On the discrete boundary value problem for anisotropic equation

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Abstract

In this paper we consider the discrete anisotropic boundary value problem using critical point theory. Thirstily we apply the direct method of the calculus of variations and the mountain pass technique in order to reach the existence of at least one non-trivial solution. Secondly we derive some version of a discrete three critical point theorem which we apply in order to get the existence of at least two non-trivial solutions.

1 Introduction

In this note we consider an anisotropic difference equation

$$\begin{cases} \Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) = \lambda f(k,u(k)), \\ u(0)=u(T+1)=0, \end{cases} \tag{1}$$

where $\lambda > 0$ is a numerical parameter, $f : [0,T+1]\times\mathbb{R}^{T+2} \to \mathbb{R}$, $[a,b]$ for $a<b$, $a,b\in\mathbb{Z}$ denotes a discrete interval $\{a,a+1,...,b\}$, $\Delta u(k-1) = u(k) - u(k-1)$ is the forward difference operator; $p : [0,T+1] \to \mathbb{R}_+$, $p^- = \min_{k\in[0,T+1]} p(k) > 1$ and $p^+ = \max_{k\in[0,T+1]} p(k) > 1$; $q : [0,T+1] \to \mathbb{R}_+$ stands for the conjugate exponent. We aim at showing that problem (1) has at least one, at least two nontrivial solutions. Our approach relies on the application of the direct method of the calculus of variations, the mountain pass technique and a modified version of a finite dimensional three critical point theorem which we derive for our special case.

Let us mention, far from being exhaustive, the following recent papers on discrete BVPs investigated via critical point theory, [1], [3], [8], [12], [13], [14], [15], [16]. These papers employ in the discrete setting the variational techniques already known for continuous problems of course with necessary modifications. The tools employed cover the Morse theory, mountain pass methodology, linking arguments. Continuous version of problems like (1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [17]), electrorheological fluids (see [11]) or image restoration (see [4]). Variational continuous anisotropic problems have been started by Fan and Zhang in [5] and later considered by many methods and authors- see [6] for an extensive survey of such boundary value problems. The research concerning the discrete anisotropic problems of type (1) have only been started, see [7], [10], where known tools from the critical point theory are applied in order to get the existence of solutions.

2 Variational framework for (1)

By a solution to (1) we mean such a function $x : [0, T+1] \to \mathbb{R}$ which satisfies the given equation and the associated boundary conditions. Solutions will be investigated in a space

$$H = \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\}$$

considered with a norm

$$||u|| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}.$$

Then $(H, ||\cdot||)$ becomes a Hilbert space. The functional corresponding to (1) is

$$J_\lambda(u) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^{T} F(k, u(k)),$$

where $F(k, u(k)) = \int_0^{u(k)} f(k,t)dt$. With any fixed $\lambda > 0$ functional J_λ is differentiable in the sense of Gâteaux and its Gâteaux derivative reads

$$\left\langle J_\lambda^{'}(u), v \right\rangle = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) - \lambda \sum_{k=1}^{T} f(k, u(k)) v\,(k)\,.$$

A critical point to J_λ, i.e. such a point $u \in E$ that

$$\left\langle J_\lambda^{'}(u), v \right\rangle = 0 \text{ for all } v \in E$$

is a weak solution to (1). Summing by parts we see that any weak solution to (1) is in fact a strong one. Hence in order to solve (1) we need to find critical points to J_λ and investigate their multiplicity.

The following auxiliary result was proved in [10].

Lemma 1 *(a) There exist two positive constants C_1 and C_2 such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 ||u||^{p^-} - C_2$$

for every $u \in H$ with $||u|| > 1$.

(b) For any $m \geq 2$ there exists a positive constant c_m such that

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m$$

for every $u \in H$.

We note that

$$2^m \sum_{k=1}^{T} |u(k)|^m \geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^m \tag{2}$$

for any $m \geq 2$. Since any two norms on a finite-dimensional Banach space are equivalent, then there is a positive constant K_m (depending on m) with

$$(K_m)^m \, ||u||^m \geq \sum_{k=1}^{T} |u(k)|^m.$$

It can be verified that

$$(T+1)^{\frac{2-m}{2m}} ||u|| \leq \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \right)^{1/m} \leq (T+1)^{\frac{1}{m}} ||u|| \tag{3}$$

for any $u \in H$ and any $m \geq 2$.

