# ON THE LACZKOVICH-KOMJÁTH PROPERTY OF SIGMA-IDEALS

MAREK BALCERZAK AND SZYMON GŁĄB

Abstract. Komjáth in 1984 proved that, for each sequence $(A_n)$ of analytic subsets of a Polish apace $X$, if $\limsup_{n\in H} A_n$ is uncountable for every $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n$ is uncountable for some $G \in [\mathbb{N}]^\omega$. This fact, by our definition, means that the $\sigma$-ideal $[X]^{\leq\omega}$ has property (LK). We prove that every $\sigma$-ideal generated by $X/E$ has property (LK), for an equivalence relation $E \subset X^2$ of type $F_\sigma$ with uncountably many equivalence classes. We also show the parametric version of this result. Finally, the invariance of property (LK) with respect to various operations is studied.

## 1. Introduction

We use standard set theoretical notation (see [Sr] or [Ke]). As usual, $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $(A_n)$ be a sequence of subsets of the real line (or a Polish space). We are interested in the following question. If the set $\limsup_{n\in H} A_n$ is large, in a given sense, for every $H \in [\mathbb{N}]^\omega$, is it true that at least one among the sets $\bigcap_{n\in H} A_n$, $n \in H$, is large in the same sense? Observe that

$$\limsup_{n\in H} A_n = \bigcap_{n\in H} \bigcup_{\substack{k\in H\\ k\geq n}} A_k = \bigcup_{G\in[H]^\omega} \bigcap_{k\in G} A_k,$$

so, we ask how strongly the largeness of all unions $\bigcup_{G\in[H]^\omega} \bigcap_{k\in G} A_k$, $H \in [\mathbb{N}]^\omega$, has the influence on the largeness of the summands $\bigcap_{k\in H} A_k$, $H \in [\mathbb{N}]^\omega$. These and related questions were discussed by Laczkovich [L] and Halmos [H]. Laczkovich in [L] proved that, for every sequence $(A_n)$ of Borel subsets of a Polish space, if $\limsup_{n\in H} A_n$ is uncountable for each $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n$ is uncountable for some $G \in [\mathbb{N}]^\omega$. This result was then generalized by Komjáth [K, Thm 1] to the case when the sets $A_n$ are analytic. Note that an uncountable analytic subset of a Polish space contains a homeomorphic copy of the Cantor set [Sr, Thm 4.3.5], so it is of cardinality of the continuum. Komjáth also proved that the result of Laczkovich cannot be generalized within ZFC to the case of coanalytic sets. Namely, if $V = L$, there is a sequence $(A_n)$ of coanalytic sets such that $|\limsup_{n\in H} A_n| > \omega$ and $|\bigcap_{n\in H} A_n| \leq \omega$ for all $H \in [\mathbb{N}]^\omega$; see [K, Thm. 4].

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From now on, let $X$ be an uncountable Polish space. In connection with the above quoted theorem of Komjáth about analytic sets, we introduce the following property of an ideal $\mathcal{J}$ of subsets of $X$. We say that $\mathcal{J}$ has *property (LK)* (*the Laczkovich-Komjáth property*) whenever for every sequence $(A_n)$ of analytic subsets of $X$, if $\limsup_{n\in H} A_n \notin \mathcal{J}$ for each $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n \notin \mathcal{J}$ for some $G \in [\mathbb{N}]^\omega$. So, the Komjáth theorem states that the ideal $[X]^{\leq\omega}$ has property (LK). While studying ideals with property (LK) we may restrict our considerations only to those ones with bases consisting of analytic sets. Recall that a family $\mathcal{F}$ is a *base* of an ideal $\mathcal{J} \subset \mathcal{P}(X)$ if $\mathcal{F} \subset \mathcal{J}$ and each set $A \in \mathcal{J}$ is contained in a set $B \in \mathcal{F}$. Namely, observe that $\mathcal{J}$ has property (LK) if and only if the ideal

$$\mathcal{J}|_{\mathbf{\Sigma}^1_1} = \{A \subset X : (\exists\, B \in \mathcal{J} \cap \mathbf{\Sigma}^1_1(X))\ A \subset B\}$$

has property (LK), and $\mathcal{J} \cap \mathbf{\Sigma}^1_1(X)$ is a base of $\mathcal{J}|_{\mathbf{\Sigma}^1_1}$ consisting of analytic sets. If a base of an ideal $\mathcal{J}$ consists of analytic sets, we say that $\mathcal{J}$ has an *analytic base*.

The next observation is due to T. Banakh (oral communication).

**Proposition 1.** *An ideal $\mathcal{J}$ with an analytic base and with property (LK) is $\sigma$-additive.*

*Proof.* Suppose that $\mathcal{J}$ is not $\sigma$-additive. Let $(A_n)$ be an increasing sequence of analytic sets from $\mathcal{J}$ whose union is not in $\mathcal{J}$. Then for each $H \in [\mathbb{N}]^\omega$ we have $\limsup_{n\in H} A_n = \bigcup_{n\in\mathbb{N}} A_n \notin \mathcal{J}$ but $\bigcap_{n\in H} = A_{\min(H)} \in \mathcal{J}$. $\square$

If we do not assume that $\mathcal{J}$ has analytic base, the assertion of Proposition 1 can be false. Namely, consider a partition $\mathcal{F} = \{B_n : n \in \mathbb{N}\}$ of the real line into pairwise disjoint Bernstein sets. Let $\mathcal{J}$ stand for the ideal generated by $\mathcal{F} \cup [\mathbb{R}]^{\leq\omega}$. Then $\mathcal{J}$ is not $\sigma$-additive but it has property (LK) since $\mathcal{J}|_{\mathbf{\Sigma}^1_1} = [\mathbb{R}]^{\leq\omega}$.

**Example 2.** Consider $X = \{0,1\}^{\mathbb{N}}$ and the sequence $(A_n)$ of clopen subsets of $X$, given by

$$A_n = \{x \in X \colon x(n) = 1\},\ n \in \mathbb{N}.$$

Let $\lambda$ stand for the standard (product) probability measure on $X$. The sets $A_n$, $n \in \mathbb{N}$, are independent with $\lambda(A_n) = 1/2$. We then have $\lambda(\bigcap_{n\in H} A_n) = 0$ for each $H \in [\mathbb{N}]^\omega$ and, by the Borel-Cantelli lemma, $\lambda(\limsup_{n\in H} A_n) = 1$ for each $H \in [\mathbb{N}]^\omega$. Hence the $\sigma$-ideal of sets of measure zero does not have property (LK). This example can be easily modified to the case of $X = [0,1]$ with Lebesgue measure – the respective versions were given by Laczkovich [L, proof of 2] and Halmos [H]. Also note that the sets $\limsup_{n\in H} A_n$, $H \in [\mathbb{N}]^\omega$, are dense of type $G_\delta$ (thus residual) while the sets $\bigcap_{n\in H} A_n$, $H \in [\mathbb{N}]^\omega$, are closed nowhere dense. Hence it follows that the $\sigma$-ideal of meager sets, and the $\sigma$-ideal generated by closed sets of measure zero, do not have property (LK).

