ON THE LACZKOVICH-KOMJÁTH PROPERTY OF SIGMA-IDEALS

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Abstract. Komjáth in 1984 proved that, for each sequence \((A_n)\) of analytic subsets of a Polish space \(X\), if \(\lim sup_{n \in H} A_n\) is uncountable for every \(H \in [\mathbb{N}]^\omega\) then \(\bigcap_{n \in G} A_n\) is uncountable for some \(G \in [\mathbb{N}]^\omega\). This fact, by our definition, means that the \(\sigma\)-ideal \(\langle X \rangle_{\leq \omega}\) has property (LK). We prove that every \(\sigma\)-ideal generated by \(X/E\) has property (LK), for an equivalence relation \(E \subset X^2\) of type \(F_\sigma\) with uncountably many equivalence classes. We also show the parametric version of this result. Finally, the invariance of property (LK) with respect to various operations is studied.

1. Introduction

We use standard set theoretical notation (see [Sr] or [Ke]). As usual, \(\mathbb{N} = \{0, 1, 2, \ldots\}\). Let \((A_n)\) be a sequence of subsets of the real line (or a Polish space). We are interested in the following question. If the set \(\lim sup_{n \in H} A_n\) is large, in a given sense, for every \(H \in [\mathbb{N}]^\omega\), is it true that at least one among the sets \(\bigcap_{n \in H} A_n, n \in H\), is large in the same sense? Observe that

\[
\lim sup_{n \in H} A_n = \bigcap_{n \in H} \bigcup_{k \geq n} A_k = \bigcup_{G \in [H]^\omega} \bigcap_{k \in G} A_k,
\]

so, we ask how strongly the largeness of all unions \(\bigcup_{G \in [H]^\omega} \bigcap_{k \in G} A_k, H \in [\mathbb{N}]^\omega\), has the influence on the largeness of the summands \(\bigcap_{k \in H} A_k, H \in [\mathbb{N}]^\omega\). These and related questions were discussed by Laczkovich [L] and Halmos [H]. Laczkovich in [L] proved that, for every sequence \((A_n)\) of Borel subsets of a Polish space, if \(\lim sup_{n \in H} A_n\) is uncountable for each \(H \in [\mathbb{N}]^\omega\) then \(\bigcap_{n \in G} A_n\) is uncountable for some \(G \in [\mathbb{N}]^\omega\). This result was then generalized by Komjáth [K, Thm 1] to the case when the sets \(A_n\) are analytic. Note that an uncountable analytic subset of a Polish space contains a homeomorphic copy of the Cantor set [Sr, Thm 4.3.5], so it is of cardinality of the continuum. Komjáth also proved that the result of Laczkovich cannot be generalized within ZFC to the case of coanalytic sets. Namely, if \(V = L\), there is a sequence \((A_n)\) of coanalytic sets such that \(|\lim sup_{n \in H} A_n| > \omega\) and \(|\bigcap_{n \in H} A_n| \leq \omega\) for all \(H \in [\mathbb{N}]^\omega\); see [K, Thm. 4].

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From now on, let $X$ be an uncountable Polish space. In connection with the above quoted theorem of Komjáth about analytic sets, we introduce the following property of an ideal $\mathcal{J}$ of subsets of $X$. We say that $\mathcal{J}$ has property (LK) (the Laczkovich-Komjáth property) whenever for every sequence $(A_n)$ of analytic subsets of $X$, if $\limsup_{n \in H} A_n \notin \mathcal{J}$ for each $H \in [\omega]^\omega$ then $\bigcap_{n \in G} A_n \notin \mathcal{J}$ for some $G \in [\omega]^\omega$. So, the Komjáth theorem states that the ideal $[X]^{\leq \omega}$ has property (LK). While studying ideals with property (LK) we may restrict our considerations only to those ones with bases consisting of analytic sets. Recall that a family $F$ is a base of an ideal $\mathcal{J} \subset \mathcal{P}(X)$ if $\mathcal{J} \subset \mathcal{J}$ and each set $A \in \mathcal{J}$ is contained in a set $B \in \mathcal{F}$. Namely, observe that $\mathcal{J}$ has property (LK) if and only if the ideal

$\mathcal{J} |_{\Sigma_1^1} = \{ A \subset X : (\exists B \in \mathcal{J} \cap \Sigma_1^1(X)) A \subset B \}$

has property (LK), and $\mathcal{J} \cap \Sigma_1^1(X)$ is a base of $\mathcal{J} |_{\Sigma_1^1}$ consisting of analytic sets. If a base of an ideal $\mathcal{J}$ consists of analytic sets, we say that $\mathcal{J}$ has an analytic base.

The next observation is due to T. Banakh (oral communication).

**Proposition 1.** An ideal $\mathcal{J}$ with an analytic base and with property (LK) is $\sigma$-additive.

**Proof.** Suppose that $\mathcal{J}$ is not $\sigma$-additive. Let $(A_n)$ be an increasing sequence of analytic sets from $\mathcal{J}$ whose union is not in $\mathcal{J}$. Then for each $H \in [\omega]^\omega$ we have $\limsup_{n \in H} A_n = \bigcup_{n \in \mathbb{N}} A_n \notin \mathcal{J}$ but $\bigcap_{n \in H} A_n = A_{\min(H)} \in \mathcal{J}$. $\square$

If we do not assume that $\mathcal{J}$ has analytic base, the assertion of Proposition 1 can be false. Namely, consider a partition $\mathcal{F} = \{ B_n : n \in \mathbb{N} \}$ of the real line into pairwise disjoint Bernstein sets. Let $\mathcal{J}$ stand for the ideal generated by $\mathcal{F} \cup [\mathbb{R}]^{\leq \omega}$. Then $\mathcal{J}$ is not $\sigma$-additive but it has property (LK) since $\mathcal{J} |_{\Sigma_1^1} = [\mathbb{R}]^{\leq \omega}$.

**Example 2.** Consider $X = \{0, 1\}^\mathbb{N}$ and the sequence $(A_n)$ of clopen subsets of $X$, given by

$A_n = \{ x \in X : x(n) = 1 \}, n \in \mathbb{N}.$

Let $\lambda$ stand for the standard (product) probability measure on $X$. The sets $A_n$, $n \in \mathbb{N}$, are independent with $\lambda(A_n) = 1/2$. We then have $\lambda(\bigcap_{n \in H} A_n) = 0$ for each $H \in [\omega]^\omega$ and, by the Borel-Cantelli lemma, $\lambda(\limsup_{n \in H} A_n) = 1$ for each $H \in [\omega]^\omega$. Hence the $\sigma$-ideal of sets of measure zero does not have property (LK). This example can be easily modified to the case of $X = [0, 1]$ with Lebesgue measure – the respective versions were given by Laczkovich [L, proof of 2] and Halmos [H]. Also note that the sets $\limsup_{n \in H} A_n$, $H \in [\omega]^\omega$, are dense of type $G_\delta$ (thus residual) while the sets $\bigcap_{n \in H} A_n$, $H \in [\omega]^\omega$, are closed nowhere dense. Hence it follows that the $\sigma$-ideal of meager sets, and the $\sigma$-ideal generated by closed sets of measure zero, do not have property (LK).