Let us recall some preliminaries from critical point theory. Let E be a reflexive Banach space. Let $J \in C^1(E, \mathbb{R})$. For any sequence $\{u_n\} \subset E$, if $\{J(u_n)\}$ is bounded and $J^{'}(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say J satisfies the Palais–Smale condition - (PS) condition for short.

Theorem 2 *(Mountain pass lemma)[9] Let J satisfy the (PS) condition. Suppose that*

1. *$J(0) = 0$;*
2. *there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $||u|| = \rho$;*
3. *there exists u_1 in E with $||u_1|| \geq \rho$ such that $J(u_1) < \alpha$.*

Then J has a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$\inf_{g \in \Gamma} \max_{u \in g([1,0])} J(u),$$

where $\Gamma = \{g \in C([1,0], E) : g(0) = 0, g(1) = u_1\}$.

Theorem 3 *[9]If functional $J : E \to \mathbb{R}$, J is weakly lower semi-continuous and coercive, i.e. $\lim_{||x|| \to \infty} J(x) = +\infty$, then there exists x_0 such that*

$$\inf_{x \in E} J(x) = J(x_0)$$

and x_0 is also a critical point of J, i.e. $J^{'}(x_0) = 0$. Moreover, if J is strictly convex, then a critical point is unique.

3 Existence of solution by a direct method and a mountain pass lemma

In this section we are concerned with the applications of Theorems 3 and 2 in order to get the existence results. We introduce firstly some assumptions.

H1 *$a : [1, T] \to \mathbb{R}_+$, $b : [1, T] \to \mathbb{R}$ are such that*

$$|f(k, t)| \leq a(k)|t|^{q(k)} + b(k) \text{ for all } t \in \mathbb{R} \text{ and all } k \in [1, T] .$$

H2 *$a_1 : [1, T] \to \mathbb{R}_+$, $b_1 : [1, T] \to \mathbb{R}$ are such that*

$$|f(k, t)| \geq a_1(k)|t|^{q(k)} + b_1(k) \text{ for all } t \in \mathbb{R} \text{ and all } k \in [1, T] .$$

H3 *$c : [1, T] \to \mathbb{R}_+$ is such that*

$$|f(k, t)| \geq c(k)|t|^{q(k)} \text{ for all } t \in \mathbb{R} \text{ and all } k \in [1, T] .$$

H4 *$c_1 : [1, T] \to \mathbb{R}_+$ is such that*

$$|f(k, t)| \leq c_1(k)|t|^{q(k)} \text{ for all } t \in \mathbb{R} \text{ and all } k \in [1, T] .$$

Theorem 4 *Assume that condition **H1** holds. Then*

(4.1) functional J_λ is coercive provided $p^- > q^+ + 1$;

(4.2) if $p^- = q^+ + 1$, then there is $\lambda^\star$ such that for any $\lambda \in (0, \lambda^\star)$ functional J_λ is coercive.

Proof. Note that

$$\left|\sum_{k=1}^T F(k, u(k))\right| \leq \sum_{k=1}^T \left|\int_0^{u(k)} f(k, t)dt\right| \leq \sum_{k=1}^T \int_0^{u(k)} |f(k, t)|dt \leq$$

$$\sum_{k=1}^T \int_0^{u(k)} (a(k)|t|^{q(k)} + b(k))dt = \sum_{k=1}^T \frac{a(k)|u(k)|^{q(k)+1}}{q(k)+1} + \sum_{k=1}^T b(k)u(k) \leq$$

$$\frac{a^+}{q^-+1} \sum_{k=1}^T |u(k)|^{q^++1} + b^+ \sum_{k=1}^T |u(k)| \leq$$

$$\frac{a^+ c_{q^++1}}{q^-+1} \sum_{k=1}^{T+1} |\Delta u(k)|^{q^++1} + b^+ c_1 \sum_{k=1}^T |\Delta u(k)|,$$

