Positive solutions for anisotropic discrete boundary value problems

Marek Galewski, Szymon Głąb, Renata Wieteska

Abstract

Using mountain pass arguments and the Karsuh-Kuhn-Tucker Theorem, we prove the existence of at least two positive solution of the anisotropic discrete Dirichlet boundary value problem. Our results generalize and improve those of [16].

Math Subject Classifications: 39A10, 34B18, 58E30.

Key Words: Discrete boundary value problem; variational methods; mountain pass theorem; Karush-Kuhn-Tucker Theorem; positive solution; anisotropic problem.

1 Introduction

In this note we consider an anisotropic difference equation with Dirichlet type boundary condition on the form

$$\begin{cases} \Delta\left(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\right) + f(k,y(k)) = 0, \ k \in [1,T], \\ y(0) = y(T+1) = 0, \end{cases} \tag{1}$$

where $T \geq 2$ is a integer, $f : [1,T] \times \mathbb{R} \to (0,+\infty)$ is a continuous function; $[1,T]$ is a discrete interval $\{1,2,...,T\}$, $\Delta y(k-1) = y(k) - y(k-1)$ is the forward difference operator; $y(k) \in \mathbb{R}$ for all $k \in [1,T]$; $p : [0,T+1] \to [2,+\infty)$. Let $p^- = \min_{k\in[0,T+1]} p(k)$; $p^+ = \max_{k\in[0,T+1]} p(k)$.

About the nonlinear term, we assume the following condition

(C.1) There exist a number $m > p^+$ and functions $\varphi_1, \varphi_2 : [1,T] \to (0,\infty)$, $\psi_1, \psi_2 : [1,T] \to (0,\infty)$ such that

$$\psi_1(k) + \varphi_1(k)|y|^{m-2}y \leq f(k,y) \leq \varphi_2(k)|y|^{m-2}y + \psi_2(k)$$

for all $y \geq 0$ and all $k \in [1,T]$.

Now, we will show the example of a function which satisfies condition **(C.1)**.

Example 1 *Let* $f : [1,T] \times \mathbb{R} \to (0,\infty)$ *be given by*

$$f(k,y) = |y|^{m-2}\, y \frac{2 + arctg(y)}{T^2 k} + \frac{\sin^2(k)e^{-|y|} + 1}{T^3}$$

for $(k,y) \in [1,T] \times \mathbb{R}$*; here* $m > p^+$*. We see that for* $y \geq 0$ *we have*

$$\frac{1}{T^3} + \frac{2}{T^2 k}\,|y|^{m-2}\,y \leq f(k,y) \leq \frac{4+\pi}{2T^2 k}\,|y|^{m-2}\,y + \frac{2}{T^3}.$$

Thus we may put

$$\varphi_1(k) = \frac{2}{T^2 k};\ \varphi_2(k) = \frac{4+\pi}{2T^2 k};\ \psi_1(k) = \frac{1}{T^3};\ \psi_2(k) = \frac{2}{T^3}.$$

Solutions to (1) will be investigated in a space

$$Y = \{y : [0, T+1] \to \mathbb{R} : y(0) = y(T+1) = 0\}$$

considered with a norm

$$\|y\| = \left(\sum_{k=1}^{T+1} |\Delta y(k-1)|^2 \right)^{1/2}$$

with which Y becomes a Hilbert space. For $y \in Y$ let

$$y_+ = \max\{y, 0\}, \quad y_- = \max\{-y, 0\}.$$

Note that

$$y_+ \geq 0 \text{ and } y_- \geq 0; \quad y = y_+ - y_-; \quad y_+ \cdot y_- = 0.$$

In order to demonstrate that problem (1) has at least two positive solutions we assume additionally the following condition

(C.2) $\qquad T^{\frac{p^+-2}{2}} \left(\frac{1}{\sqrt{T+1}} \right)^{p^+} > \sum_{k=1}^{T} \left(\varphi_2(k) + \psi_2(k) \right).$

Example 2 *We show how assumption **(C.2)** is verified in Example 1. Taking $p^+ = 18$ and $T = 200$ we see that*

$$T^{\frac{p^+-2}{2}} \left(\frac{1}{\sqrt{T+1}} \right)^{p^+} = 0.009 > 0.002 = \sum_{k=1}^{T} \left(\varphi_2(k) + \psi_2(k) \right).$$

Theorem 3 *Suppose that assumptions **(C.1)** and **(C.2)** hold. Then* (1) *has at least two positive solutions.*

Discrete BVPs received some attention lately. Let us mention, far from being exhaustive, the following recent papers on discrete BVPs investigated via critical point theory, [1], [3], [4], [11], [14], [15], [18], [19], [20]. The tools employed cover the Morse theory, mountain pass methodology, linking arguments, i.e. methods usually applied in continuous problems.