Denote by $\sigma(\Sigma_1^1)$ the $\sigma$-algebra generated by all analytic subsets of $X$. Recall that a Boolean algebra $A$ is said to be atomic if for each positive element $x \in A$, there is
an atom $a \in A$ such that $a \leq x$. If $\mathcal{J} \subset \mathcal{P}(X)$ is an ideal, the symbol $\sigma(\Sigma_1^1)/\mathcal{J}$ will abbreviate the quotient Boolean algebra $\sigma(\Sigma_1^1)/(\mathcal{J} \cap \sigma(\Sigma_1^1))$.

**Proposition 3.** Let $\mathcal{J}$ be a $\sigma$-ideal, with analytic base, such that $\sigma(\Sigma_1^1)/\mathcal{J}$ is an atomic Boolean algebra. Then $\mathcal{J}$ has property (LK).

**Proof.** For $A \in \sigma(\Sigma_1^1)$ let $[A]$ denote the respective element of $\sigma(\Sigma_1^1)/\mathcal{J}$. Let $(A_n)$ be a sequence of analytic sets. Since $\mathcal{J}$ is a $\sigma$-ideal, we have $[\limsup_n A_n] = \bigwedge_k \bigvee_{n \geq k} [A_n] \neq 0$.

Since $\sigma(\Sigma_1^1)/\mathcal{J}$ is atomic, pick an atom $a \leq [\limsup_n A_n]$. It follows that $a \leq \bigvee_{n \geq k} [A_n]$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ pick $n_k \geq k$ such that $a \wedge [A_{n_k}] \neq 0$, thus $a = [A_{n_k}]$.

Consequently, the set $H = \{n_k \in \mathbb{N} : a = [A_{n_k}]\}$ is infinite, and $a = \bigwedge_{n \in H} [A_n]$. Hence $\bigcap_{n \in H} A_n \notin \mathcal{J}$.

To show a simple application of Proposition 3, consider an analytic set $A \subset X$, $A \neq X$, and the ideal $\mathcal{P}(A)$. Then $\mathcal{P}(A)$ has property (LK) since the atoms of $\sigma(\Sigma_1^1)/\mathcal{P}(A)$ are of the form $\{\{x\}\}$, $x \in X \setminus A$.

2. A generalization of the Komjáth theorem

If $A \subset X \times Y$ and $x \in X$, we denote by $A(x) = \{y \in Y : (x, y) \in A\}$; this is the section of $A$ generated by $x$.

Assume that $E \subset X^2$ is an equivalence relation such that the family $X/E$ of all equivalence classes $E(x) = \{y \in X : (x, y) \in E\}$, $x \in X$, is uncountable. Next, consider the $\sigma$-ideal $\mathcal{J}_E$ generated by $X/E$, that is, $A \in \mathcal{J}_E$ if and only if $A \subset \bigcup_{n \in \mathbb{N}} E(x_n)$ for a sequence $(x_n) \in X^{\mathbb{N}}$. A set $B$ is called a partial transversal for $E$ if $|B \cap E(x)| \leq 1$, for each $x \in X$. Note that, if a partial transversal $B$ is uncountable then $B \notin \mathcal{J}_E$.

We are going to prove the following generalization of the Komjáth theorem.

**Theorem 4.** Let $E \subset X^2$ be an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$. Then for every sequence $(A^{(n)})$ of analytic subsets of $X$, such that $\limsup_{n \in H} A^{(n)} \notin \mathcal{J}_E$ for all $H \in [\mathbb{N}]^\omega$, there are sets $G \subset [\mathbb{N}]^\omega$ and $P \subset \bigcap_{n \in G} A^{(n)}$ such that $P$ is a partial transversal for $E$, homeomorphic with $\{0, 1\}^{\mathbb{N}}$. In particular, the $\sigma$-ideal $\mathcal{J}_E$ possesses property (LK).

The proof of Theorem 4 combines original ideas from the paper by Komjáth [K] with a demonstration of the fact that every relation $E$ satisfying assumptions of Theorem 3 admits a partial transversal homeomorphic with $\{0, 1\}^{\mathbb{N}}$ (cf. [Sr, 2.6.7, 2.6.8]; this fact remains true if $E$ is $\Pi_1^1$, by the Silver theorem [Ke, 35.20]).

The following three lemmas are counterparts of the respective lemmas in [K]. Before we will formulate them, we give some auxiliary terminology modified respectively in comparision with [K].

Fix a proper $\sigma$-ideal $\mathcal{J} \subset \mathcal{P}(X)$ containing all singletons, and a sequence $(A^{(n)})$ of analytic subsets of $X$ such that $\limsup_{n \in H} A^{(n)} \notin \mathcal{J}$ for all $H \in [\mathbb{N}]^\omega$. Next, fix $H \in [\mathbb{N}]^\omega$. 

We say that a set \( Y \subset X \) is good with respect to \( H \) if \( Y \cap \limsup_{n \in G} A^{(n)} \neq \emptyset \) for all \( G \in [H]^\omega \). Observe that, if \( Y \) is good with respect to \( H \), and \( Z \subset Y \), \( Z \in \mathcal{J} \), then \( Y \setminus Z \) is good with respect to \( H \). In particular, if \( Y \) is closed and good with respect to \( H \), then the perfect kernel of \( Y \) (cf. [Sr, 2.6.2]) is good with respect to \( H \); we will use this fact several times. For \( H_1, H_2 \in [\mathbb{N}]^\omega \) we say that \( H_1 \) is almost contained in \( H_2 \) if \( |H_1 \setminus H_2| < \omega \).

**Lemma 5.** If a set \( Y = \bigcup_{i \in \mathbb{N}} Y_i \) is good with respect to \( H \in [\mathbb{N}]^\omega \) then there are \( i \in \mathbb{N} \) and \( H' \in [H]^\omega \) such that \( Y_i \) is good with respect to \( H' \).

The proof is analogous to that given in [K, Lemma 1].

