mathcal{I} \)-convergent if there
is $A \in \mathcal{I}$ such that $\sum_{n \in \mathbb{N} \setminus A} a_n$ converges. Clearly the series $\sum_{n=1}^{\infty} a_n$ satisfies the (*-$\mathcal{I}$-Cauchy condition if and only if $\sum_{n \in \mathbb{N} \setminus A} a_n$ is (*-$\mathcal{I}$-convergent.

Now, we will show how these two definitions of Cauchy conditions are related to each other. The following is a counterpart of [1, Proposition 3].

**Lemma 7.** Let $\mathcal{I}$ be an admissible ideal. If $\sum_{n=1}^{\infty} a_n$ satisfies the (*)-$\mathcal{I}$-Cauchy condition, then it satisfies the $\mathcal{I}$-Cauchy condition.

**Proof.** Since $\sum_{n=1}^{\infty} a_n$ satisfies the (*)-$\mathcal{I}$-Cauchy condition, there is $A \in \mathcal{I}$ such that $\sum_{n \in \mathbb{N} \setminus A} a_n$ satisfies the Cauchy condition. Let $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N} \setminus A$ such that $\left| \sum_{m \in \{l,...,k\} \setminus A} a_m \right| < \varepsilon$ for any $k > l > n_\varepsilon$. Put $A_\varepsilon = A \cup \{1,...,n_\varepsilon\}$. Then $A_\varepsilon \in \mathcal{I}$ and $\left| \sum_{m \in \{l,...,k\} \setminus A_\varepsilon} a_m \right| < \varepsilon$ for any $k > l > n_\varepsilon$. □

It turns out that the reverse implication is true if and only if $\mathcal{I}$ is a $P$-ideal. This is a counterpart of [14, Theorem 3.2].

**Theorem 8.** Let $\mathcal{I}$ be an admissible ideal. Then the following are equivalent:

1. $\mathcal{I}$ is a $P$-ideal,
2. $\sum_{n=1}^{\infty} a_n$ satisfies the $\mathcal{I}$-Cauchy condition if and only if satisfies the (*)-$\mathcal{I}$-Cauchy condition.

**Proof.** Let $\mathcal{I}$ be a $P$-ideal and assume that $\sum_{n=1}^{\infty} a_n$ satisfies the $\mathcal{I}$-Cauchy condition. Then for every $j \in \mathbb{N}$ there exist $A_j \in \mathcal{I}$ and $q$ such that for any $k > l > q$ we have $\left| \sum_{m \in \{l,...,k\} \setminus A_j} a_m \right| < \frac{1}{j}$. Since $\mathcal{I}$ is a $P$-ideal, there exists $A_\infty \in \mathcal{I}$ such that $A_j \setminus A_\infty$ is finite for all $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ and let $p \in \mathbb{N}$ be such that $A_j \setminus A_\infty \subset \{1,...,p\}$. Thus for any $k > l > p$ if $k,l \notin A_j$, then $k,l \notin A_\infty$ and therefore $\left| \sum_{m \in \{l,...,k\} \setminus A_\infty} a_m \right| < \frac{1}{j}$.

Assume now that $\mathcal{I}$ is not a $P$-ideal. Then by Lemma 1 there is a sequence $A_1, A_2, \ldots$ of pairwise disjoint infinite sets in $\mathcal{I}$ such that for any $A \in \mathcal{I}$ there is $n$ such that $A_n \setminus A$ is infinite. Let $\{k_1^1 < k_2^1 < \ldots\}$ be an increasing enumeration of $A_n$. Define $a_n = (-1)^{k_n^1}/2^n$ and $a_m = 0$ if $m \notin \bigcup_{n=1}^{\infty} A_n$. 

Let $\varepsilon > 0$. There is $n$ with $1/2^n < \varepsilon$. Take $m < k$ and consider

$$t := \sum_{m=1, m \notin A_1 \cup \cdots \cup A_n}^k a_m.$$

By the construction of series $\sum_{m=1}^\infty a_m$, we have

$$|t| < \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^n} < \varepsilon.$$

Hence $\sum_{m=1}^\infty a_m$ fulfills the $I$-Cauchy condition.

Let $A \in I$. Let $n_0$ be such that $A_{n_0} \setminus A$ is infinite. Then the set $\{s_m : m \notin A\}$ contains infinitely many elements of the form $(-1)^i/2^{n_0}$. Therefore $\sum_{m \notin \mathbb{N} \setminus A} a_m$ does not converge, and thus $\sum_{m \notin \mathbb{N} \setminus A} a_m$ does not fulfill the Cauchy condition.