Continuous versions of problems like (1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [17]), electrorheological fluids (see [13]) or image restoration (see [5]). Variational continuous anisotropic problems have been started by Fan and Zhang in [7] and later considered by many methods and authors (see [9] for an extensive survey of such boundary value problems). The research concerning the discrete anisotropic problems of type (1) have only been started (see [10], [12] where known tools from the critical point theory are applied in order to get the existence of solutions).

When compared with [16] we see that our problem is more general since we consider variable exponent case instead of a constant one. While we do not include term depending on $\Phi_{p^-}(y) = |y|^{p^- -2}y$ in the nonlinear part as is the case in [16], it is apparent that our results would also hold should we have made our nonlinearity more complicated. We note that term $\Phi_{p^-}(y) = |y|^{p^- -2}y$ does not influence the growth of the nonlinearity.

2 Auxiliary results

We connect positive solutions to (1) with critical points of suitably chosen action functional. Let

$$F(k,y)=\int_0^y f(k,s)ds \text{ for } y\in\mathbb{R} \text{ and } k\in[1,T].$$

Let us define a functional $J:Y\to R$ by the formula

$$J(y)=\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta y(k-1)|^{p(k-1)}-\sum_{k=1}^{T}F(k,y_+(k)).$$

Functional J is slightly different from functionals applied in investigating the existence of positive solutions, compare with [15]. Thus we indicate its properties. The functional J is continuously Gâteaux differentiable and its Gâteaux derivative J' at y reads

$$\begin{aligned}\langle J'(y),v\rangle=\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta v(k-1)-\\ \sum_{k=1}^{T}f(k,y_+(k))v(k)\end{aligned}\tag{2}$$

for all $v\in Y$. Suppose that y is a critical point to J, i.e. $\langle J'(y),v\rangle=0$ for all $v\in Y$. Summing by parts and taking boundary values into account, see [8], we observe that

$$\begin{aligned}0=-\sum_{k=1}^{T+1}\Delta(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1))v(k)-\\ \sum_{k=1}^{T}f(k,y_+(k))v(k).\end{aligned}$$

Since $v\in Y$ is arbitrary we see that y satisfies (1).

Now, we recall some auxiliary materials which we use later on, **(A.1)-(A.3)** see [12], **(A.4)**, **(A.5)** see [8], **(A.6)** see [15]:

(A.1) *For every $y\in Y$ with $\|y\|>1$ we have*

$$\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)}\geq T^{\frac{2-p^-}{2}}\|y\|^{p^-}-T.$$

(A.2) *For every $y\in Y$ with $\|y\|\leq 1$ we have*

$$\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)}\geq T^{\frac{p^+-2}{2}}\|y\|^{p^+}.$$

(A.3) *For every $y\in Y$ and for any $m\geq 2$ we have*

$$(T+1)^{\frac{2-m}{2}}\|y\|^m\leq\sum_{k=1}^{T+1}|\Delta y(k-1)|^m\leq(T+1)\|y\|^m.$$

(A.4) *If* $p^+ \geq 2$, *there exists* $C_{p^+} > 0$ *such that for every* $y \in Y$

$$\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)} \leq 2^{p^+}(T+1)\left(C_{p^+}\|y\|^{p^+}+1\right).$$

(A.5) *For every* $y \in Y$ *and for any* $m \geq 2$ *we have*

$$\sum_{k=1}^{T+1} |\Delta y(k-1)|^{m} \leq 2^{m}\sum_{k=1}^{T} |y(k)|^{m}.$$

(A.6) *For every* $y \in Y$ *and for any* $p, q > 1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *we have*

$$\|y\|_C = \max_{k\in[1,T]} |y(k)| \leq (T+1)^{\frac{1}{q}}\left(\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p}\right)^{1/p}.$$

Let E be a real Banach space. We say that a functional $J : E \to \mathbb{R}$ satisfies Palais-Smale condition if every sequence (y_n) such that $\{J(y_n)\}$ is bounded and $J'(y_n) \to 0$, has a convergent subsequence.