**Lemma 6.** Let \( F, P, A \subset X \) where \( F \) and \( P \) are closed, \( F \in \mathcal{J} \) and \( P \cap A \) is good with respect to a given \( H \in [\mathbb{N}]^\omega \). Then there are \( x \in P \setminus F \) and \( H' \in [H]^\omega \) such that \( (P \setminus F) \cap A \subset U \) is good with respect to \( H' \), for each neighbourhood \( U \) of \( x \).

**Proof.** (cf. [K, Lemma 2]) Since \( F \in \mathcal{J} \), the set \( (P \setminus F) \cap A \) is good with respect to \( H \). The set \( P \setminus F \) is of type \( F_\sigma \), so we can express it as \( \bigcup_{i \in \mathbb{N}} P_i \) where every \( P_i \) is closed and \( \text{diam} \, P_i < 1 \). By Lemma 5 there are \( i_0 \in \mathbb{N} \) and \( H_0 \in [H]^\omega \) such that \( A \cap P_{i_0} \) is good with respect to \( H_0 \). Let \( P_{i_0} = \bigcup_{i \in \mathbb{N}} P_{i_0i} \) where every \( P_{i_0i} \) is closed and \( \text{diam} \, P_{i_0i} < 1/2 \). By Lemma 5 there are \( i_1 \in \mathbb{N} \) and \( H_1 \in [H_0]^\omega \) such that \( A \cap P_{i_0i_1} \) is good with respect to \( H_1 \). We continue this process and find a sequence \( P \setminus F \supset P_{i_0} \supset P_{i_0i_1} \supset \ldots \) of closed sets with diameters tending to zero, and a sequence \( H_0 \supset H_1 \supset H_2 \supset \ldots \) such that \( H_0 \in [H]^\omega \) and \( A \cap P_{i_0 \ldots i_n} \) is good for \( H_n \), for every \( n \). Pick a point \( x \in \bigcap_{n \in \mathbb{N}} P_{i_0 \ldots i_n} \) and \( H' \in [H]^\omega \) almost contained in every \( H_n \). Then \( A \cap P_{i_0 \ldots i_n} \) is good with respect to \( H' \). For each neighbourhood \( U \) of \( x \), pick \( P_{i_0 \ldots i_n} \subset U \) and note that \( (P \setminus F) \cap A \subset U \) is good with respect to \( H' \). \( \square \)

**Lemma 7.** Let \( E \subset X^2 \) be as in Theorem 4, let \( E = \bigcup_{k \in \mathbb{N}} E_k \) where \( E_k \) are closed sets, and let \( \mathcal{J} = \mathcal{J}_E \). Fix \( H \in [\mathbb{N}]^\omega \), \( \varepsilon > 0 \), \( n \in \mathbb{N} \) and \( P_0, P_1, A_0, A_1 \subset X \) where \( P_0, P_1 \) are closed. If \( P_1 \cap A_1 \) is good with respect to \( H \), there are \( H' \in [H]^\omega \) and disjoint closed sets \( P_0 \subset P_0, P_1 \subset P_1 \) such that \( \text{diam} \, P_0 < \varepsilon \), \( \text{diam} \, P_1 < \varepsilon \), \( (P_0 \times P_1) \cap E_n = \emptyset \) and \( P_0 \cap A_0, P_1 \cap A_1 \) are good with respect to \( H' \).

**Proof.** Applying Lemma 6 to \( F = \emptyset \) we obtain a point \( x_0 \in P_0 \) and a set \( H_0 \in [H]^\omega \) such that \( P_0 \cap A_0 \subset U \) is good with respect to \( H_0 \), for each neighbourhood \( U \) of \( x_0 \). Since \( E_n(x_0) = \mathcal{J}_E \), applying Lemma 6 to \( F = E_n(x_0) \) we obtain \( x_1 \in P_1 \setminus E_n(x_0) \) and \( H_1 \in [H_0]^\omega \) such that \( (P_1 \setminus E_n(x_0)) \cap A_1 \subset U \) is good with respect to \( H_1 \), for each neighbourhood \( U \) of \( x_1 \). Since \( (x_0, x_1) \notin E_n \), by the closedness of \( E_n \) we can find open neighbourhoods \( U_0 \) and \( U_1 \) of \( x_0 \) and \( x_1 \) (respectively) such that \( \text{cl} \, U_0 \cap \text{cl} \, U_1 = \emptyset \), \( (\text{cl} \, U_0 \times \text{cl} \, U_1) \cap E_n = \emptyset \) and \( \text{diam} \, U_0 < \varepsilon \), \( \text{diam} \, U_1 < \varepsilon \). Put \( H' = H_1 \) and \( P_i' = P_i \cap \text{cl} \, U_i \) for \( i = 0, 1 \). \( \square \)
Proof of Theorem 4. We will follow the scheme used in the proof of the Komjáth
theorem [K]. We may suppose that $X$ is perfect (replacing it by its perfect kernel) and
that $\text{diam } X < 1$. Assume that $E = \bigcup_{n \in \mathbb{N}} E_n$ where sets $(E_n)$ is an increasing sequence
of closed sets. Fix a sequence $(A^{(j)})$ of analytic subsets of $X$ such that $\limsup_{j \in H} A^{(j)} \notin \mathcal{J}_E$ for every $H \in [\mathbb{N}]^\omega$. Express each set $A^{(j)}$ as the result of the Souslin operation (cf.
[Ke, 25.7]), that is

$$A^{(j)} = \bigcup_{z \in \mathbb{N}^\omega} \bigcap_{n \in \mathbb{N}} F^{(j)}_{z|n}$$

where sets $F^{(j)}_{z|n}$ are closed, $\text{diam } F^{(j)}_{z|n} < 1/(n+1)$ and for any $z \in \mathbb{N}^\omega$, $m, n \in \mathbb{N}$, if $n > m$ then $F^{(j)}_{z|m} \subset F^{(j)}_{z|n}$. For $t \in \mathbb{N}^\omega$ we put

$$A^{(j)}_t = \bigcup_{z \in \mathbb{N}^\omega, n = t \in \mathbb{N}} F^{(j)}_{z|k}.$$ 

We may assume that $A^{(0)} = X$. By recursion, for each $n \in \mathbb{N}$ we define a number $j_n \in \mathbb{N}$,
perfect sets $P_s$ (with $s \in \{0,1\}^n$), finite sequences $t(k, s) \in \mathbb{N}^n$ (with $k \leq n$, $s \in \{0,1\}^n$)
and a set $H_n \in [\mathbb{N}]^\omega$ with the following properties:

(W1) $j_n > j_{n-1}$, $H_n \in [H_{n-1}]^\omega$, $j_n \in H_{n-1}$;
(W2) $\text{diam } P_s < \frac{1}{n+1}$, $P_{s,0} \cup P_{s,1} \subset P_s$, $P_{s,0} \cap P_{s,1} = \emptyset$;
(W3) if $s, s' \in \{0,1\}^{n+1}$ and $s \neq s'$ then $(P_s \times P_{s'}) \cap E_n = \emptyset$;
(W4) $P_s \cap A^{(j_n)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)}$ is good with respect to $H_n$;
(W5) $P_s \subset F^{(j_n)}_{t(0,s)} \cap \ldots \cap F^{(j_n)}_{t(n,s)}$;
(W6) $t(k, s) \subset t(k, s'0) \cap t(k, s'1)$.