Denote by $\sigma(\mathbf{\Sigma}^1_1)$ the $\sigma$-algebra generated by all analytic subsets of $X$. Recall that a Boolean algebra $A$ is said to be *atomic* if for each positive element $x \in A$, there is an atom $a \in A$ such that $a \leq x$. If $\mathcal{J} \subset \mathcal{P}(X)$ is an ideal, the symbol $\sigma(\mathbf{\Sigma}^1_1)/\mathcal{J}$ will abbreviate the quotient Boolean algebra $\sigma(\mathbf{\Sigma}^1_1)/(\mathcal{J} \cap \sigma(\mathbf{\Sigma}^1_1))$.

**Proposition 3.** *Let $\mathcal{J}$ be a $\sigma$-ideal, with analytic base, such that $\sigma(\mathbf{\Sigma}^1_1)/\mathcal{J}$ is an atomic Boolean algebra. Then $\mathcal{J}$ has property (LK).*

*Proof.* For $A \in \sigma(\mathbf{\Sigma}^1_1)$ let $[A]$ denote the respective element of $\sigma(\mathbf{\Sigma}^1_1)/\mathcal{J}$. Let $(A_n)$ be a sequence of analytic sets. Since $\mathcal{J}$ is a $\sigma$-ideal, we have $[\limsup_{n\in\mathbb{N}} A_n] = \bigwedge_k \bigvee_{n\geq k} [A_n] \neq 0$. Since $\sigma(\mathbf{\Sigma}^1_1)/\mathcal{J}$ is atomic, pick an atom $a \leq [\limsup_{n\in\mathbb{N}} A_n]$. It follows that $a \leq \bigvee_{n\geq k} [A_n]$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ pick $n_k \geq k$ such that $a \wedge [A_{n_k}] \neq 0$, thus $a = [A_{n_k}]$. Consequently, the set $H = \{n_k \in \mathbb{N} : a = [A_{n_k}]\}$ is infinite, and $a = \bigwedge_{n\in H}[A_n]$. Hence $\bigcap_{n\in H} A_n \notin \mathcal{J}$. □

To show a simple application of Proposition 3, consider an analytic set $A \subset X$, $A \neq X$, and the ideal $\mathcal{P}(A)$. Then $\mathcal{P}(A)$ has property (LK) since the atoms of $\sigma(\mathbf{\Sigma}^1_1)/\mathcal{P}(A)$ are of the form $[\{x\}]$, $x \in X \setminus A$.

## 2. A generalization of the Komjáth theorem

If $A \subset X \times Y$ and $x \in X$, we denote by $A(x) = \{y \in Y : (x, y) \in A\}$; this is the section of $A$ generated by $x$.

Assume that $E \subset X^2$ is an equivalence relation such that the family $X/E$ of all equivalence classes $E(x) = \{y \in X : (x, y) \in E\}$, $x \in X$, is uncountable. Next, consider the $\sigma$-ideal $\mathcal{J}_E$ generated by $X/E$, that is, $A \in \mathcal{J}_E$ if and only if $A \subset \bigcup_{n\in\mathbb{N}} E(x_n)$ for a sequence $(x_n) \in X^{\mathbb{N}}$. A set $B$ is called a *partial transversal* for $E$ if $|B \cap E(x)| \leq 1$, for each $x \in X$. Note that, if a partial transversal $B$ is uncountable then $B \notin \mathcal{J}_E$.

We are going to prove the following generalization of the Komjáth theorem.

**Theorem 4.** *Let $E \subset X^2$ be an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$. Then for every sequence $(A^{(n)})$ of analytic subsets of $X$, such that $\limsup_{n\in H} A^{(n)} \notin \mathcal{J}_E$ for all $H \in [\mathbb{N}]^\omega$, there are sets $G \in [\mathbb{N}]^\omega$ and $P \subset \bigcap_{n\in G} A^{(n)}$ such that $P$ is a partial transversal for $E$, homeomorphic with $\{0,1\}^{\mathbb{N}}$. In particular, the $\sigma$-ideal $\mathcal{J}_E$ possesses property (LK).*

The proof of Theorem 4 combines original ideas from the paper by Komjáth [K] with a demonstration of the fact that every relation $E$ satisfying assumptions of Theorem 3 admits a partial transversal homeomorphic with $\{0,1\}^{\mathbb{N}}$ (cf. [Sr, 2.6.7, 2.6.8]; this fact remains true if $E$ is $\mathbf{\Pi}^1_1$, by the Silver theorem [Ke, 35.20]).

The following three lemmas are counterparts of the respective lemmas in [K]. Before we will formulate them, we give some auxiliary terminology modified respectively in comparision with [K].

Fix a proper $\sigma$-ideal $\mathcal{J} \subset \mathcal{P}(X)$ containing all singletons, and a sequence $(A^{(n)})$ of analytic subsets of $X$ such that $\limsup_{n\in H} A^{(n)} \notin \mathcal{J}$ for all $H \in [\mathbb{N}]^\omega$. Next, fix $H \in [\mathbb{N}]^\omega$.

We say that a set $Y \subset X$ is *good* with respect to $H$ if $Y \cap \limsup_{n\in G} A^{(n)} \notin \mathfrak{J}$ for all $G \in [H]^\omega$. Observe that, if $Y$ is good with respect to $H$, and $Z \subset Y$, $Z \in \mathfrak{J}$, then $Y \setminus Z$ is good with respect to $H$. In particular, if $Y$ is closed and good with respect to $H$, then the perfect kernel of $Y$ (cf. [Sr, 2.6.2]) is good with respect to $H$; we will use this fact several times. For $H_1, H_2 \in [\mathbb{N}]^\omega$ we say that $H_1$ is *almost contained* in $H_2$ if $|H_1 \setminus H_2| < \omega$.

**Lemma 5.** *If a set $Y = \bigcup_{i\in\mathbb{N}} Y_i$ is good with respect to $H \in [\mathbb{N}]^\omega$ then there are $i \in \mathbb{N}$ and $H' \in [H]^\omega$ such that $Y_i$ is good with respect to $H'$.*

The proof is analogous to that given in [K, Lemma 1].