Lemma 4 *[6] Let E be a Banach space and $J \in C^1(E,\mathbb{R})$ satisfy Palais-Smale condition. Assume that there exist $x_0, x_1 \in E$ and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*

$$\max\{J(x_0), J(x_1)\} < \inf_{x\in\partial\Omega} J(x).$$

Let

$$\Gamma = \{h \in C([0,1],E) : h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma}\max_{s\in[0,1]} J(h(s)).$$

Then c is a critical value of J; that is, there exists $x^\star \in E$ such that $J'(x^\star) = 0$ and $J(x^\star) = c$, where $c > \max\{J(x_0), J(x_1)\}$.

Finally we recall the Karush-Kuhn-Tucker theorem with Slater qualification conditions (for one constraint), see [2]:

Theorem 5 *Let X be a finite-dimensional Euclidean space, $\eta, \mu : X \to \mathbb{R}$ be differentiable functions, with μ convex and $\inf_X \mu < 0$, and $S = \{x \in X : \mu(x) \leq 0\}$. Moreover, let $\overline{x} \in S$ be such that $\eta(\overline{x}) = \inf_S \eta$. Then, there exists $\sigma \geq 0$ such that*

$$\eta'(\overline{x}) + \sigma\mu'(\overline{x}) = 0 \quad \text{and} \quad \sigma\mu(\overline{x}) = 0.$$

We will provide now some results which are used in the proof of the Main Theorem. The following lemma may be viewed as a kind of a discrete maximum principle.

Lemma 6 *Assume that $y \in Y$ is a solution of the equation*

$$\begin{cases} \Delta\left(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\right) + f(k, y_+(k)) = 0, k \in [1,T], \\ y(0) = y(T+1) = 0, \end{cases} \tag{3}$$

then $y(k) > 0$ for all $k \in [1,T]$ and moreover y is a solution of (1).

Proof. We will show that

$$\Delta y(k-1)\Delta y_-(k-1) \le 0 \ \ \textit{for every} \ \ k \in [1, T+1].$$

Indeed,

$$\begin{aligned}
&\Delta y(k-1)\Delta y_-(k-1) = (y(k) - y(k-1))(y_-(k) - y_-(k-1)) = \\
&[(y_+(k) - y_+(k-1)) - (y_-(k) - y_-(k-1))]\,(y_-(k) - y_-(k-1)) = \\
&(y_+(k) - y_+(k-1))\,(y_-(k) - y_-(k-1)) - (y_-(k) - y_-(k-1))^2 = \\
&y_+(k)y_-(k) - y_+(k)y_-(k-1) - y_+(k-1)y_-(k) + y_+(k-1)y_-(k-1) - \\
&(y_-(k) - y_-(k-1))^2 = \\
&-\left[y_+(k)y_-(k-1) + y_+(k-1)y_-(k) + (y_-(k) - y_-(k-1))^2\right] \le 0.
\end{aligned}$$

Assume that $y \in Y$ is a solution of (3). Taking $v = y_-$ in (2) we obtain

$$\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta y_-(k-1) = \sum_{k=1}^{T} f(k, y_+(k))y_-(k).$$

Since the term on the left is non-positive and the one on the right is non-negative, so this equation holds true if the both terms are equal zero, which leads to $y_-(k) = 0$ for all $k \in [1, T]$. Then $y = y_+$. Therefore y is a positive solution of (1). Arguing by contradiction, assume that there exists $k \in [1, T]$ such that $y(k) = 0$, while we can assume $y(k-1) > 0$. Then, by (3) we have

$$|y(k+1)|^{p(k)-2}y(k+1) = -y(k-1)^{p(k-1)-1} - f(k, 0) < 0,$$

which implies $y(k+1) < 0$, a contradiction. So $y(k) > 0$ for all $k \in [1, T]$. ■

Finally we prove that J satisfies Palais-Smale condition.