where constants c_{q^++1} and c_1 in the last inequality follow from lemma 1(b). Using the above estimation we obtain

$$J_\lambda(u) \geq \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \frac{a^+ c_{q^++1}}{q^-} \sum_{k=1}^{T+1} |\Delta u(k)|^{q^++1}$$

$$-\lambda b^+ c_1 \sum_{k=1}^T |\Delta u(k)|$$

Using Lemma 1(a) we have

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 ||u||^{p^-} - C_2.$$

Hence

$$J_\lambda(u) \geq \tfrac{C_1}{p^+}||u||^{p^-} - C_2 - \lambda\tfrac{a^+c_{q^++1}}{q^-+1}\sum_{k=1}^{T+1}|\Delta u(k)|^{q^++1} - \lambda b^+ c_1 \sum_{k=1}^{T}|\Delta u(k)|.$$

Thus $J_\lambda(u) \to \infty$ as $||u|| \to \infty$ in case $p^- > q^+ + 1$.

Let us now assume that $p^- = q^+ + 1$. Then $J_\lambda(u) \to \infty$ as $||u|| \to \infty$ in case

$$\frac{C_1}{p^+} - \lambda\frac{a^+c_{q^++1}}{q^- + 1} > 0.$$

Thus there is $\lambda^\star$ such that for any $\lambda \in (0, \lambda^\star)$ the functional $J(u)$ is coercive. In fact we may take $\lambda^\star = \frac{C_1(q^-+1)}{p^+a^+c_{q^++1}}$. ■

Now, we immediately obtain the following existence result.

Theorem 5 *Assume that condition **H1** holds and let $f(k, 0) \neq 0$ for at least one $k \in [1, T]$. Then problem (1) has at least one nontrivial solution for all $\lambda > 0$ provided $p^- > q^+ + 1$. When $p^- = q^+ + 1$, there is $\lambda^\star$ such that for any $\lambda \in (0, \lambda^\star)$ problem (1) has at least one nontrivial solution.*

Proof. Indeed, functional J_λ is continuous differentiable in the sense of Gâteaux. The assertion follows then by Theorem 4 and Theorem 3. ■

Theorem 6 *Assume that condition **H2** holds. Then*

(6.1) functional J_λ is anti-coercive provided $p^+ < q^- + 1$;

(6.2) if $p^+ = q^- + 1$, then there is $\lambda^\star$ such that for any $\lambda > \lambda^\star$ the functional J_λ is anti-coercive.

Proof. As in the proof of Theorem 4 we see by (2) that

$$\sum_{k=1}^{T} F(k, u(k)) \geq$$

$$\frac{a_1^-}{(q^++1)2^{q^-+1}}\sum_{k=1}^{T}|\Delta u(k-1)|^{q^-+1} + \frac{b_1^-}{2}\sum_{k=1}^{T}|\Delta u(k-1)|.$$

Using the above we have for some constant $\overline{c}$

$$J_\lambda(u) \leq \frac{1}{p^+}||u||_{p^+}^{p^+} - \lambda\frac{a_1^-}{(q^+ + 1)2^{q^-+1}}||u||_{q^-+1}^{q^-+1} + \overline{c}||u||.$$

Hence by (3) we see that

$$J_\lambda(u) \le \frac{T+1}{p^-}||u||^{p^+} - \frac{\lambda a_1^-(T+1)^{\frac{2}{2-q^-+1}}}{(q^++1)2^{q^-+1}}||u||^{q^-+1} + \bar{c}||u||. \tag{4}$$

This inequality provides the assertion. ■

The existence result immediately follows.

Theorem 7 *Assume that condition **H2** holds and let $f(k,0) \neq 0$ for at least one $k \in [1,T]$. Then problem (1) has at least one nontrivial solution for all $\lambda > 0$ provided $p^+ < q^- + 1$. When $p^+ = q^- + 1$, there is $\lambda^\star$ such that for any $\lambda > \lambda^\star$ problem (1) has at least one nontrivial solution.*

When $f(k,0) = 0$ for all $k \in [1,T]$ Theorem 3 yields the existence of at least one solution which in this case may become trivial. So that we will use some different methodology pertaining to Theorem 2. This however requires some additional assumptions.