**Lemma 6.** *Let $F, P, A \subset X$ where $F$ and $P$ are closed, $F \in \mathfrak{J}$ and $P \cap A$ is good with respect to a given $H \in [\mathbb{N}]^\omega$. Then there are $x \in P \setminus F$ and $H' \in [H]^\omega$ such that $(P \setminus F) \cap A \cap U$ is good with respect to $H'$, for each neighbourhood $U$ of $x$.*

*Proof.* (cf. [K, Lemma 2]) Since $F \in \mathfrak{J}$, the set $(P \setminus F) \cap A$ is good with respect to $H$. The set $P \setminus F$ is of type $F_\sigma$, so we can express it as $\bigcup_{i\in\mathbb{N}} P_i$ where every $P_i$ is closed and $\operatorname{diam} P_i < 1$. By Lemma 5 there are $i_0 \in \mathbb{N}$ and $H_0 \in [H]^\omega$ such that $A \cap P_{i_0}$ is good with respect to $H_0$. Let $P_{i_0} = \bigcup_{i\in\mathbb{N}} P_{i_0 i}$ where every $P_{i_0 i}$ is closed and $\operatorname{diam} P_{i_0 i} < 1/2$. By Lemma 5 there are $i_1 \in \mathbb{N}$ and $H_1 \in [H_0]^\omega$ such that $A \cap P_{i_0 i_1}$ is good with respect to $H_1$. We continue this process and find a sequence $P \setminus F \supset P_{i_0} \supset P_{i_0 i_1} \supset \dots$ of closed sets with diameters tending to zero, and a sequence $H \supset H_0 \supset H_1 \supset \dots$ such that $H_n \in [H]^\omega$ and $A \cap P_{i_0 \dots i_n}$ is good for $H_n$, for every $n$. Pick a point $x \in \bigcap_{n\in\mathbb{N}} P_{i_0\dots i_n}$ and $H' \in [H]^\omega$ almost contained in every $H_n$. Then $A \cap P_{i_0\dots i_n}$ is good with respect to $H'$. For each neighbourhood $U$ of $x$, pick $P_{i_0\dots i_n} \subset U$ and note that $(P \setminus F) \cap A \cap U$ is good with respect to $H'$. $\square$

**Lemma 7.** *Let $E \subset X^2$ be as in Theorem 4, let $E = \bigcup_{k\in\mathbb{N}} E_k$ where $E_k$ are closed sets, and let $\mathfrak{J} = \mathfrak{J}_E$. Fix $H \in [\mathbb{N}]^\omega$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $P_0, P_1, A_0, A_1 \subset X$ where $P_0, P_1$ are closed. If $P_i \cap A_i$ is good with respect to $H$, there are $H' \in [H]^\omega$ and disjoint closed sets $\overline{P_0} \subset P_0$, $\overline{P_1} \subset P_1$ such that $\operatorname{diam} \overline{P_0} < \varepsilon$, $\operatorname{diam} \overline{P_1} < \varepsilon$, $(\overline{P_0} \times \overline{P_1}) \cap E_n = \emptyset$ and $\overline{P_0} \cap A_0$, $\overline{P_1} \cap A_1$ are good with respect to $H'$.*

*Proof.* Applying Lemma 6 to $F = \emptyset$ we obtain a point $x_0 \in P_0$ and a set $H_0 \in [H]^\omega$ such that $P_0 \cap A_0 \cap U$ is good with respect to $H_0$, for each neighbourhood $U$ of $x_0$. Since $E_n(x_0) \in \mathfrak{J}_E$, applying Lemma 6 to $F = E_n(x_0)$ we obtain $x_1 \in P_1 \setminus E_n(x_0)$ and $H_1 \in [H_0]^\omega$ such that $(P_1 \setminus E_n(x_0)) \cap A_1 \cap U$ is good with respect to $H_1$, for each neighbourhood $U$ of $x_1$. Since $(x_0, x_1) \notin E_n$, by the closedness of $E_n$ we can find open neighbourhoods $U_0$ and $U_1$ of $x_0$ and $x_1$ (respectively) such that $\operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$, $(\operatorname{cl} U_0 \times \operatorname{cl} U_1) \cap E_n = \emptyset$ and $\operatorname{diam} U_0 < \varepsilon$, $\operatorname{diam} U_1 < \varepsilon$. Put $H' = H_1$ and $\overline{P_i} = P_i \cap \operatorname{cl} U_i$ for $i = 0, 1$. $\square$

*Proof of Theorem 4.* We will follow the scheme used in the the proof of the Komjáth theorem [K]. We may suppose that $X$ is perfect (replacing it by its perfect kernel) and that $\operatorname{diam} X < 1$. Assume that $E = \bigcup_{n\in\mathbb{N}} E_n$ where sets $(E_n)$ is an increasing sequence of closed sets. Fix a sequence $(A^{(j)})$ of analytic subsets of $X$ such that $\limsup_{j\in H} A^{(j)} \notin \mathfrak{I}_E$ for every $H \in [\mathbb{N}]^\omega$. Express each set $A^{(j)}$ as the result of the Souslin operation (cf. [Ke, 25.7]), that is

$$A^{(j)} = \bigcup_{z\in\mathbb{N}^\mathbb{N}} \bigcap_{n\in\mathbb{N}} F^{(j)}_{z|n}$$

where sets $F^{(j)}_{z|n}$ are closed, $\operatorname{diam} F^{(j)}_{z|n} < 1/(n+1)$ and for any $z \in \mathbb{N}^\mathbb{N}$, $m, n \in \mathbb{N}$, if $n > m$ then $F^{(j)}_{z|n} \subset F^{(j)}_{z|m}$. For $t \in \mathbb{N}^n$ we put

$$A^{(j)}_t = \bigcup_{z\in\mathbb{N}^\mathbb{N}, z|n=t} \bigcap_{k\in\mathbb{N}} F^{(j)}_{z|k}.$$

We may assume that $A^{(0)} = X$. By recursion, for each $n \in \mathbb{N}$ we define a number $j_n \in \mathbb{N}$, perfect sets $P_s$ (with $s \in \{0,1\}^n$), finite sequences $t(k,s) \in \mathbb{N}^n$ (with $k \le n, s \in \{0,1\}^n$) and a set $H_n \in [\mathbb{N}]^\omega$ with the following properties:

(W1) $j_n > j_{n-1}, H_n \in [H_{n-1}]^\omega, j_n \in H_{n-1}$;
(W2) $\operatorname{diam} P_s < \frac{1}{n+1}$, $P_{s\hat{\ }0} \cup P_{s\hat{\ }1} \subset P_s$, $P_{s\hat{\ }0} \cap P_{s\hat{\ }1} = \emptyset$;
(W3) if $s, s' \in \{0,1\}^{n+1}$ and $s \ne s'$ then $(P_s \times P_{s'}) \cap E_n = \emptyset$;
(W4) $P_s \cap A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)}$ is good with respect to $H_n$;
(W5) $P_s \subset F^{(j_0)}_{t(0,s)} \cap \ldots \cap F^{(j_n)}_{t(n,s)}$;
(W6) $t(k,s) \subset t(k,s\hat{\ }0) \cap t(k,s\hat{\ }1)$.