Having these objects defined, by (W2) we infer that $P = \bigcap_{n \in \mathbb{N}} \bigcup_{s \in \{0,1\}^n} P_s$ is a
set homeomorphic with $\{0,1\}^\mathbb{N}$ (cf. [Sr, 2.6]). For $z \in \{0,1\}^\mathbb{N}$ and $k \in \mathbb{N}$ denote
$z|k = (z(0), \ldots, z(k-1))$. Let $x, y \in P$, $x \neq y$, and consider $z, w \in \{0,1\}^\mathbb{N}$ such that $x \in \bigcap_{n \in \mathbb{N}} P_{z|n}$, $y \in \bigcap_{n \in \mathbb{N}} P_{w|n}$. Pick the minimal $k \in \mathbb{N}$ with $z(k) \neq w(k)$. Then by (W3)
we have $(P_{z|k} \times P_{w|k}) \cap E_i = \emptyset$ for all $i \geq k$, and also $(P_{z|k+1} \times P_{w|k+1}) \cap E_i = \emptyset$
for all $i < k$ since $E_0 \subset E_1 \subset \ldots \subset E_k$. So, $(x, y) \notin E$ and consequently, $P$ is a partial
transversal for $E$. Let $G = \{j_0, j_1, \ldots, j_n, \ldots\}$. Then by (W5) and (W6), we have that
$P \subset \bigcap_{n \in \mathbb{N}} A^{(j_n)} = \bigcap_{j \in G} A^{(j)}$ which yields the assertion.

The rest of proof consists of a construction of objects fulfilling (W1)–(W6) – the idea
is quite similar to that given in [K]. However, some details are more involved and we give
them for the reader’s convenience. First put $j_0 = 0$, $P_0 = X$, $H_0 = \mathbb{N}$ and $t(0, 0) = 0$.
Next assume that, for a fixed $n \in \mathbb{N}$, we have chosen $j_k$ (for $k \leq n$), $P_s$ (for $s \in \{0,1\}^k$,
$k \leq n$), $t(k, s)$ (for $k \leq l \leq n, s \in \{0,1\}^l$) and $H_k$ (for $k \leq n$).

First, we shall prove that there are a number $j \in H_n$ such that $j > j_n$ and a set
$H'_n \in [H_n]^{\omega}$ fulfilling the condition

(W7) $(\forall s \in \{0,1\}^n) P_s \cap A^{(j_n)}_{t(0,s)} \cap \ldots \cap A^{(j_n)}_{t(n,s)} \cap A^{(j)}$ is good wrt $H'_n$.
If it is not so, for each \( j \in H_n \), \( j > j_n \), and for each \( H \in [H_n]^\omega \) there are \( s \in \{0, 1\}^n \) and \( G \in [H]^\omega \) such that
\[
P_s \cap A_{t(0, s)}^{(j_0)} \cap \ldots \cap A_{t(n, s)}^{(j_n)} \cap A^{(j)} \cap \limsup_{r \in G} A^{(r)} \in \mathcal{J}_E.
\]

Proceeding inductively, we find numbers \( k \) such that for each \( m \in \mathbb{N} \) we have \( k_m \in G_m \in [\mathbb{N}]^\omega \) and we can fix an \( s_m \in \{0, 1\}^n \) with
\[
P_{s_m} \cap A_{t(0, s_m)}^{(j_0)} \cap \ldots \cap A_{t(n, s_m)}^{(j_n)} \cap A^{(k_m)} \cap \limsup_{r \in G_{m+1}} A^{(r)} \in \mathcal{J}_E.
\]

Then pick an \( s \in \{0, 1\}^n \) such that \( \Gamma = \{ k_m : s_m = s \} \) is infinite. Observe that \( \Gamma \) is almost contained in every \( G_m \). Hence
\[
P_s \cap A_{t(0, s)}^{(j_0)} \cap \ldots \cap A_{t(n, s)}^{(j_n)} \cap \left( \bigcup_{m \in \Gamma} A^{(m)} \right) \cap \limsup_{r \in \Gamma} A^{(r)} \in \mathcal{J}_E.
\]

Since \( \limsup_{r \in \Gamma} A^{(r)} \subset \bigcup_{m \in \Gamma} A^{(m)} \), the union in the above condition can be deleted, and so, we obtain a contradiction with (W4).

Consequently, the respective \( j \in H_n \), \( j > j_n \), and \( H_n' \in [H_n]^\omega \) fulfilling (W7) do exist, and we put \( j_{n+1} = j \). For each \( s \in \{0, 1\}^n \), write in short
\[
A^*_s = A_{t(0, s)}^{(j_0)} \cap \ldots \cap A_{t(n, s)}^{(j_n)} \cap A^{(j_{n+1})}.
\]

Now, we will show how to construct sets \( P_s \) with \( s \in \{0, 1\}^{n+1} \). Because of condition (W3), the construction is divided into several steps using Lemma 7. List all distinct pairs in \( \{0, 1\}^n \times \{0, 1\}^n \) as \( (s_i, s'_i) \) \((i = 1, \ldots, p_n)\). We decreases sets \( P_s, s \in \{0, 1\}^n \), in \( p_n \) steps as follows. Put \( H^{(0)}_n = H'_n \) and \( P^{(0)}_s = P_s \) for \( s \in \{0, 1\}^n \). In the \( i \)th step \((i = 1, \ldots, p_n)\) applying Lemma 7 (and (W7) if \( i = 1 \)) we find \( H^{(i)}_n \in [H^{(i-1)}_n]^\omega \) and closed sets \( P^{(i)}_{s_1} \subset P^{(i-1)}_{s_1}, P^{(i)}_{s'_1} \subset P^{(i-1)}_{s'_1} \) such that \( (P^{(i)}_{s_1} \times P^{(i)}_{s'_1}) \cap E_n = \emptyset \) and \( P^{(i)}_{s_1} \cap A^*_{s_1} \) are good wrt \( H^{(i)}_n \); we also put \( P^{(i)}_s = P^{(i-1)}_s \) for all \( s \in \{0, 1\}^n \setminus \{s_i, s'_i\} \). If this process is finished, we define \( H''_n = H^{(p_n)}_n \) and \( P^*_s = P^{(p_n)}_s \) for all \( s \in \{0, 1\}^n \).