Corollary 8 *(i) Assume that conditions **H1**, **H3** hold and let $p^- > q^+ + 1$. Then for any $M < 0$ there is positive $\lambda^\star$ such that for any $\lambda > \lambda^\star$ and for all $||u|| = 1$ we have $J_\lambda(u) \le M$.*

*(ii) Assume that conditions **H2**, **H4** hold and let $q^- + 1 > p^+$. Then there is $\lambda^\star$, $t > 0$ and $M > 0$ such that for any $\lambda \in (0, \lambda^\star)$ and for all $||u|| = t$ we have $J_\lambda(u) \ge M$.*

Proof. We will prove only the first assertion. Let us fix any $M > 0$. Note that by the proof of Theorem 6, see relation (4), we obtain that for all $||u|| = 1$

$$J_\lambda(u) \le \frac{T+1}{p^-} - \frac{\lambda c^-(T+1)^{\frac{2}{2-q^-+1}}}{(q^++1)2^{q^-+1}} \le M$$

when $\lambda \ge \lambda^* = \frac{\left(-M+\frac{T+1}{p^-}\right)(q^++1)2^{q^-+1}}{c^-(T+1)^{\frac{2}{2-q^-+1}}}$. ■

Corollary 9 •

*(i) Assume that conditions **H1**,**H3** hold and let $p^- > q^+ + 1$ and $f(k,0) = 0$ for $k \in [1,T]$. Then problem (1) has at least one nontrivial solution for all $\lambda > \lambda^* = \frac{\left(-M+\frac{T+1}{p^-}\right)(q^++1)2^{q^-+1}}{c^-(T+1)^{\frac{2}{2-q^-+1}}}$.*

*(ii) Assume that conditions **H2**, **H4** hold and let $q^- + 1 > p^+$. Then there is $\lambda^\star$ such that (1) has at least one nontrivial solution for all $\lambda \in (0, \lambda^\star)$.*

Proof. We will show that all assumptions of Theorem 2 are satisfied in case of assertion (i). We put $E_\lambda = -J_\lambda$. Now we see that E_λ is anti-coercive. Next we note that $E_\lambda(0) = 0$. Thus assumptions 1. and 3. of Theorem 2 are satisfied. Assumption 2. follows by Lemma 8. Hence the assertion of the theorem follows. ■

4 General multiplicity result

In this section we are concerned with the existence of multiple solutions. We will use the version of the three critical point theorem. In order to do so we shall somehow generalize the main result, namely Theorem 3 from [2] which is proved for some special functional containing term $\|\cdot\|^p$. We note however, that this term can be replaced by any nonnegative, coercive function being 0 at 0. Our proof in fact follows the ideas employed in [2]. Let us start from the following abstract result which is used in the proof Theorem 3 from [2] and which we also use.

Theorem 10 *Let (X, τ) be a Hausdorff space and $\Phi, J : X \to \mathbb{R}$ be functionals; moreover, let M be the (possibly empty) set of all the global minimizers of J and define*

$$\alpha = \inf_{x \in X} \Phi(x),$$

$$\beta = \begin{cases} \inf_{x \in M} \Phi(x), & \text{if } M \neq \emptyset, \\ \sup_{x \in X} \Phi(x), & \text{if } M = \emptyset. \end{cases}$$

Assume that the following conditions are satisfied:

(10.1) for every $\sigma > 0$ and every $\rho \in \mathbb{R}$ the set $\{x \in X : \Phi(x) + \sigma J(x) \leq \rho\}$ is sequentially compact (if not empty);

(10.2) $\alpha < \beta$.

Then at least one of the following conditions holds:

(10.3) there is a continuous mapping $h : (\alpha, \beta) \to X$ with the following property: for every $t \in (\alpha, \beta)$, one has

$$\Phi(h(t)) = t$$

and for every $x \in \Phi^{-1}(t)$ with $x \neq h(t)$,

$$J(x) > J(h(t));$$

(10.4) there is $\sigma^\star > 0$ such that the functional $\Phi + \sigma^\star J$ admits at least two global minimizers in X.