Having these objects defined, by (W2) we infer that $P = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in\{0,1\}^n} P_s$ is a set homeomorphic with $\{0,1\}^\mathbb{N}$ (cf. [Sr, 2.6]). For $z \in \{0,1\}^\mathbb{N}$ and $k \in \mathbb{N}$ denote $z|k = (z(0), \ldots, z(k-1))$. Let $x, y \in P$, $x \ne y$, and consider $z, w \in \{0,1\}^\mathbb{N}$ such that $x \in \bigcap_{n\in\mathbb{N}} P_{z|n}$, $y \in \bigcap_{n\in\mathbb{N}} P_{w|n}$. Pick the minimal $k \in \mathbb{N}$ with $z(k) \ne w(k)$. Then by (W3) we have $(P_{z|(i+1)} \times P_{w|(i+1)}) \cap E_i = \emptyset$ for all $i \ge k$, and also $(P_{z|(k+1)} \times P_{w|(k+1)}) \cap E_i = \emptyset$ for all $i < k$ since $E_0 \subset E_1 \subset \ldots \subset E_k$. So, $(x,y) \notin E$ and consequently, $P$ is a partial transversal for $E$. Let $G = \{j_0, j_1, \ldots, j_n, \ldots\}$. Then by (W5) and (W6), we have that $P \subset \bigcap_{n\in\mathbb{N}} A^{(j_n)} = \bigcap_{j\in G} A^{(j)}$ which yields the assertion.

The rest of proof consists of a construction of objects fulfilling (W1)–(W6) – the idea is quite similar to that given in [K]. However, some details are more involved and we give them for the reader's convenience. First put $j_0 = 0$, $P_\emptyset = X$, $H_0 = \mathbb{N}$ and $t(0,\emptyset) = \emptyset$. Next assume that, for a fixed $n \in \mathbb{N}$, we have chosen $j_k$ (for $k \le n$), $P_s$ (for $s \in \{0,1\}^k$, $k \le n$), $t(k,s)$ (for $k \le l \le n, s \in \{0,1\}^l$) and $H_k$ (for $k \le n$).

First, we shall prove that there are a number $j \in H_n$ such that $j > j_n$ and a set $H'_n \in [H_n]^\omega$ fulfilling the condition

(W7) $\qquad (\forall s \in \{0,1\}^n)\ P_s \cap A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap A^{(j)} \quad$ is good wrt $H'_n$.

If it is not so, for each $j \in H_n$, $j > j_n$, and for each $H \in [H_n]^\omega$ there are $s \in \{0,1\}^n$ and $G \in [H]^\omega$ such that

$$P_s \cap A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap A^{(j)} \cap \limsup_{r \in G} A^{(r)} \in \mathfrak{J}_E.$$

Proceeding inductively, we find numbers $k_0 < k_1 < \ldots$ and sets $G_0 \supset G_1 \supset \ldots$ with $G_0 = H_n$, such that for each $m \in \mathbb{N}$ we have $k_m \in G_m \in [\mathbb{N}]^\omega$ and we can fix an $s_m \in \{0,1\}^n$ with

$$P_{s_m} \cap A^{(j_0)}_{t(0,s_m)} \cap \ldots \cap A^{(j_n)}_{t(n,s_m)} \cap A^{(k_m)} \cap \limsup_{r \in G_{m+1}} A^{(r)} \in \mathfrak{J}_E.$$

Then pick an $s \in \{0,1\}^n$ such that $\Gamma = \{k_m \colon s_m = s\}$ is infinite. Observe that $\Gamma$ is almost contained in every $G_m$. Hence

$$P_s \cap A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap (\bigcup_{m \in \Gamma} A^{(m)}) \cap \limsup_{r \in \Gamma} A^{(r)} \in \mathfrak{J}_E.$$

Since $\limsup_{r\in\Gamma} A^{(r)} \subset \bigcup_{m\in\Gamma} A^{(m)}$, the union in the above condition can be deleted, and so, we obtain a contradiction with (W4).

Consequently, the respective $j \in H_n$, $j > j_n$, and $H'_n \in [H_n]^\omega$ fulfilling (W7) do exist, and we put $j_{n+1} = j$. For each $s \in \{0,1\}^n$, write in short

$$A^*_s = A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap A^{(j_{n+1})}.$$

Now, we will show how to construct sets $P_s$ with $s \in \{0,1\}^{n+1}$. Because of condition (W3), the construction is divided into several steps using Lemma 7. List all distinct pairs in $\{0,1\}^n \times \{0,1\}^n$ as $(s_i, s'_i)$ $(i = 1, \ldots, p_n)$. We decrease sets $P_s$, $s \in \{0,1\}^n$, in $p_n$ steps as follows. Put $H^{(0)}_n = H'_n$ and $P^{(0)}_s = P_s$ for $s \in \{0,1\}^n$. In the $i$th step $(i = 1, \ldots, p_n)$ applying Lemma 7 (and (W7) if $i = 1$), we find $H^{(i)}_n \in [H^{(i-1)}_n]^\omega$ and closed sets $P^{(i)}_{s_1} \subset P^{(i-1)}_{s_1}$, $P^{(i)}_{s'_1} \subset P^{(i-1)}_{s'_1}$ such that $(P^{(i)}_{s_1} \times P^{(i)}_{s'_1}) \cap E_n = \emptyset$ and $P^{(i)}_{s_1} \cap A^*_{s_1}$, $P^{(i)}_{s'_1} \cap A^*_{s'_1}$ are good wrt $H^{(i)}_n$; we also put $P^{(i)}_s = P^{(i-1)}_s$ for all $s \in \{0,1\}^n \setminus \{s_i, s'_i\}$. If this process is finshed, we define $H^*_n = H^{(p_n)}_n$ and $P^*_s = P^{(p_n)}_s$ for all $s \in \{0,1\}^n$.

Next by Lemma 7, for every $s \in \{0,1\}^n$ we find disjoint closed sets $\overline{P_{s\hat{}0}}, \overline{P_{s\hat{}1}} \subset P^*_s$ such that $\operatorname{diam} \overline{P_{s\hat{}0}} < 1/(n+2)$, $\operatorname{diam} \overline{P_{s\hat{}1}} < 1/(n+2)$, $(\overline{P_{s\hat{}0}} \times \overline{P_{s\hat{}1}}) \cap E_n = \emptyset$, and we find $H''_n \in [H^*_n]^\omega$ such that for all $s \in \{0,1\}^n$ and $i \in \{0,1\}$

$$\text{(W8)} \qquad \overline{P_{s\hat{}i}} \cap A^*_s = \overline{P_{s\hat{}i}} \cap A^{(j_0)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap A^{(j_{n+1})} \quad \text{is good wrt } H''_n.$$