Next by Lemma 7, for every \( s \in \{0, 1\}^n \) we find disjoint closed sets \( P^{(0)}_{s_0}, P^{(1)}_{s_1} \subset P^*_s \) such that \( \text{diam} P^{(0)}_{s_0} < 1/(n+2) \), \( \text{diam} P^{(1)}_{s_1} < 1/(n+2) \), \( (P^{(0)}_{s_0} \times P^{(1)}_{s_1}) \cap E_n = \emptyset \), and we find \( H''_n \in [H''_n]^\omega \) such that for all \( s \in \{0, 1\}^n \) and \( i \in \{0, 1\} \)
\[
(P^{(i)}_{s_0} \cap A^*_{s_1}) = (P^{(i)}_{s_0} \cap A_{t(0, s)}^{(j_0)} \cap \ldots \cap A_{t(n, s)}^{(j_n)} \cap A^{(j_{n+1})}) \text{ is good wrt } H''_n.
\]

Fix, \( s \in \{0, 1\}^n \), \( i \in \{0, 1\} \). Since the set in (W8) is contained in the union
\[
\bigcup_{z_0 \in \mathbb{N}^{n+1}, z_0 \supseteq (0, s)} \ldots \bigcup_{z_n \in \mathbb{N}^{n+1}, z_n \supseteq (n, s)} \bigcup_{z_{n+1} \in \mathbb{N}^{n+1}} P^{(i)}_{z_0} \cap A_{z_0}^{(j_0)} \cap \ldots \cap A_{z_{n+1}}^{(j_{n+1})},
\]
by Lemma 5, one of the components of this union is good. Moreover, if we use \( 2^{n+1} \) times Lemma 5, we obtain \( H''_n \in [H''_n]^\omega \) witnessing this fact simultaneously for all \( s \in \{0, 1\}^n \).
and \( i \in \{0,1\} \). Choose \( t(0, s^i), \ldots, t(n+1, s^i) \) as the sequence corresponding to \( s^i \).

Observe that the sets
\[
Q_{s^i} = \overline{P_{s^i}} \cap F_{t(0, s^i)} \cap \ldots \cap F_{t(n+1, s^i)}
\]
are good for \( H_n \) since \( Q_{s^i} \) and \( \overline{P_{s^i}} \) have the same intersections with \( A_{t(0, s^i)} \cap \ldots \cap A_{t(n+1, s^i)} \). Finally, define \( P_{s^i} \) as the perfect kernel of \( Q_{s^i} \) and let \( H_{n+1} = \overline{H_n} \).

If \( E \) is the equality relation, Theorem 4 yields exactly the Komjáth theorem. If \( A_n = A \) for every \( n \in \mathbb{N} \), we obtain the following corollary which also can be derived from a deep result of Silver (cf. [Ke, 35.20]).

**Corollary 8.** If \( E \subset X^2 \) is an equivalence relation of type \( F_\sigma \) with \( |X/E| > \omega \), and \( A \not\subset \mathcal{J}_E \) is an analytic set, then there is a set \( P \subset A \) homeomorphic with \( \{0,1\}^\mathbb{N} \) and being a partial transversal for \( E \).

3. Parametric Laczkovich-Komjáth property

Several combinatorial results have their parametric versions which in fact generalize them in a nice way, see e.g. [Mi], [Pa]. A parametric version of the Komjáth theorem was proved in [G]. Here, by the use of similar methods, we shall prove a parametric version of Theorem 3. Moreover, we give a condition which guarantees that a \( \sigma \)-ideal \( \mathcal{J} \) with property (LK) has parametric property (LK).

By the Mazurkiewicz-Sierpiński theorem [Ke, 29.19], if \( X, Y \) are Polish spaces then for each analytic set \( A \subset X \times Y \), the set \( \{x \in X : |A(x)| > \omega \} \) is also analytic. We say that an ideal \( \mathcal{J} \subset \mathcal{P}(Y) \) has the Mazurkiewicz-Sierpiński property if for any Polish space \( X \) and analytic set \( A \subset X \times Y \), the set \( \{x \in X : A(x) \notin \mathcal{J} \} \) is analytic. This property holds true for, besides the ideal of countable sets, the ideal of meager sets in \( Y \) [Ke, 29.22] and the ideal of Lebesgue null sets in \( \mathbb{R} \) [Ke, 29.26].

We say that an ideal \( \mathcal{J} \subset \mathcal{P}(Y) \) has parametric property (LK), whenever for every uncountable Polish space \( X \) and every sequence \( \{A_n \} \) of analytic subsets of \( X \times Y \), if \( \limsup_{n \in H} A_n(x) \notin \mathcal{J} \) for all \( x \in X \) and \( H \in [\mathbb{N}]^\omega \) then there are a perfect set \( P \subset X \) and \( G \in [\mathbb{N}]^\omega \) such that \( \bigcap_{j \in G} A_j(x) \notin \mathcal{J} \) for each \( x \in P \). Since \( X \) contains a homeomorphic copy of the Cantor space, we may assume that \( X = \{0,1\}^\mathbb{N} \). Clearly, parametric property (LK) is stronger than property (LK). In [G], it was proved that the ideal \( [\mathbb{N}]^{\leq \omega} \) of all countable subsets of \( \mathbb{N} \) has parametric property (LK). The same scheme of a proof will work for Proposition 9.

Recall some definitions. For any \( \alpha \in [\mathbb{N}]^\omega \) and \( H \in [\mathbb{N}]^{<\omega} \) with \( \max(\alpha) < \min(H) \), the set of the form \( [\alpha, H] = \{ G \in [\mathbb{N}]^{<\omega} : \alpha \subset G \subset \alpha \cup H \} \) is said to be an **Ellentuck neighbourhood**, and the topology generated by all Ellentuck neighbourhoods is called the **Ellentuck topology** on \( [\mathbb{N}]^\omega \). According to [Pa], a set \( A \subset \{0,1\}^\omega \times [\mathbb{N}]^\omega \) is called **perfectly Ramsey** if for every perfect set \( P \subset \{0,1\}^\mathbb{N} \) and every Ellentuck neighbourhood
There are a perfect set $Q \subset P$ and $G \in [H]^{\omega}$ such that either $Q \times [\alpha, G] \subset A$ or $(Q \times [\alpha, G]) \cap A = \emptyset$. (All the considered perfect sets are nonempty.) If we identify sets $H \in [N]^{\omega}$ with their indicators in $\{0, 1\}^N$, the space $[N]^{\omega}$ is Polish. From [Pa, Thm 1.1] it follows that every analytic set $A \subset \{0, 1\}^N \times [N]^{\omega}$ is perfectly Ramsey.