Theorem 11 *Let H be a finite dimensional Banach space. Assume that $J, \mu \in C^1(H)$, $\mu(x) \geq 0$ for all $x \in H$, $\mu(0) = 0$ and μ is coercive. Let $0 < r < s$ be constants. Assume that*

(11.1) $\limsup_{\mu(u) \to \infty} \frac{J(u)}{\mu(u)} \geq 0$;

(11.2) $\inf_{u \in H} J(u) < \inf_{\mu(u) \leq s} J(u)$;

(11.3) $J(0) \leq \inf_{r \leq \mu(u) \leq s} J(u)$.

Then there is $\lambda^\star > 0$ such that the functional $u \mapsto \mu(u) + \lambda^\star J(u)$ has at least three critical points in H.

Proof. Define a continuous functional $\Phi : H \to \mathbb{R}$ in the following way

$$\Phi(u) = \begin{cases} \mu(u), \text{ if } \mu(u) < r, \\ r, \text{ if } r \leq \mu(u) \leq s, \\ \mu(u) - s + r, \text{ if } \mu(u) > s. \end{cases}$$

Let M, α and β be defined as in Theorem 10. At first we show that $\beta > r$. We consider two cases:

Case 1. Assume that $M \neq \emptyset$. The set M of minimizers of continuous functional J is closed, and therefore there is $\overline{u} \in M$ with $\Phi(\overline{u}) = \beta$. Hence by (11.2) we obtain that $\mu(\overline{u}) > s$, and thus

$$\beta = \Phi(\overline{u}) = \mu(\overline{u}) - s + r > r.$$

Case 2. Assume that $M = \emptyset$. Then $\beta = \infty > r$.

Now we will show that the assumptions of Theorem 10 are fulfilled. Note that for any $\sigma > 0$ we have

$$\liminf_{\mu(u)\to\infty} (\Phi(u) + \sigma J(u)) = \liminf_{\mu(u)\to\infty} (\mu(u) - s + r + \sigma J(u)) =$$

$$\liminf_{\mu(u)\to\infty} \mu(u) \left(1 - \tfrac{s-r}{\mu(u)} + \sigma \tfrac{J(u)}{\mu(u)}\right) \geq$$

$$\liminf_{\mu(u)\to\infty} \mu(u) \cdot \liminf_{\mu(u)\to\infty} \left(1 - \tfrac{s-r}{\mu(u)} + \sigma \tfrac{J(u)}{\mu(u)}\right) \geq \liminf_{\mu(u)\to\infty} \mu(u) = \infty.$$

Hence by the above and by the continuity of Φ and J we obtain that the set $\{u \in X : \Phi(u) + \sigma J(u) \leq \rho\}$ is closed and bounded, and consequently compact.

By the definition of Φ we have $\alpha = \inf_{u\in H} \Phi(u) = 0$. Thus since $\beta > r > 0$ we obtain that $\beta > \alpha$. By Theorem 10 one of the two conditions (10.3) and (10.4) holds. We will show that (10.3) cannot hold. Suppose to the contrary that (10.3) holds true, i.e. there is a mapping $h : (0, \beta) \to H$ with $\Phi(h(t)) = t$ for every $t \in (0, \beta)$. Since $r \in (0, \beta)$, then

$$t < r \iff \Phi(h(t)) < r \iff \mu(h(t)) < r,$$

$$t = r \iff \Phi(h(t)) = r \iff r \leq \mu(h(t)) \leq s,$$

$$t > r \iff \Phi(h(t)) > r \iff \mu(h(t)) > s.$$

Therefore

$$\limsup_{t\to r^-} \mu(h(t)) \leq r = \mu(h(r)),$$

$$\liminf_{t\to r^+} \mu(h(t)) \geq s > \mu(h(r)).$$

But this contradicts the continuity of h. Hence the condition (10.4) holds, that is there is $\sigma^\star > 0$ such that the functional $\Phi + \sigma^\star J$ has at least two minimizers, say u_1 and u_2 ($u_1 \neq u_2$).