Fix, $s \in \{0,1\}^n$, $i \in \{0,1\}$. Since the set in (W8) is contained in the union

$$\bigcup_{z_0 \in \mathbb{N}^{n+1}, z_0 \supset t(0,s)} \cdots \bigcup_{z_n \in \mathbb{N}^{n+1}, z_n \supset t(n,s)} \bigcup_{z_{n+1} \in \mathbb{N}^{n+1}} \overline{P_{s\hat{}i}} \cap A^{(j_0)}_{z_0} \cap \ldots \cap A^{(j_{n+1})}_{z_{n+1}},$$

by Lemma 5, one of the components of this union is good. Moreover, if we use $2^{n+1}$ times Lemma 5, we obtain $\overline{H_n} \in [H''_n]^\omega$ witnessing this fact simultaneously for all $s \in \{0,1\}^n$

and $i \in \{0,1\}$. Choose $t(0, s\hat{\ }i), \ldots, t(n+1, s\hat{\ }i)$ as the sequence corresponding to $s\hat{\ }i$. Observe that the sets

$$Q_{s\hat{\ }i} = \overline{P_{s\hat{\ }i}} \cap F^{(j_0)}_{t(0,s\hat{\ }i)} \cap \ldots \cap F^{(j_{n+1})}_{t(n+1,s\hat{\ }i)}$$

are good for $\overline{H_n}$ since $Q_{s\hat{\ }i}$ and $\overline{P_{s\hat{\ }i}}$ have the same intersections with $A^{(j_0)}_{t(0,s\hat{\ }i)} \cap \ldots \cap A^{(j_{n+1})}_{t(n+1,s\hat{\ }i)}$. Finally, define $P_{s\hat{\ }i}$ as the perfect kernel of $Q_{s\hat{\ }i}$ and let $H_{n+1} = \overline{H_n}$. $\square$

If $E$ is the equality relation, Theorem 4 yields exactly the Komjáth theorem. If $A_n = A$ for every $n \in \mathbb{N}$, we obtain the following corollary which also can be derived from a deep result of Silver (cf. [Ke, 35.20]).

**Corollary 8.** *If $E \subset X^2$ is an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$, and $A \notin \mathcal{J}_E$ is an analytic set, then there is a set $P \subset A$ homeomorphic with $\{0,1\}^{\mathbb{N}}$ and being a partial transveral for $E$.*

## 3. Parametric Laczkovich-Komjáth property

Several combinatorial results have their parametric versions which in fact generalize them in a nice way, see e.g. [Mi], [Pa]. A parametric version of the Komjáth theorem was proved in [G]. Here, by the use of similar methods, we shall prove a parametric version of Theorem 3. Moreover, we give a condition which guarantees that a $\sigma$-ideal $\mathcal{J}$ with property (LK) has parametric property (LK).

By the Mazurkiewicz-Sierpiński theorem [Ke, 29.19], if $X, Y$ are Polish spaces then for each analytic set $A \subset X \times Y$, the set $\{x \in X : |A(x)| > \omega\}$ is also analytic. We say that an ideal $\mathcal{J} \subset \mathcal{P}(Y)$ has the *Mazurkiewicz-Sierpiński property* if for any Polish space $X$ and analytic set $A \subset X \times Y$, the set $\{x \in X \colon A(x) \notin \mathcal{J}\}$ is analytic. This property holds true for, besides the ideal of countable sets, the ideal of meager sets in $Y$ [Ke, 29.22] and the ideal of Lebesgue null sets in $\mathbb{R}$ [Ke, 29.26].

We say that an ideal $\mathcal{J} \subset \mathcal{P}(Y)$ has *parametric* property (LK), whenever for every uncountable Polish space $X$ and every sequence $(A_n)$ of analytic subsets of $X \times Y$, if $\limsup_{n \in H} A_n(x) \notin \mathcal{J}$ for all $x \in X$ and $H \in [\mathbb{N}]^\omega$ then there are a perfect set $P \subset X$ and $G \in [\mathbb{N}]^\omega$ such that $\bigcap_{j \in G} A_j(x) \notin \mathcal{J}$ for each $x \in P$. Since $X$ contains a homeomorphic copy of the Cantor space, we may assume that $X = \{0,1\}^{\mathbb{N}}$. Clearly, parametric property (LK) is stronger than property (LK). In [G], it was proved that the ideal $[Y]^{\le\omega}$ of all countable subsets of $Y$ has parametric property (LK). The same scheme of a proof will work for Proposition 9.

Recall some definitions. For any $\alpha \in [\mathbb{N}]^\omega$ and $H \in [\mathbb{N}]^{<\omega}$ with $\max(\alpha) < \min(H)$, the set of the form $[\alpha, H] = \{G \in [\mathbb{N}]^{<\omega} : \alpha \subset G \subset \alpha \cup H\}$ is said to be an *Ellentuck neighbourhood*, and the topology generated by all Ellentuck neighbourhoods is called the *Ellentuck topology* on $[\mathbb{N}]^\omega$. According to [Pa], a set $A \subset \{0,1\}^\omega \times [\mathbb{N}]^\omega$ is called *perfectly Ramsey* if for every perfect set $P \subset \{0,1\}^{\mathbb{N}}$ and every Ellentuck neighbourhood

$[\alpha, H]$ there are a perfect set $Q \subset P$ and $G \in [H]^\omega$ such that either $Q \times [\alpha, G] \subset A$ or $(Q \times [\alpha, G]) \cap A = \emptyset$. (All the considered perfect sets are nonempty.) If we identify sets $H \in [\mathbb{N}]^\omega$ with their indicators in $\{0,1\}^{\mathbb{N}}$, the space $[\mathbb{N}]^\omega$ is Polish. From [Pa, Thm 1.1] it follows that every analytic set $A \subset \{0,1\}^{\mathbb{N}} \times [\mathbb{N}]^\omega$ is perfectly Ramsey.

**Proposition 9.** *Let $Y$ be an uncountable Polish space and let $\mathcal{J} \subset \mathcal{P}(Y)$ be a $\sigma$-ideal with property (LK) and with the Mazurkiewicz-Sierpiński property. Then $\mathcal{J}$ has parametric property (LK).*

*Proof.* Put $X = \{0,1\}^{\mathbb{N}}$. Let $A_j \subset X \times Y$, $j \in \mathbb{N}$, be analytic sets such that $\limsup_{j \in H} A_j(x) \notin \mathcal{J}$ for $x \in X$ and $H \in [\mathbb{N}]^\omega$. Define

$$A = \{(x, H) \in X \times [\mathbb{N}]^\omega \colon \bigcap_{j \in H} A_j(x) \notin \mathcal{J}\}.$$

Consider

$$\begin{aligned} B =& \{(x, H, y) \in X \times [\mathbb{N}]^\omega \times Y \colon (x, y) \in \bigcap_{j \in H} A_j\} \\ =& \{(x, H, y) \in X \times [\mathbb{N}]^\omega \times Y \colon \forall j \in \mathbb{N}\ (j \notin H \text{ or } (x, y) \in A_j)\} \end{aligned}$$

and observe that $B$ is analytic. Hence the set

$$A = \{(x, H) \in X \times [\mathbb{N}]^\omega \colon B(x, H) \notin \mathcal{J}\}$$

is analytic, since $\mathcal{J}$ has the Mazurkiewicz-Sierpiński property. Now, by the Pawlikowski theorem, $A$ is perfectly Ramsey. Then pick a perfect set $P \subset X$ and $H \in [\mathbb{N}]^\omega$ such that either $P \times [\emptyset, H] \subset A$ or $(P \times [\emptyset, H]) \cap A = \emptyset$. The latter case is impossible since, by property (LK) of $\mathcal{J}$, for each $x \in P$ there exists $G \in [H]^\omega$ such that $\bigcap_{j \in G} A_j(x) \notin \mathcal{J}$. The former case yields $\bigcap_{j \in H} A_j(x) \notin \mathcal{J}$ for all $x \in P$. □

By $\mathcal{K}(X)$ we denote the hyperspace of all nonempty compact subsets of $X$, equipped with the Vietoris topology (or, equivalently with the Hausdorff metric); cf. [Ke, 4.7] and [Sr, pp. 66–69]. In the sequel, a perfect set which is a partial transversal for an equivalence relation $E$ will be called a *perfect partial transversal for $E$* (in short, $E$-ppt).