Hence $\sum_{m=1}^\infty a_m$ does not fulfill the $(\ast)$-$I$-Cauchy condition. \hfill $\Box$

4. When $I$-$\lim_{n \to \infty} a_n = 0$ implies $I$-convergence of $\sum_{n=1}^\infty a_n$

In this section we will prove two facts. The first fact states that for a large class of ideals, namely analytic $P$-ideals $I$, there is an $I$ divergent series $\sum_{n=1}^\infty a_n$ such that $I$-$\lim_{n \to \infty} a_n = 0$. The second fact states that there is a maximal $P$-ideal $I$ such that $I$-$\lim_{n \to \infty} a_n = 0$ implies the $I$-convergence of $\sum_{n=1}^\infty a_n$. But first let us note the following basic fact.

**Proposition 9.** Assume that $\sum_{n=1}^\infty a_n$ is $I$-convergent. Then $(a_n)$ is $I$-convergent to zero.

**Proof.** Since $\sum_{n=1}^\infty a_n$ is $I$-convergent, then

$$\forall \varepsilon > 0 \ \exists A_\varepsilon \in I \exists n_\varepsilon \ \forall k > n_\varepsilon \left| \sum_{m \in \{1, \ldots, k \} \setminus A_\varepsilon} a_m \right| < \varepsilon.$$

Thus $|a_m| \leq \varepsilon$ for every $m > n_\varepsilon$, $m \notin A_\varepsilon$. Hence $(a_n)$ is $I$-convergent to zero. \hfill $\Box$

**Theorem 10.** For any analytic $P$-ideal $I$ there exists an $I$-divergent series $\sum_{n=1}^\infty a_n$ such that $(a_n)_{n=1}^\infty$ is $I$-convergent to zero.
Let $\varphi$ be a submeasure witnessing that $\mathcal{I}$ is an analytic $P$-ideal. Let $M' = \lim_{n \to \infty} \varphi(\mathbb{N} \setminus n) > 0$. Since $\mathcal{I}$ does not contain $\mathbb{N}$, then $M'$ is a positive real number or $M' = \infty$. By $M$ denote $M'/2$ if $M'$ is finite or 1 if $M' = \infty$.

Let $0 = n_0 < n_1 < n_2 < ...$ be such that $\varphi(n_{k+1} \setminus n_k) \geq M$. Let $A_k = \{n \in \mathbb{N} : n_{k-1} \leq n < n_k\}$ and for every index $n$ from $A_k$ define $a_n = 1/k$. It can be easily seen that $\bigcup_{k=0}^{\infty} A_k = \mathbb{N}$ and the sequence $(a_n)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent to zero.

Let $A \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus A \in \mathcal{I}$. Therefore $\lim_{n \to \infty} \varphi((\mathbb{N} \setminus A) \setminus n) = 0$. Now, we want to show that there exists $l \in \mathbb{N}$ such that for all $n > l$ we have $A_n \cap A \neq \emptyset$.

Suppose to the contrary that for any $l \in \mathbb{N}$ there exists $k > l$ that $A_k \cap A = \emptyset$.

This means that $\lim_{n \to \infty} \varphi((\mathbb{N} \setminus A) \setminus n) \geq M$ which is a contradiction. Hence series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-divergent since it contains a subseries of the form $\sum_{n=k}^{\infty} 1/n$. 

Now, we will prove that Theorem 10 is not true for all $P$-ideals. To do that we will need the following set-theoretic statement proved by Bartoszewicz, Głąb and Wachowicz in [2]. We refer the reader to [2] for the notation used in this section.

**Theorem 11.** Assume that $p = c$. Let $\tau < p$. Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are two properties of sequences $x \in \mathbb{R}^\mathbb{N}$ such that:

- (a) for all $x \in \mathbb{R}^\mathbb{N}$ and $K \in [\mathbb{N}]^\mathbb{N}$, if $x_{\upharpoonright K}$ has $\mathcal{B}_1$, then there is $L \in [\mathbb{N}]^\mathbb{N}$, $L \subseteq K$, such that $x_{\upharpoonright L}$ has $\mathcal{B}_2$;
- (b) $\mathcal{B}_1$ is closed under taking subsequences, i.e. for all $x \in \mathbb{R}^\mathbb{N}$, $L, K \in [\mathbb{N}]^\mathbb{N}$, if $L \cap K$ and $x_{\upharpoonright K}$ has $\mathcal{B}_1$, then $x_{\upharpoonright L}$ has $\mathcal{B}_1$.

If a filter $\mathcal{F}$ is $\tau$-generated, then $\mathcal{F}$ can be extended to a filter $\mathcal{F}'$ such that for any $x \in \mathbb{R}^\mathbb{N}$ and $K \in \mathcal{F}'$, if $x_{\upharpoonright K}$ has $\mathcal{B}_1$, then there is $L \in \mathcal{F}'$, $L \subseteq K$, such that $x_{\upharpoonright L}$ has $\mathcal{B}_2$.

**Theorem 12.** Assume that $p = c$. There exists a $P$-ideal $\mathcal{I}$ such that if $(a_n)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent to zero then the series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}$-convergent.