Lemma 7 *Assume that* ***(C.1)*** *holds. Then the functional J satisfies Palais-Smale condition.*

Proof. Assume that $\{y_n\}$ is such that $\{J(y_n)\}$ is bounded and $J'(y_n) \to 0$. Since Y is finitely dimensional, it is enough to show that $\{y_n\}$ is bounded. Note that

$$\Delta y_+(k)\Delta y_-(k) \le 0 \quad \text{for every} \quad k \in [0, T].$$

Using the above inequality we obtain

$$\begin{aligned}
&-\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta y_-(k-1) = \\
&-\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta(y_+(k-1) - y_-(k-1))\Delta y_-(k-1) = \\
&-\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y_+(k-1)\Delta y_-(k-1) + \qquad (4) \\
&\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y_-(k-1)\Delta y_-(k-1) \ge \\
&\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\left(\Delta y_-(k-1)\right)^2 \ge \sum_{k=1}^{T+1} |\Delta y_-(k-1)|^{p(k-1)}.
\end{aligned}$$

Since $y_n = (y_n)_+ - (y_n)_-$ we will show that $\{(y_n)_-\}$ and $\{(y_n)_+\}$ are bounded. Suppose that $\{(y_n)_-\}$ is unbounded. Then we may assume that there exists $N_0 > 0$ such that for $n \geq N_0$ we have $\|(y_n)_-\| \geq T \geq 2$. Using (4) we obtain

$$\langle J'(y_n), (y_n)_-\rangle = \sum_{k=1}^{T+1} |\Delta y_n(k-1)|^{p(k-1)-2}\Delta y_n(k-1)\Delta(y_n)_-(k-1)$$

$$-\sum_{k=1}^{T} f(k,(y_n)_+(k))(y_n)_-(k) \leq -\sum_{k=1}^{T+1} |\Delta(y_n)_-(k-1)|^{p(k-1)}.$$

So by **(A.1)** we obtain

$$T^{\frac{2-p^-}{2}}\|(y_n)_-\|^{p^-} - T \leq \sum_{k=1}^{T+1} |\Delta(y_n)_-(k-1)|^{p(k-1)} \leq$$

$$\langle J'(y_n), -(y_n)_-\rangle \leq \|J'(y_n)\| \cdot \|(y_n)_-\|.$$

Next, we see

$$T^{\frac{2-p^-}{2}}\|(y_n)_-\|^{p^-} \leq \|J'(y_n)\| \cdot \|(y_n)_-\| + T \leq$$

$$\|J'(y_n)\| \cdot \|(y_n)_-\| + \|(y_n)_-\| \leq (\|J'(y_n)\| + 1)\,\|(y_n)_-\|$$

and

$$T^{\frac{2-p^-}{2}}\|(y_n)_-\|^{p^- -1} \leq \left(\|J'(y_n)\| + 1\right).$$

Since for a fixed $\varepsilon > 0$ there exists some $N_1 \geq N_0$ such that $\|J'(y_n)\| < \varepsilon$ for every $n \geq N_1$, we get

$$\|(y_n)_-\|^{p^- -1} \leq \frac{(\varepsilon+1)}{T^{\frac{2-p^-}{2}}}.$$

This means that $\{(y_n)_-\}$ is bounded.

Now, we will show that $\{(y_n)_+\}$ is bounded. Suppose that $\{(y_n)_+\}$ is unbounded. We may assume that $\|(y_n)_+\| \to \infty$. Since

$$f(k,y) \geq \varphi_1(k)|y|^{m-2}y + \psi_1(k) \quad \text{ for all } k \in [1,T],$$

then

$$F(k,y) \geq \frac{\varphi_1(k)}{m}|y|^m + \psi_1(k)y.$$

Thus by **(A.3)** and **(A.5)** we obtain

$$\sum_{k=1}^{T} F(k,(y_n)_+(k)) \geq \frac{\varphi_1^-}{m}\sum_{k=1}^{T}|(y_n)_+(k)|^m \geq \frac{\varphi_1^-}{m}2^{-m}(T+1)^{\frac{2-m}{2}}\|(y_n)_+\|^m,$$

where $\varphi_1^- = \min_{k\in[1,T]} \varphi_1(k)$. Therefore by **(A.4),** we have

$$J(y_n) = \sum_{k=1}^{T+1} \left[\frac{1}{p(k-1)}|\Delta y_n(k-1)|^{p(k-1)} - F(k,(y_n)_+(k))\right] \leq$$