\textbf{Proposition 9.} Let $Y$ be an uncountable Polish space and let $\mathcal{J} \subset \mathcal{P}(Y)$ be a $\sigma$-ideal with property (LK) and with the Mazurkiewicz-Sierpiński property. Then $\mathcal{J}$ has parametric property (LK).

\textbf{Proof.} Put $X = \{0, 1\}^N$. Let $A_j \subset X \times Y$, $j \in \mathbb{N}$, be analytic sets such that $\limsup_{j \in H} A_j(x) \notin \mathcal{J}$ for $x \in X$ and $H \in [N]^{\omega}$. Define

$$A = \{(x, H) \in X \times [N]^{\omega} : \bigcap_{j \in H} A_j(x) \notin \mathcal{J}\}.$$ 

Consider

$$B = \{(x, H, y) \in X \times [N]^{\omega} \times Y : (x, y) \in \bigcap_{j \in H} A_j\}$$

$$= \{(x, H, y) \in X \times [N]^{\omega} \times Y : \forall j \in \mathbb{N} \ (j \notin H \text{ or } (x, y) \in A_j)\}$$

and observe that $B$ is analytic. Hence the set

$$A = \{(x, H) \in X \times [N]^{\omega} : B(x, H) \notin \mathcal{J}\}$$

is analytic, since $\mathcal{J}$ has the Mazurkiewicz-Sierpiński property. Now, by the Pawlikowski theorem, $A$ is perfectly Ramsey. Then pick a perfect set $P \subset X$ and $H \in [N]^{\omega}$ such that either $P \times \{0, H\} \subset A$ or $(P \times \{0, H\}) \cap A = \emptyset$. The latter case is impossible since, by property (LK) of $\mathcal{J}$, for each $x \in P$ there exists $G \in [H]^{\omega}$ such that $\bigcap_{j \in G} A_j(x) \notin \mathcal{J}$. The former case yields $\bigcap_{j \in H} A_j(x) \notin \mathcal{J}$ for all $x \in P$. \hfill \Box

By $\mathcal{K}(X)$ we denote the hyperspace of all nonempty compact subsets of $X$, equipped with the Vietoris topology (or, equivalently with the Hausdorff metric); cf. [Ke, 4.7] and [Sr, pp. 66–69]. In the sequel, a perfect set which is a partial transversal for an equivalence relation $E$ will be called a \textit{perfect partial transversal for $E$} (in short, $E$-ppt).

\textbf{Lemma 10.} Let $Y$ be an uncountable Polish space. If $E \subset Y^2$ is an equivalence relation of type $F_\sigma$ with $|Y/E| > \omega$ then the family of all sets $L \in \mathcal{K}(Y)$ containing a perfect partial transversal for $E$ is analytic.

\textbf{Proof.} Let $E = \bigcup_{n \in \mathbb{N}} E_n$ where $(E_n)$ is an increasing sequence of closed sets. Fix a countable base $\{U_i : i \in \mathbb{N}\}$ of the topology in $Y$. For $L \in \mathcal{K}(Y)$ we have the following equivalence

$$(*) \quad L \text{ contains an } E\text{-ppt} \iff (\exists K \in \mathcal{K}(L)) (\forall i, j, n \in \mathbb{N}) (U_i \cap K \neq \emptyset \neq U_j \cap K) \Rightarrow (\exists k, l \in \mathbb{N}) (\text{cl} U_k \subset U_i, \text{cl} U_l \subset U_j, \text{cl} U_k \cap \text{cl} U_l = \emptyset, \text{diam} U_k < \frac{1}{n + 1}, \text{diam} U_l < \frac{1}{n + 1}, U_k \cap K \neq \emptyset \neq U_l \cap K, (U_k \times U_l) \cap E_n = \emptyset).$$
Hence, in a standard way (cf. [Ke, 4.29], [Sr, 2.4.11]) we show that the family of all sets $L \in \mathcal{K}(Y)$ containing an $E$-ppt is analytic. Thus to finish the proof it suffices to show that $(\ast)$ does hold.

If $L \in \mathcal{K}(Y)$ contains an $E$-ppt $K$, we easily conclude that $K$ satisfies the right hand side of the equivalence $(\ast)$. Conversely, if $K \in \mathcal{K}(L)$ satisfies the right hand side of the equivalence $(\ast)$, we can define by recursion a family $\{V_s : s \in \{0, 1\}^{<\mathbb{N}}\} \subset \{U_i : i \in \mathbb{N}\}$ such that for each $s \in \{0, 1\}^{<\mathbb{N}}$ the following conditions hold:

(i) $V_s \cap K \neq \emptyset$;
(ii) $\text{cl}_{V_s} 0 \cup \text{cl}_{V_s} 1 \subset V_s$, $\text{cl}_{V_s} 0 \cap \text{cl}_{V_s} 1 = \emptyset$;
(iii) $\text{diam} V_s < 1/(|s| + 1)$;

and additionally,

(iv) $(V_s \times V_{s'}) \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and $s, s' \in \{0, 1\}^{n+1}$, $s \neq s'$.

The construction is similar to that given in the proof of Theorem 4 (cf. conditions (W1)–(W3)). Then $\bigcap_{n \in \mathbb{N}} \bigcup_{s \in \{0, 1\}^n} (K \cap \text{cl} V_s)$ is an $E$-ppt contained in $L$. \hfill $\square$

**Theorem 11.** Let $E \subset X^2$ be an equivalence relation of type $F_\sigma$ with $|X/E| > \omega$. Then the $\sigma$-ideal $\mathcal{E}$ has the Mazurkiewicz-Sierpiński property.