**Lemma 10.** *Let $Y$ be an uncountable Polish space. If $E \subset Y^2$ is an equivalence relation of type $F_\sigma$ with $|Y/E| > \omega$ then the family of all sets $L \in \mathcal{K}(Y)$ containing a perfect partial transversal for $E$ is analytic.*

*Proof.* Let $E = \bigcup_{n \in \mathbb{N}} E_n$ where $(E_n)$ is an increasing sequence of closed sets. Fix a countable base $\{U_i \colon i \in \mathbb{N}\}$ of the topology in $Y$. For $L \in \mathcal{K}(Y)$ we have the following equivalence

$$\begin{aligned} (*) \qquad & L \text{ contains an } E\text{-ppt} \iff (\exists K \in \mathcal{K}(L))(\forall i, j, n \in \mathbb{N})(U_i \cap K \neq \emptyset \neq U_j \cap K) \Rightarrow \\ & (\exists k, l \in \mathbb{N})(\operatorname{cl} U_k \subset U_i,\ \operatorname{cl} U_l \subset U_j,\ \operatorname{cl} U_k \cap \operatorname{cl} U_l = \emptyset,\ \operatorname{diam} U_k < \frac{1}{n+1},\ \operatorname{diam} U_l < \frac{1}{n+1}, \\ & \qquad U_k \cap K \neq \emptyset \neq U_l \cap K,\ (U_k \times U_l) \cap E_n = \emptyset). \end{aligned}$$

Hence, in a standard way (cf. [Ke, 4.29], [Sr, 2.4.11]) we show that the family of all sets $L \in \mathcal{K}(Y)$ containing an $E$-ppt is analytic. Thus to finish the proof it suffices to show that $(*)$ does hold.

If $L \in \mathcal{K}(Y)$ contains an $E$-ppt $K$, we easily conclude that $K$ satisfies the right hand side of the equivalence $(*)$. Conversely, if $K \in \mathcal{K}(L)$ satisfies the right hand side of the equivalence $(*)$, we can define by recursion a family $\{V_s\colon s \in \{0,1\}^{<\mathbb{N}}\} \subset \{U_i\colon i \in \mathbb{N}\}$ such that for each $s \in \{0,1\}^{<\mathbb{N}}$ the following conditions hold:

(i) $V_s \cap K \neq \emptyset$;
(ii) $\operatorname{cl} V_{s\hat{\ }0} \cup \operatorname{cl} V_{s\hat{\ }1} \subset V_s$, $\operatorname{cl} V_{s\hat{\ }0} \cap \operatorname{cl} V_{s\hat{\ }1} = \emptyset$;
(iii) $\operatorname{diam} V_s < 1/(|s|+1)$;

and additionally,

(iv) $(V_s \times V_{s'}) \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and $s, s' \in \{0,1\}^{n+1}$, $s \neq s'$.

The construction is similar to that given in the proof of Theorem 4 (cf. conditions (W1)–(W3)). Then $\bigcap_{n\in\mathbb{N}} \bigcup_{s\in\{0,1\}^n} (K \cap \operatorname{cl} V_s)$ is an $E$-ppt contained in $L$. □

**Theorem 11.** *Let $E \subset X^2$ be an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$. Then the $\sigma$-ideal $\mathcal{J}_E$ has the Mazurkiewicz-Sierpiński property.*

*Proof.* Set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. For an analytic set $B \subset Y$ pick a closed set $F \subset Y \times \mathcal{N}$ such that $\operatorname{pr}_Y(F) = B$ where $\operatorname{pr}_Y$ stands for the projection from $Y \times \mathcal{N}$ to $Y$. Observe that

$$(**) \qquad B \notin \mathcal{J}_E \iff (\exists K \in \mathcal{K}(Y \times \mathcal{N}))(K \subset F \text{ and } \operatorname{pr}_Y(K) \text{ contains an } E\text{-ppt}).$$

Indeed, to show "⇒" assume that $B \notin \mathcal{J}_E$. By Corollary 8, $B$ contains an $E$-ppt $P$. Note that $P = \operatorname{pr}_Y((P \times \mathcal{N}) \cap F)$. By [Ke, 29.20] there is a set $K \subset (P \times \mathcal{N}) \cap F$ such that the both $K$ and $\operatorname{pr}_Y(K)$ are homeomorphic with $\{0,1\}^{\mathbb{N}}$. Since $\operatorname{pr}_Y(K) \subset P$ so $\operatorname{pr}_Y(K)$ is an $E$-ppt with $K \subset F$. Implication "⇐" is obvious.

Now, let $A \subset X \times Y$ be an analytic set and pick a closed set $F \subset X \times Y \times \mathcal{N}$ such that $\operatorname{pr}_{X\times Y}(F) = A$. Then $A(x) = \operatorname{pr}_Y(F(x))$ and $F(x) \subset Y \times \mathcal{N}$ is closed for each $x \in X$. By $(**)$, for each $x \in X$ we have

$$(**') \quad A(x) \notin \mathcal{J}_E \iff (\exists K \in \mathcal{K}(Y \times \mathcal{N}))(K \subset F(x) \text{ and } \operatorname{pr}_Y(K) \text{ contains an } E\text{-ppt}).$$

Observe that the set $\{(x,K) \in X \times \mathcal{K}(Y \times \mathcal{N})\colon K \subset F(x)\}$ is closed and note that the mapping $K \mapsto \operatorname{pr}_Y(K)$ from $\mathcal{K}(Y \times \mathcal{N})$ to $\mathcal{K}(Y)$ is continuous [Ke, 4.29(vi)]. Hence by Lemma 10 and $(**')$ the assertion follows. □

From Proposition 9, by Theorems 4 and 11, we deduce immediately the following fact.

**Theorem 12.** *Let $Y$ be an uncountable Polish space. If $E \subset Y^2$ is an equivalence relation of type $F_\sigma$ with $|Y/E| > \omega$ then $\mathcal{J}_E$ has parametric property (LK).*

## 4. Some results about invariance

In this section we study how property (LK) is preserved by various operations.