$$2^{p^+}(T+1)\left(C_{p^+}\left\|(y_n)_+ - (y_n)_-\right\|^{p^+} + 1\right) - \frac{\varphi_1^-}{m}2^{-m}(T+1)^{\frac{2-m}{2}}\|(y_n)_+\|^m \leq$$

$$2^{p^+}(T+1)\left(C_{p^+}2^{p^+-1}\left(\left\|(y_n)_+\right\|^{p^+} + \left\|(y_n)_-\right\|^{p^+}\right) + 1\right) -$$

$$\frac{\varphi_1^-}{m}2^{-m}(T+1)^{\frac{2-m}{2}}\|(y_n)_+\|^m.$$

Since $p^+ < m$ and $\{(y_n)_+\}$ is unbounded and $\{(y_n)_-\}$ is bounded, so $J(y_n) \to -\infty$. Thus we obtain a contradiction with the assumption $\{J(y_n)\}$ is bounded, so $\{(y_n)_+\}$ is bounded. It follows that $\{y_n\}$ is bounded. ■

3 Proof of the main result

In this section we present the proof of Theorem 3.

Proof. Assume that $y_0 \in Y$ is a local minimizer of J in

$$B := \{y \in Y : \mu(y) \leq 0\},$$

where $\mu(y) = \frac{\|y\|^2}{2} - \frac{1}{2(T+1)}$. Note that for $y \in B$ by **(A.6)** it follows that for all $k \in [1,T]$

$$|y(k)| \leq \max_{s\in[1,T]} |y(s)| \leq \sqrt{T+1}\,\|y\| \leq \frac{1}{\sqrt{T+1}}\sqrt{T+1} = 1.$$

We prove that $y_0 \in IntB$, by contradiction. Thus suppose otherwise, i.e. we suppose that $y_0 \in \partial B$. Then by Theorem 5 there exists $\sigma \geq 0$ such that for all $v \in Y$

$$\langle J'(y_0), v\rangle + \sigma\langle y_0, v\rangle = 0.$$

Hence

$$\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)-2}\Delta y_0(k-1)\Delta v(k-1) -$$

$$\sum_{k=1}^{T} f(k,(y_0)_+(k))v(k) + \sigma\sum_{k=1}^{T}\langle y_0(k), v(k)\rangle = 0.$$

Taking $v = y_0$, we see that

$$\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} + \sigma\|y_0\|^2 = \sum_{k=1}^{T} f(k,(y_0)_+(k))y_0(k).$$

Since $y_0 \in \partial B$, we see that $\|y_0\| = \frac{1}{\sqrt{T+1}}$. Thus by **(A.2)** we have

$$\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} + \sigma\|y_0\|^2 \geq \sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} \geq T^{\frac{p^+-2}{2}}\left(\frac{1}{\sqrt{T+1}}\right)^{p^+}.$$

On the other hand

$$\sum_{k=1}^{T} f(k,(y_0)_+(k))y_0(k) =$$
$$\sum_{k=1}^{T} f(k,(y_0)_+(k))(y_0)_+(k) - \sum_{k=1}^{T} f(k,(y_0)_+(k))(y_0)_-(k) \leq$$
$$\sum_{k=1}^{T} \varphi_2(k)\left|(y_0)_+(k)\right|^m + \sum_{k=1}^{T} \psi_2(k)\left|(y_0)_+(k)\right| \leq \sum_{k=1}^{T} \varphi_2(k) + \sum_{k=1}^{T} \psi_2(k).$$

Thus

$$T^{\frac{p^+-2}{2}}\left(\frac{1}{\sqrt{T+1}}\right)^{p^+} \leq \sum_{k=1}^{T}\left(\varphi_2(k)+\psi_2(k)\right).$$

A contradiction with the assumption **(C.2)**. Hence $y_0 \in IntB$ and y_0 is a local minimizer of J. Thus $J(y_0) < \min_{y\in\partial B} J(y)$. We will show that there exists y_1 such that $y_1 \in Y \setminus B$ and $J(y_1) < \min_{y\in\partial B} J(y)$. Let $y_\lambda \in Y$ be define as follows: $y_\lambda(k) = \lambda$ for $k = 1, ..., T$ and $y_\lambda(0) = y_\lambda(T+1) = 0$. Then for $\lambda > 1$ we have

$$J(y_\lambda) \leq \frac{\lambda^{p(0)}}{p(0)} + \frac{\lambda^{p(T)}}{p(T)} - \sum_{k=1}^{T}\frac{\varphi_1(k)\lambda^m}{m} \leq \frac{\lambda^{p^+}}{p(0)} + \frac{\lambda^{p^+}}{p(T)} - \frac{\varphi_1^-\lambda^m}{m}T - \psi_1^-\lambda T.$$