**Proof.** Set $N = \mathbb{N}^\mathbb{N}$. For an analytic set $B \subset Y$ pick a closed set $F \subset Y \times N$ such that $\text{pr}_Y(F) = B$ where $\text{pr}_Y$ stands for the projection from $Y \times N$ to $Y$. Observe that

$$\tag{**} B \notin \mathcal{E} \iff (\exists K \in \mathcal{K}(Y \times N))(K \subset F \text{ and } \text{pr}_Y(K) \text{ contains an } E\text{-ppt}).$$

Indeed, to show “$\Rightarrow$” assume that $B \notin \mathcal{E}$. By Corollary 8, $B$ contains an $E$-ppt $P$. Note that $P = \text{pr}_Y((P \times N) \cap F)$. By [Ke, 29.20] there is a set $K \subset (P \times N) \cap F$ such that the both $K$ and $\text{pr}_Y(K)$ are homeomorphic with $\{0, 1\}^N$. Since $\text{pr}_Y(K) \subset P$ so $\text{pr}_Y(K)$ is an $E$-ppt with $K \subset F$. Implication “$\Leftarrow$” is obvious.

Now, let $A \subset X \times Y$ be an analytic set and pick a closed set $F \subset X \times Y \times N$ such that $\text{pr}_{X \times Y}(F) = A$. Then $A(x) = \text{pr}_Y(F(x))$ and $F(x) \subset Y \times N$ is closed for each $x \in X$.

By (**), for each $x \in X$ we have

$$\tag{**'} A(x) \notin \mathcal{E} \iff (\exists K \in \mathcal{K}(Y \times N))(K \subset F(x) \text{ and } \text{pr}_Y(K) \text{ contains an } E\text{-ppt}).$$

Observe that the set $\{(x, K) \in X \times \mathcal{K}(Y \times N) : K \subset F(x)\}$ is closed and note that the mapping $K \mapsto \text{pr}_Y(K)$ from $\mathcal{K}(Y \times N)$ to $\mathcal{K}(Y)$ is continuous [Ke, 4.29(vi)]. Hence by Lemma 10 and (**') the assertion follows. \hfill $\square$

From Proposition 9, by Theorems 4 and 11, we deduce immediately the following fact.

**Theorem 12.** Let $Y$ be an uncountable Polish space. If $E \subset Y^2$ is an equivalence relation of type $F_\sigma$ with $|Y/E| > \omega$ then $\mathcal{E}$ has parametric property (LK).
4. Some results about invariance

In this section we study how property (LK) is preserved by various operations.

It is well known that for any two uncountable Polish spaces there is a Borel isomorphism between them; see [Ke, 15.6] and [Sr, 3.3.13]. Observe that if \( X \) and \( Y \) are uncountable Polish spaces and a \( \sigma \)-ideal \( J \subseteq \mathcal{P}(X) \) has property (LK) then, for every Borel isomorphism \( \varphi: X \rightarrow Y \), the \( \sigma \)-ideal \( \{ \varphi(A): A \in J \} \subseteq \mathcal{P}(Y) \) has property (LK).

From Example 2 we know that the \( \sigma \)-ideals of meager subsets of \( \{0,1\}^\mathbb{N} \) and of measure zero subsets of \( \{0,1\}^\mathbb{N} \) do not have property (LK). These facts can be generalized. Since between any two perfect Polish spaces there is a Borel isomorphism preserving the Baire category (see e.g. [CKW, 3.15]), the \( \sigma \)-ideal of meager subsets of a perfect Polish space does not have property (LK). Similarly, using a special Borel isomorphism (cf. [Ke, 17.41]) we infer that the \( \sigma \)-ideal of null sets with respect to a finite continuous Borel measure on an uncountable Polish space does not have property (LK).

For uncountable Polish spaces \( X, Y \) and for \( \sigma \)-ideals \( J \subseteq \mathcal{P}(X), \, J \subseteq \mathcal{P}(Y) \), put
\[
J \otimes J = \{ A \subseteq X \times Y: \{ x \in X: A(x) \notin J \} \in J \}.
\]

Then \( J \otimes J \) forms a \( \sigma \)-ideal.

**Example 13.** Let \( E \subseteq X^2 \) be an equivalence relation of type \( F_\sigma \) with \( |X/E| > \omega \). Consider \( \{ \emptyset \} \), the trivial \( \sigma \)-ideal of subsets of \( Y \). We will show that \( J \subseteq \mathcal{P}(X) \) has property (LK). To this aim define \( E \subseteq (X \times Y)^2 \) by
\[
(x,y) \in E \iff xEx', \quad \text{for } (x,y), (x',y') \in X \times Y.
\]

Clearly, \( E \) is of type \( F_\sigma \). Also \( (x,y) \in E \) for each \( (x,y) \in X \times Y \). It is easy to check that \( J_E = J \subseteq \mathcal{P}(X) \) has property (LK) by Theorem 4. In particular, \([X]^{<\omega} \otimes \{ \emptyset \} \) has property (LK).

The case of the \( \sigma \)-ideal \( \{ \emptyset \} \otimes [X]^{<\omega} \) is more interesting. The problem whether this \( \sigma \)-ideal has property (LK) remains open. We have only a partial result which can shed some light on the problem.

Let \( J, J \) be \( \sigma \)-ideals such that \( J \subseteq J \subseteq \mathcal{P}(X) \). We say that the pair \( (J, J) \) has property (LK) whenever for every sequence \( (A_n) \) of analytic subsets of \( X \), condition \( \limsup_{n \in H} A_n \notin J \) for every \( H \subseteq \mathbb{N}^\omega \) implies \( \bigcap_{n \in G} A_n \notin J \) for some \( G \subseteq \mathbb{N}^\omega \). Clearly, if \( J \) has property (LK) then \( (J, J) \) has property (LK), and \( J \) has property (LK) iff \( (J, J) \) has property (LK).

**Proposition 14.** For uncountable Polish spaces \( X \) and \( Y \), consider the \( \sigma \)-ideals \( [X]^{<\omega}, \{ \emptyset \} \subseteq \mathcal{P}(X) \) and a fixed \( \sigma \)-ideal \( J \subseteq \mathcal{P}(Y) \) with property (LK). Then \( \{ \emptyset \} \otimes J \) has property (LK) if and only if \( \{ \emptyset \otimes J \} \subseteq [X]^{<\omega} \otimes J \) has property (LK).

**Proof.** "\( \Rightarrow \)" Let \( (A_n) \) be a sequence of analytic subsets of \( X \times Y \) such that \( \limsup_{n \in H} A_n \notin [X]^{<\omega} \otimes J \) for each \( H \subseteq \mathbb{N}^\omega \). Then also \( \limsup_{n \in H} A_n \notin \{ \emptyset \} \otimes J \) for
each $H \in [N]^{\omega}$. By the assumption, $\{\emptyset\} \otimes J$ has property (LK). So, $\bigcap_{n \in G} A_n \notin \{\emptyset\} \otimes J$ for some $G \in [N]^{\omega}$.