It is well known that for any two uncountable Polish spaces there is a Borel isomorphism between them; see [Ke, 15.6] and [Sr, 3.3.13]. Observe that if $X$ and $Y$ are uncountable Polish spaces and a $\sigma$-ideal $\mathcal{J} \subset \mathcal{P}(X)$ has property (LK) then, for every Borel isomorphism $\varphi\colon X \to Y$, the $\sigma$-ideal $\{\varphi(A)\colon A \in \mathcal{J}\} \subset \mathcal{P}(Y)$ has property (LK). From Example 2 we know that the $\sigma$-ideals of meager subsets of $\{0,1\}^{\mathbb{N}}$ and of measure zero subsets of $\{0,1\}^{\mathbb{N}}$ do not have property (LK). These facts can be generalized. Since between any two perfect Polish spaces there is a Borel isomorphism preserving the Baire category (see e.g. [CKW, 3.15]), the $\sigma$-ideal of meager subsets of a perfect Polish space does not have property (LK). Similarly, using a special Borel isomorphism (cf. [Ke, 17.41]) we infer that the $\sigma$-ideal of null sets with respect to a finite continuous Borel measure on an uncountable Polish space does not have property (LK).

For uncountable Polish spaces $X, Y$ and for $\sigma$-ideals $\mathcal{I} \subset \mathcal{P}(X)$, $\mathcal{J} \subset \mathcal{P}(Y)$, put

$$\mathcal{I} \otimes \mathcal{J} = \{A \subset X \times Y\colon \{x \in X\colon A(x) \notin \mathcal{J}\} \in \mathcal{I}\}.$$

Then $\mathcal{I} \otimes \mathcal{J}$ forms a $\sigma$-ideal.

**Example 13.** Let $E \subset X^2$ be an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$. Consider $\{\emptyset\}$, the trivial $\sigma$-ideal of subsets of $Y$. We will show that $\mathcal{J}_E \otimes \{\emptyset\}$ has property (LK). To this aim define $\overline{E} \subset (X \times Y)^2$ by

$$(x,y)\overline{E}(x',y') \iff xEx', \quad \text{for } (x,y),(x',y') \in X \times Y.$$

Clearly, $\overline{E}$ is of type $F_\sigma$. Also $\overline{E}(x,y) = E(x) \times Y$ for each $(x,y) \in X \times Y$. It is easy to check that $\mathcal{J}_{\overline{E}} = \mathcal{J}_E \otimes \{\emptyset\}$. Hence $\mathcal{J}_E \otimes \{\emptyset\}$ has property (LK) by Theorem 4. In particular, $[X]^{\le\omega} \otimes \{\emptyset\}$ has property (LK).

The case of the $\sigma$-ideal $\{\emptyset\} \otimes [X]^{\le\omega}$ is more interesting. The problem whether this $\sigma$-ideal has property (LK) remains open. We have only a partial result which can shed some light on the problem.

Let $\mathcal{I}, \mathcal{J}$ be $\sigma$-ideals such that $\mathcal{I} \subset \mathcal{J} \subset \mathcal{P}(X)$. We say that *the pair* $(\mathcal{I}, \mathcal{J})$ *has property* (LK) whenever for every sequence $(A_n)$ of analytic subsets of $X$, condition $\limsup_{n\in H} A_n \notin \mathcal{J}$ for every $H \in [\mathbb{N}]^\omega$ implies $\bigcap_{n\in G} A_n \notin \mathcal{I}$ for some $G \in [\mathbb{N}]^\omega$. Clearly, if $\mathcal{J}$ has property (LK) then $(\mathcal{I}, \mathcal{J})$ has property (LK), and $\mathcal{J}$ has property (LK) iff $(\mathcal{J}, \mathcal{J})$ has property (LK).

**Proposition 14.** *For uncountable Polish spaces $X$ and $Y$, consider the $\sigma$-ideals $[X]^{\le\omega}$, $\{\emptyset\} \subset \mathcal{P}(X)$ and a fixed $\sigma$-ideal $\mathcal{J} \subset \mathcal{P}(Y)$ with property (LK). Then $\{\emptyset\} \otimes \mathcal{J}$ has property (LK) if and only if $(\{\emptyset\} \otimes \mathcal{J}, [X]^{\le\omega} \otimes \mathcal{J})$ has property (LK).*

*Proof.* "$\Rightarrow$" Let $(A_n)$ be a sequence of analytic subsets of $X \times Y$ such that $\limsup_{n\in H} A_n \notin [X]^{\le\omega} \otimes \mathcal{J}$ for each $H \in [\mathbb{N}]^\omega$. Then also $\limsup_{n\in H} A_n \notin \{\emptyset\} \otimes \mathcal{J}$ for each $H \in [\mathbb{N}]^\omega$. By the assumption, $\{\emptyset\} \otimes \mathcal{J}$ has property (LK). So, $\bigcap_{n\in G} A_n \notin \{\emptyset\} \otimes \mathcal{J}$ for some $G \in [\mathbb{N}]^\omega$.

"$\Leftarrow$" Suppose that $\{\emptyset\}\otimes\mathcal{J}$ does not have property (LK). Thus there is a sequence $(A_n)$ of analytic subsets of $X \times Y$ such that for all $H \in [\mathbb{N}]^\omega$ we have $\limsup_{n\in H} A_n \notin \{\emptyset\}\otimes\mathcal{J}$ and

$$(\triangle) \qquad (\forall G \in [H]^\omega) \bigcap_{n\in G} A_n \in \{\emptyset\} \otimes \mathcal{J}.$$

Define

$$B_H = \{x \in X\colon \limsup_{n\in H} A_n(x) \notin \mathcal{J}\}, \;\; H \in [\mathbb{N}]^\omega.$$

Consider two cases:

$1^0$ $|B_H| > \omega$ for all $H \in [\mathbb{N}]^\omega$. Hence $\limsup_{n\in H} A_n \notin [X]^{\le\omega} \otimes \mathcal{J}$ for all $H \in [\mathbb{N}]^\omega$, and by the assumption $\bigcap_{n\in G} A_n \notin \{\emptyset\} \otimes \mathcal{J}$ for some $G \in [\mathbb{N}]^\omega$, a contradiction with $(\triangle)$.