Since $m > p^+$, then $\lim_{\lambda\to\infty} J(y_\lambda) = -\infty$. Thus there exists λ_0 with $J(y_{\lambda_0}) < \min_{y\in\partial B} J(y)$. By Lemma 4 and Lemma 7 we obtain a critical value of the functional J for some $y^\star \in Y \setminus \partial B$. Then y_0 and $y^\star$ are two different critical points of J and therefore by Lemma 6 these are positive solutions of problem (1). ■

Acknowledgement 8 *The Authors would like to thank anonymous Referee for suggestions which allowed us to improve both the results and their presentation.*

References

[1] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Difference Equ. 2 (2005) 93–99.

[2] J.M. Borwein, A.S. Lewis, Convex analysis and nonlinear optimization. Theory and examples. 2nd ed., CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 3. New York, NY: Springer. xii, 2006.

[3] A. Cabada, A. Iannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems. J. Math. Anal. Appl. 356 (2009), 418–428.

[4] X. Cai, J. Yu, Existence theorems of periodic solutions for second-order nonlinear difference equations, Adv. Difference Equ. 2008 (2008) Article ID 247071.

[5] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, SIAM J. Appl. Math. 66 (2006), 1383-1406.

[6] D. Guo, Nonlinear Functional Analysis, Shandong science and technology Press, 1985.

[7] X.L. Fan , H. Zhang, Existence of Solutions for $p(x)-$Lapacian Dirichlet Problem, Nonlinear Anal., Theory Methods Appl. 52 (2003), 1843-1852.

[8] M. Galewski, R. Wieteska, On the system of anisotropic discrete BVPs, to appear J. Difference Equ. Appl., DOI:10.1080/10236198.2012.709508.

[9] P. Harjulehto, P. Hästö, U. V. Le and M. Nuortio, Overview of differential equations with non-standard growth, Nonlinear Anal. 72 (2010), 4551-4574

[10] B. Kone. S. Ouaro, Weak solutions for anisotropic discrete boundary value problems, J. Difference Equ. Appl. 17 (2011), 1537-1547.

[11] J.Q. Liu, J.B. Su, Remarks on multiple nontrivial solutions for quasi-linear resonant problemes, J. Math. Anal. Appl. 258 (2001) 209–222.

[12] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems. J. Difference Equ. Appl. 15 (2009), 557–567.

[13] M. Růžička, Electrorheological fluids: Modelling and Mathematical Theory, in: Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.

[14] P. Stehlík, On variational methods for periodic discrete problems, J. Difference Equ. Appl. 14 (3) (2008) 259–273.

[15] Y. Tian, Z. Du, W. Ge, Existence results for discrete Sturm-Liouville problem via variational methods, J. Difference Equ. Appl. 13 (6) (2007) 467–478.

[16] Y. Tian, W. Ge, Existence of multiple positive solutions for discrete problems with p-Laplacian via variational methods. Electron. J. Differential Equations 45 (2011), 8 pp.

[17] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR Izv. 29 (1987), 33-66.

[18] Y. Yang and J. Zhang, Existence of solution for some discrete value problems with a parameter, Appl. Math. Comput. 211 (2009), 293–302.

[19] G. Zhang, S.S. Cheng, Existence of solutions for a nonlinear system with a parameter, J. Math. Anal. Appl. 314 (1) (2006) 311–319.

[20] G. Zhang, Existence of non-zero solutions for a nonlinear system with a parameter, Nonlinear Anal. 66 (6) (2007) 1400–1416.

Marek Galewski, Szymon Głąb, Renata Wieteska
Institute of Mathematics,
Technical University of Lodz,
Wolczanska 215, 90-924 Lodz, Poland,
marek.galewski@p.lodz.pl, szymon.glab@p.lodz.pl, renata.wieteska@p.lodz.pl