We will show that $\mu(u_i) < r$ or $\mu(u_i) > s$ for $i = 1, 2$. Suppose to the contrary, that $r \leq \mu(u_i) \leq s$ for some $i = 1, 2$. Then

$$\Phi(u_i) + \sigma^\star J(u_i) = r + \sigma^\star J(u_i) > \sigma^\star J(u_i) = \Phi(0) + \sigma^\star J(0),$$

which contradicts with the fact that u_i is a minimizer of $\Phi + \sigma^\star J$. Put

$$E(u) = \mu(u) + \sigma^\star J(u).$$

Note that u_1 and u_2 are *local* minimizers of E. Moreover $E \in C^1(H, \mathbb{R})$. We need only to show that E has at least one critical point $u_3 \in H \setminus \{u_1, u_2\}$. If both u_1 and u_2 are strict local minimizers of E, then using Mountain Pass technique (see [2, Theorem 2]) we will find a critical point $u_3 \notin \{u_1, u_2\}$. If one of u_1 and u_2 is not strict local minimizer, then E admits infinitely many local minimizers at the same level. ■

5 Applications of multiplicity result for anisotropic problems

In order to apply Theorem 11 for (1) we introduce the following notation. Let $\mu(u) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}$. Clearly $\mu \in C^1(H, \mathbb{R})$, $\mu(0) = 0$, $\mu \geq 0$ and μ is coercive. For numbers $r, s > 0$ we put $r' = \inf\{||u||_\infty : \mu(u) \geq r\}$ and $s' = \sup\{||u||_\infty : \mu(u) \leq s\}$. Hence

$$\mu(u) \geq r \implies ||u||_\infty \geq r',$$

$$\mu(u) \leq s \implies ||u||_\infty \leq s'.$$

Let $F(k,t) = \int_0^t f(k,\tau)d\tau$, $J(u) = -\sum_{k=1}^{T} F(k, u(k))$, and $E_\lambda(u) = \mu(u) + \lambda J(u)$.

Theorem 12 *Let $0 < r < s$. Assume that*

(12.1) $\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^{p^-}} \leq 0$ *for any* $k \in [1, T]$*;*

(12.2) $\sum_{k=1}^{T} \sup_{|t|\leq s'} F(k,t) < \sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t)$*;*

(12.3) $\sup_{r'\leq t\leq s'} F(k,t) \leq -\sum_{h\neq k} \sup_{|t|\leq s'} F(h,t)$ *for any* $k \in [1, T]$*.*

Then there exists $\lambda^\star$ such that the functional $E_{\lambda^\star}$ has at least three critical points, two of which must be nontrivial.

Proof. By Lemma 1(a), $\mu(u) \geq c_1 ||u||^{p^-} - c_2$ for some positive constants c_1, c_2 and $||u|| > 1$. Since there is a positive c_3 with

$$||u||^{p^-} \geq c_3 \sum_{k=1}^{T} |u(k)|^{p^-},$$

then

$$\mu(u) \geq c_1 c_3 \sum_{k=1}^{T} |u(k)|^{p^-} - c_2 \geq \frac{c_1 c_3}{2} \sum_{k=1}^{T} |u(k)|^{p^-}$$

provided $\mu(u)$ is sufficiently large.

Let $\varepsilon > 0$. Using (12.1) we will find $K > 0$ with

$$\frac{F(k,t)}{|t|^{p^-}} < \frac{c_1 c_3 \varepsilon}{2T}$$

for any $k \in [1,T]$ and $|t| > K$. Let $M = \max\{F(k,t) : k \in [1,T], |t| \leq K\}$. Then for $\mu(u) > MT/\varepsilon$ with $||u|| \geq 1$ we obtain

$$\frac{J(u)}{\mu(u)} = \frac{-\sum_{k=1}^{T} F(k,u(k))}{\mu(u)} \geq -\frac{\sum_{k=1}^{T} |F(k,u(k))|}{\mu(u)} \geq$$

$$-\frac{\sum_{|u(k)| \leq K} |F(k,u(k))|}{\mu(u)} - \frac{\sum_{|u(k)| > K} |F(k,u(k))|}{\frac{c_1 c_3}{2} \sum_{k=1}^{T} |u(k)|^{p^-}} \geq$$

$$-\sum_{|u(k)| \leq K} \frac{M}{MT/\varepsilon} - \sum_{|u(k)| > K} \frac{|F(k,u(k))|}{\frac{c_1 c_3}{2} |u(k)|^{p^-}} \geq$$

$$-\sum_{|u(k)| \leq K} \frac{\varepsilon}{T} - \sum_{|u(k)| > K} \frac{\varepsilon}{T} = -\varepsilon.$$

Hence we get condition (11.1).