**Proof.** We say that a sequence $(a_n)$ has the property $\mathcal{B}_1$ if $(a_n)$ is bounded, and we say that a sequence $(a_n)$ has the property $\mathcal{B}_2$ if $(a_n)$ is convergent and $\sum_{n=1}^{\infty} (a_n - \lim_k a_k)$ is finite. Clearly conditions (a) and (b) of Theorem 11 are fulfilled. Let $\mathcal{F}$ be a Frechet filter, i.e. a filter which consists of cofinite subsets of $\mathbb{N}$. Then by Theorem 11 there is a filter $\mathcal{F}' \supseteq \mathcal{F}$ such that if $(a_n)$ is bounded on a set $K \in \mathcal{F}'$, then $\sum_{n=1}^{\infty} (a_n - \lim_k a_k)$ converges on a set $L \in \mathcal{F}'$. By $\mathcal{T}'$ denote the dual ideal to $\mathcal{F}'$. In particular we obtain that $l^{\infty}(\mathcal{T}') = c^*(\mathcal{T}')$ and therefore by [14, Theorem 3.2] and [6, Proposition 3] $\mathcal{T}'$ is a maximal $P$-ideal.
Let \((a_n)\) be \(I'\)-convergent to zero. Since \(I'\) is \(P\)-ideal there is \(L \in \mathcal{F}'\) such that \(\lim_{n \in L} a_n = 0\). By the \(I'\)-boundedness of \((a_n)\) there is \(K \in \mathcal{F}'\) such that \(\sum_{k \in K} a_n\) is finite, which means that \(\sum_{n=1}^{\infty} a_n\) is \(I'\)-convergent. □

The anonymous referee suggested that the notion of rapid filter is crucial for the property of ideals studied in this section. A filter \(\mathcal{F}\) on \(\mathbb{N}\) is called a rapid filter, if for any sequence \((\varepsilon_n)\) such that \(\varepsilon_n \to 0\), there exists \(X \in \mathcal{F}\) such that \(\sum_{n \in X} \varepsilon_n < \infty\). (There are several equivalent definitions of rapid filters, see e.g. [3, Lemma 4.6.2].)

**Proposition 13.** Let \(\mathcal{I}\) be an ideal on \(\mathbb{N}\). The \((*)-I\)-convergence of \((a_n)\) to zero implies the \((*)-I\)-convergence of series \(\sum_{n=1}^{\infty} a_n\) for every sequence \((a_n)\) of real numbers if and only if the filter \(\mathcal{F}\) dual to \(I\) is a rapid filter.

**Proof.** Assume that \(\mathcal{F}\) is not a rapid filter. Then there is a sequence \((\varepsilon_n)\) tending to zero such that \(\sum_{n \in X} \varepsilon_n = \infty\) for every \(X \in \mathcal{F}\). Note that \((\varepsilon_n)\) is \((*)-I\)-convergent to zero while \(\sum_{n=1}^{\infty} a_n\) is not \((*)-I\)-convergent.

Assume now that \(\mathcal{F}\) is a rapid filter. Let \((a_n)\) be a sequence of real numbers which is \((*)-I\)-convergent. Thus there is a set \(A \in \mathcal{F}\) such that \(\lim_{n \in A} a_n = 0\). Put \(a'_n = a_n\) if \(n \in A\) and \(a'_n = 0\) otherwise. Then \((a'_n)\) tends to zero. Since \(\mathcal{F}\) is a rapid filter, there is \(B \in \mathcal{F}\) such that \(\sum_{n \in B} a'_n < \infty\). Note that \(A \cap B \in \mathcal{F}\) and

\[
\sum_{n \in B \cap A} a_n = \sum_{n \in B \cap A} a'_n < \infty.
\]

Therefore \(\sum_{n=1}^{\infty} a_n\) is \((*)-I\)-convergent. □

In the light of Proposition 13 what we proved in Theorem 12 is that under the assumption \(p = c\) there is a rapid filter. However this is a known fact (see e.g. [11]). Theorem 10 can be read as follows – there are no analytic rapid \(P\)-ideals. On the other hand, by the result of Judah and Shelah [13], there is a model of \(\text{ZFC}\) in which there are no rapid filters. Therefore we have the following.

**Corollary 14.** It is independent of \(\text{ZFC}\) that there exists an ideal \(\mathcal{I}\) on \(\mathbb{N}\) such that the \((*)-I\)-convergence of \((a_n)\) to zero implies the \((*)-I\)-convergence of \(\sum_{n=1}^{\infty} a_n\) for every sequence \((a_n)\) of real numbers. In particular, it is independent of \(\text{ZFC}\) that there exists a \(P\)-ideal on \(\mathbb{N}\) such that the \(I\)-convergence of \((a_n)\) to zero implies the \(I\)-convergence of \(\sum_{n=1}^{\infty} a_n\) for every sequence \((a_n)\) of real numbers.
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