$\leftarrow$ Suppose that $\{\emptyset\} \otimes J$ does not have property (LK). Thus there is a sequence $(A_n)$ of analytic subsets of $X \times Y$ such that for all $H \in [N]^{\omega}$ we have $\limsup_{n \in H} A_n \notin \{\emptyset\} \otimes J$ and

$$(\triangle) \quad (\forall G \in [H]^{\omega}) \bigcap_{n \in G} A_n \in \{\emptyset\} \otimes J.$$ 

Define 

$$B_H = \{x \in X : \limsup_{n \in H} A_n(x) \notin J\}, \quad H \in [N]^{\omega}.$$ 

Consider two cases:

1. $|B_H| > \omega$ for all $H \in [N]^{\omega}$. Hence $\limsup_{n \in H} A_n \notin [X]^{<\omega} \otimes J$ for all $H \in [N]^{\omega}$, and by the assumption $\bigcap_{n \in G} A_n \notin \{\emptyset\} \otimes J$ for some $G \in [N]^{\omega}$, a contradiction with $(\triangle)$.

2. $|B_{H_0}| \leq \omega$ for some $H_0 \in [N]^{\omega}$. Firstly note that if $H, H' \in [N]^{\omega}$ and $H$ is almost contained in $H'$ then $B_H \subseteq B_{H'}$. Secondly note that it is not possible to have $B_G = B_{H_0}$ for all $G \in [H_0]^{\omega}$ since in this case, for each $x \in B_{H_0}$ (by property (LK) for $J$) we would pick $G_x \in [H_0]^{\omega}$ such that $\bigcap_{n \in G_x} A_n(x) \notin J$, a contradiction with $(\triangle)$. Hence there exists a set $G \in [H_0]^{\omega}$ such that $B_G \neq B_{H_0}$. Proceeding inductively, we define a sequence $(H_\alpha)_{\alpha < \omega_1}$ such that $H_{\alpha+1} \in [H_\alpha]^{\omega}$, $B_{H_{\alpha+1}} \neq B_{H_\alpha}$ ($\alpha < \omega_1$), and for a limit ordinal $\alpha < \omega_1$, we pick $H_\alpha \in [H_0]^{\omega}$ almost contained in every $H_\beta$, $\beta < \alpha$. So, $(B_{H_\alpha})_{\alpha < \omega_1}$ is a strictly descending sequence of countable sets, a contradiction. $\square$

Another interesting question concerns the intersection of $\sigma$-ideals: What is the (possibly large) cardinality of a family of $\sigma$-ideals with property (LK) such that the intersection of the family has also property (LK)?

Let $r_0$ stand for the ideal of nowhere dense sets in the Ellentuck topology on $[N]^{\omega}$. Put

$$\text{cov}(r_0) = \min\{|D| : D \subseteq r_0 \text{ or } \bigcup D = [N]^{\omega}\}.$$ 

Plewik [Pl] proved that $\text{cov}(r_0) = h$ where $h$ is the cardinal introduced by Balcar, Pelant and Simon [BPS]. It is known that $\omega_1 \leq h \leq 2^{\omega}$, and either or both inequalities can be strict in some models of ZFC (see [V]). We offer the following result connected with the above-mentioned question.

**Proposition 15.** Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of size $|\mathcal{F}| < h$, of $\sigma$-ideals with property (LK) on an uncountable Polish space $X$. Then $\bigcap \mathcal{F}$ has property (LK).

**Proof.** Let $(A_n)$ be a sequence of analytic subsets of $X$ such that $\limsup_{n \in H} A_n \notin \bigcap \mathcal{F}$ for each $H \in [N]^{\omega}$. Put

$$B_J = \{H \in [N]^{\omega} : \limsup_{n \in H} A_n \notin J\} \quad \text{for } J \in \mathcal{F}.$$
We have $[\mathbb{N}]^\omega = \bigcup_{\mathcal{B} \in \mathcal{F}} \mathcal{B}_\mathcal{B}$. Since $|\mathcal{F}| < \mathfrak{b} = \text{cov}(\mathcal{R}_0)$, pick a $\sigma$-ideal $\mathcal{J} \in \mathcal{F}$ such that $\mathcal{B}_\mathcal{J}$ is dense in some Ellentuck neighbourhood $[\alpha, H]$ with $\alpha \in [\mathbb{N}]^\omega$, $H \in [\mathbb{N}]^\omega$, $\max(\alpha) < \min(H)$. Hence for every $G \in [H]^\omega$ we can find $G' \in \mathcal{B}_\mathcal{J} \cap [\alpha, G]$. Consequently, $\limsup_{n \in G'} A_n \notin \mathcal{J}$ and since $G' \subset \alpha \cup G$, we have $\limsup_{n \in G} A_n \notin \mathcal{J}$. By property (LK) of $\mathcal{J}$, there is $G_0 \in [H]^\omega$ such that $\bigcap_{n \in G_0} A_n \notin \mathcal{J}$. Thus $\bigcap_{n \in G_0} A_n \notin \bigcap \mathcal{F}$.

The following example shows a family $\mathcal{F}$, of size of the continuum, of $\sigma$-ideals with property (LK) such that $\bigcap \mathcal{F}$ has property (LK) and $\bigcap \mathcal{F}$ is different from the $\sigma$-ideal of countable sets.

**Example 16.** For a fixed $z \in [0, 1]$, let $\mathcal{J}^{(z)}$ stand for the $\sigma$-ideal of subsets of $[0, 1]^2$ generated by the family

$$\{\{0\} \times [0, 1]\} \cup \{\{z\} \times [0, 1]\} \cup ([0, 1]^2)^{<\omega}.$$

Then $\mathcal{J}^{(z)} = \mathcal{J}_{E_z}$ where $E_z \subset ([0, 1]^2)^2$ is the equivalence relation given by

$$(x, y)E_z(x', y') \iff ((x = x') \text{ and } (x = 0 \text{ or } x = z \text{ or } y = y')).$$

Since $E_z$ is closed and $|[0, 1]^2/E_z| = 2^\omega$, the $\sigma$-ideal $\mathcal{J}^{(z)}$ has property (LK) by Theorem 4. Let $\mathcal{F} = \{\mathcal{J}^{(z)}: z \in [0, 1]\}$. Then $|\mathcal{F}| = 2^\omega$ and $\bigcap \mathcal{F} = \mathcal{J}^{(0)}$, so $\bigcap \mathcal{F}$ has property (LK).

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