To show condition (11.2) we consider two cases.

Case 1. Suppose that $\inf_{u \in H} J(u) > -\infty$. We will show that for any $\sigma > 0$ the following equality holds

$$\inf_{||u||_\infty \leq \sigma} J(u) = -\sum_{k=1}^{T} \sup_{|t| \leq \sigma} F(k,t). \tag{5}$$

Note that for any $||u||_\infty \leq \sigma$ we have

$$J(u) = -\sum_{k=1}^{T} F(k,u(k)) \geq -\sum_{k=1}^{T} \sup_{|t| \leq \sigma} F(k,t)$$

On the other hand for any $\varepsilon > 0$ and $k \in [1,T]$ there is $|t_k| \leq \sigma$ with

$$F(k,t_k) > \sup_{|t| \leq \sigma} F(k,t) - \frac{\varepsilon}{T}.$$

Define $\tilde{u} \in H$ by $\tilde{u}(k) = t_k$ for $k \in [1,T]$. Then $||\tilde{u}||_\infty \leq \sigma$ and

$$J(\tilde{u}) = -\sum_{k=1}^{T} F(k,\tilde{u}(k)) < -\sum_{k=1}^{T} \sup_{|t| \leq \sigma} F(k,t) - \varepsilon.$$

Hence we obtain (5). Similarly one can show that

$$\inf_{u\in H} J(u) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t). \tag{6}$$

Now, using (12.2), (5) and (6) we obtain

$$\inf_{u\in H} J(u) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t) < -\sum_{k=1}^{T} \sup_{|t|\le s'} F(k,t) = \inf_{||u||_\infty \le s'} J(u) \le \inf_{\mu(u)\le s} J(u).$$

Case 2. Suppose that $\inf_{u\in H} J(u) = -\infty$. Then (11.2) holds since continuous functional J attains its minimum on compact set $\{u : ||u||_\infty \le s'\}$.

Now we will show (11.3). Note that $J(0) = 0$. For any $u \in H$ with $r \le \mu(u) \le s$ we have $r' \le ||u||_\infty \le s'$. Take $k \in [1, T]$ with $||u||_\infty = u(k)$. By (12.3) we obtain

$$J(u) = -\sum_{h=1}^{T} F(h,u(h)) = -F(k,u(k)) - \sum_{h\neq k} F(h,u(h)) \ge$$

$$-\sup_{r'\le|t|\le s'} F(k,t) - \sum_{h\neq k} \sup_{|t|\le s'} F(h,t) \ge 0.$$

Finally, by Theorem 11 there is $\lambda^\star > 0$ such that the functional $E_{\lambda^\star}$ has at least three critical points in H. ∎

Corollary 13 *Let $0 < r' < s'$. Suppose that there is a positive constant c such that*

$$r' < \frac{1}{2}\left(\frac{cp^-}{T+1}\right)^{\frac{1}{p^+}} \quad \text{and } s' > \frac{T+1}{2}(cp^+)^{\frac{1}{p^-}}$$

and conditions (12.1)–(12.3) are fulfilled. Then there is $\lambda^\star > 0$ such that $E_{\lambda^\star}$ has at least three critical points, two of which must be nontrivial.