$2^0$ $|B_{H_0}| \le \omega$ for some $H_0 \in [\mathbb{N}]^\omega$. Firstly note that if $H, H' \in [\mathbb{N}]^\omega$ and $H$ is almost contained in $H'$ then $B_H \subset B_{H'}$. Secondly note that it is not possible to have $B_G = B_{H_0}$ for all $G \in [H_0]^\omega$ since in this case, for each $x \in B_{H_0}$ (by property (LK) for $\mathcal{J}$) we would pick $G_x \in [H_0]^\omega$ such that $\bigcap_{n\in G_x} A_n(x) \notin \mathcal{J}$, a contradiction with $(\triangle)$. Hence there exists a set $G \in [H_0]^\omega$ such that $B_G \ne B_{H_0}$. Proceeding inductively, we define a sequence $(H_\alpha)_{\alpha<\omega_1}$ such that $H_{\alpha+1} \in [H_\alpha]^\omega$, $B_{H_{\alpha+1}} \ne B_{H_\alpha}$ ($\alpha < \omega_1$), and for a limit ordinal $\alpha < \omega_1$, we pick $H_\alpha \in [H_0]^\omega$ almost contained in every $H_\beta$, $\beta < \alpha$. So, $(B_{H_\alpha})_{\alpha<\omega_1}$ is a strictly descending sequence of countable sets, a contradiction. $\square$

Another interesting question concerns the intersection of $\sigma$-ideals: What is the (possibly large) cardinality of a family of $\sigma$-ideals with property (LK) such that the intersection of the family has also property (LK)?

Let $r_0$ stand for the ideal of nowhere dense sets in the Ellentuck topology on $[\mathbb{N}]^\omega$. Put

$$\mathrm{cov}(r_0) = \min\{|\mathcal{D}|\colon \mathcal{D} \subset r_0 \text{ or } \bigcup \mathcal{D} = [\mathbb{N}]^\omega\}.$$

Plewik [Pl] proved that $\mathrm{cov}(r_0) = \mathfrak{h}$ where $\mathfrak{h}$ is the cardinal introduced by Balcar, Pelant and Simon [BPS]. It is known that $\omega_1 \le \mathfrak{h} \le 2^\omega$, and either or both inequalities can be strict in some models of ZFC (see [V]). We offer the following result connected with the above-mentioned question.

**Proposition 15.** *Let $\mathcal{F} \subset \mathcal{P}(X)$ be a family of size $|\mathcal{F}| < \mathfrak{h}$, of $\sigma$-ideals with property (LK) on an uncountable Polish space $X$. Then $\bigcap \mathcal{F}$ has property (LK).*

*Proof.* Let $(A_n)$ be a sequence of analytic subsets of $X$ such that $\limsup_{n\in H} A_n \notin \bigcap \mathcal{F}$ for each $H \in [\mathbb{N}]^\omega$. Put

$$B_{\mathcal{J}} = \{H \in [\mathbb{N}]^\omega\colon \limsup_{n\in H} A_n \notin \mathcal{J}\} \text{ for } \mathcal{J} \in \mathcal{F}.$$

We have $[\mathbb{N}]^\omega = \bigcup_{\mathcal{J}\in\mathcal{F}} B_{\mathcal{J}}$. Since $|\mathcal{F}| < \mathfrak{h} = \mathrm{cov}(r_0)$, pick a $\sigma$-ideal $\mathcal{J} \in \mathcal{F}$ such that $B_{\mathcal{J}}$ is dense in some Ellentuck neighbourhood $[\alpha, H]$ with $\alpha \in [\mathbb{N}]^{<\omega}$, $H \in [\mathbb{N}]^\omega$, $\max(\alpha) < \min(H)$. Hence for every $G \in [H]^\omega$ we can find $G' \in B_{\mathcal{J}} \cap [\alpha, G]$. Consequently, $\limsup_{n\in G'} A_n \notin \mathcal{J}$ and since $G' \subset \alpha \cup G$, we have $\limsup_{n\in G} A_n \notin \mathcal{J}$. By property (LK) of $\mathcal{J}$, there is $G_0 \in [H]^\omega$ such that $\bigcap_{n\in G_0} A_n \notin \mathcal{J}$. Thus $\bigcap_{n\in G_0} A_n \notin \bigcap \mathcal{F}$. □

The following example shows a family $\mathcal{F}$, of size of the continuum, of $\sigma$-ideals with property (LK) such that $\bigcap \mathcal{F}$ has property (LK) and $\bigcap \mathcal{F}$ is different from the $\sigma$-ideal of countable sets.

**Example 16.** For a fixed $z \in [0,1]$, let $\mathcal{J}^{(z)}$ stand for the $\sigma$-ideal of subsets of $[0,1]^2$ generated by the family

$$\{\{0\} \times [0,1]\} \cup \{\{z\} \times [0,1]\} \cup ([0,1]^2)^{\leq\omega}.$$

Then $\mathcal{J}^{(z)} = \mathcal{J}_{E_z}$ where $E_z \subset ([0,1]^2)^2$ is the equivalence relation given by

$$(x,y)E_z(x',y') \iff ((x = x') \text{ and } (x = 0 \text{ or } x = z \text{ or } y = y')).$$

Since $E_z$ is closed and $|[0,1]^2/E_z| = 2^\omega$, the $\sigma$-ideal $\mathcal{J}^{(z)}$ has property (LK) by Theorem 4. Let $\mathcal{F} = \{\mathcal{J}^{(z)} \colon z \in [0,1]\}$. Then $|\mathcal{F}| = 2^\omega$ and $\bigcap \mathcal{F} = \mathcal{J}^{(0)}$, so $\bigcap \mathcal{F}$ has property (LK).

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## References

[BPS] B. Balcar, J. Pelant, P. Simon, *The space of ultrafilters on* $\mathbb{N}$ *covered by nowhere dense sets*, Fund. Math. **110** (1980), 11–24.

[CKW] J. Cichoń, A. B. Kharazishvili, B. Węglorz, *Subsets of the Real Line*, Łódź University Press, Łódź 1995.

[G] Sz. Głąb, *On parametric limit superior of a sequence of analytic sets*, Real Anal. Exchange **31** (2005/06), no.1, 285–289.

[H] P. R. Halmos, *Large intersections of large sets*, Amer. Math. Monthly **99** (1992), no. 4, 307–312.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York 1995.

[K] P. Komjáth, *On the limit superior of analytic sets*, Anal. Math. **10** (1984), 283–293.

[Mi] A. W. Miller, *Infinite combinatorics and definability*, Ann. Pure Appl. Logic **41** (1989), no. 2, 179–203.

[L] M. Laczkovich, *On the limit superior of sequence of sets*, Anal. Math. **3** (1977), 199–206.

[Pa] J. Pawlikowski, *Parametrized Ellentuck theorem*, Topology Appl. **37** (1990), 65–73.

[Pl] Sz. Plewik, *On completely Ramsey sets*, Fund. Math. **127** (1987), 127–132.

[Sr] S. M. Srivastava, A Course of Borel Sets, Springer, New York 1998.

[V] J. E. Vaughan, *Small uncountable cardinals and topolgy*, in: Open Problems in Topology (J. van Mill and G. M. Reed, eds), Elsevier, Amsterdam 1990, 195–218.

Institute of Mathematics, Łódź Technical University, ul. Wólczańska 215, 93-005 Łódź, Poland

*E-mail address*: mbalce@p.lodz.pl

Mathematical Institute, Polish Academy of Sciences, Śniadeckich 8 00-956 Warszawa, Poland

*E-mail address*: szymon_glab@yahoo.com