Proof. Let $r > 0$. Then $\mu(u) \ge r$ is equivalent to

$$\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \ge r.$$

Therefore there is $k \in [1, T+1]$ with

$$\frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \ge \frac{r}{T+1}.$$

Hence

$$|\Delta u(k-1)| \ge \left(\frac{rp(k-1)}{T+1}\right)^{\frac{1}{p(k-1)}}.$$

Thus there is $k \in [1, T]$ with

$$|u(k)| \geq \frac{1}{2}\left(\frac{rp(k-1)}{T+1}\right)^{\frac{1}{p(k-1)}} \geq \frac{1}{2}\left(\frac{rp^-}{T+1}\right)^{\frac{1}{p^+}}.$$

Consequently

$$\inf\{||u||_\infty : \mu(u) \geq r\} \geq \frac{1}{2}\left(\frac{rp^-}{T+1}\right)^{\frac{1}{p^+}}. \tag{7}$$

Now, let $s > 0$. Then $\mu(u) \leq s$ is equivalent to

$$\sum_{k=1}^{T+1} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \leq s.$$

Hence

$$\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \leq s$$

for every $k \in [1, T+1]$. Thus for every $k \in [1, T+1]$ we obtain

$$|\Delta u(k-1)| \leq (sp(k-1))^{\frac{1}{p(k-1)}}.$$

Note that if $|\Delta u(k-1)| \leq a$ for some a and every $k \in [1, T+1]$, then $|u(k)| \leq \frac{T+1}{2}a$. Using this observation we obtain that

$$|u(k)| \leq \frac{T+1}{2}(sp(k-1))^{\frac{1}{p(k-1)}}.$$

Thus

$$||u||_\infty \leq \frac{T+1}{2}(sp^+)^{\frac{1}{p^-}}.$$

Therefore

$$\sup\{||u||_\infty : \mu(u) \leq s\} \leq \frac{T+1}{2}(sp^+)^{\frac{1}{p^-}}.$$

Suppose that $c > 0$ is such that

$$r' < \frac{1}{2}\left(\frac{cp^-}{T+1}\right)^{\frac{1}{p^+}} \quad \text{and } s' > \frac{T+1}{2}(cp^+)^{\frac{1}{p^-}}$$

Since the following functions $t \mapsto \inf\{||u||_\infty : \mu(u) \geq t\}$ and $t \mapsto \sup\{||u||_\infty : \mu(u) \leq t\}$ are continuous and their ranges are equal to $[0, \infty)$, then there are $r < c < s$ with

$$\sup\{||u||_\infty : \mu(u) \leq s\} = s' \text{ and } \inf\{||u||_\infty : \mu(u) \geq r\} = r'.$$

■

Example. Consider the following equation

$$\begin{cases} \Delta\left(|\Delta u(0)|^2 \Delta u(0)\right) = 2\lambda\left(\frac{u^3(1)}{10} - u(1)\right), \\ \Delta\left(|\Delta u(1)|^3 \Delta u(1)\right) = 4\lambda\left(u(2) - \frac{1}{10}\right)^3, \\ u(0) = u(3) = 0. \end{cases}$$

Then $T = 2,\ p(0) = 4,\ p(1) = 5,\ p^- = 4,\ p^+ = 5,\ f(1,t) = 2\left(\frac{t^3}{10} - t\right)$ and $f(2,t) = 4\left(t - \frac{1}{10}\right)^3$. Thus

$$F(1,t) = \frac{t^4}{20} - t^2 \text{ and } F(2,t) = \frac{1}{10^4} - \left(t - \frac{1}{10}\right)^4.$$

Taking $c = 1$ we obtain

$$\frac{1}{2}\left(\frac{cp^-}{T+1}\right)^{\frac{1}{p^+}} = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{5}} \approx 0.5296 \text{ and } \frac{T+1}{2}(cp^+)^{\frac{1}{p^-}} = \frac{3}{2}5^{\frac{1}{4}} \approx 2.2430.$$

Take $r' = 0.2$ and $s' = 3$. Note that $\sup_{0.2\le|t|\le3} F(1,t) < 0$, $\sup_{|t|\le3} F(1,t) = 0$ and $\sup_{t\in\mathbb{R}} F(1,t) = \infty$. Moreover $\sup_{0.2\le|t|\le3} F(2,t) = 0$ and $\sup_{|t|\le3} F(1,t) = \sup_{t\in\mathbb{R}} F(1,t) = \frac{1}{10^4}$. Hence by Corollary 13 the equation has at least three solutions for some $\lambda^\star > 0$.

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