QUALITATIVE PROPERTIES OF IDEAL CONVERGENT SUBSEQUENCES AND REARRANGEMENTS

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Abstract. We investigate the Baire category of $\mathcal{I}$-convergent subsequences and rearrangements of a divergent sequence $s = (s_n)$ of reals, if $\mathcal{I}$ is an ideal on $\mathbb{N}$ having the Baire property. We also discuss the measure of the set of $\mathcal{I}$-convergent subsequences for some classes of ideals on $\mathbb{N}$. Our results generalize theorems due to H. Miller and C. Orhan (2001).

1. Introduction

Denote $\mathbb{N} := \{1, 2, \dots\}$. Let $s = (s_n) \in \mathbb{R}^{\mathbb{N}}$ be a given sequence. Firstly, we are interested in subsequences of s. Let $T \subset \{0,1\}^{\mathbb{N}}$ denote the set of all sequences with infinite number of ones. For $x = (x_n) \in T$ we generate a subsequence sx of s in such a way that, if t_n is a position of the nth "1" in the sequence x, then $(sx)_n := s_{t_n}$ for $n \in \mathbb{N}$. Clearly, all subsequences of s can be coded in this manner (and the coding is one-to-one). It is known that the Cantor space $\{0,1\}^{\mathbb{N}}$ equipped with the product topology is complete. Observe that T is a G_δ subset of $\{0,1\}^{\mathbb{N}}$, therefore by the Alexandrov theorem, it is completely metrizable. So, we may apply the Baire category theorem to the space T while studying sets of subsequences of s.

Secondly, we are interested in rearrangements of the sequence s. Recall that $\mathbb{N}^{\mathbb{N}}$ equipped with the product topology is a complete space, and the set $P \subset \mathbb{N}^{\mathbb{N}}$ of all bijections from $\mathbb{N}$ onto itself (permutations of $\mathbb{N}$) is a G_δ subspace [19, p. 66], so it is completely metrizable. Hence the Baire category theorem works in P. Rearrangements of s are sequences of the form $(s_{p(n)})$ for $p \in P$.

Let us recall some useful definitions and facts on ideals of subsets of $\mathbb{N}$ (cf. [14],[8]). We say that an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is *admissible* if $\mathbb{N} \notin \mathcal{I}$ and the ideal Fin of all finite subsets of $\mathbb{N}$ is contained in $\mathcal{I}$. From now on, we will consider only admissible ideals; we will simply call them *ideals on* $\mathbb{N}$. Ideals on $\mathbb{N}$ can be treated (via the characteristic functions) as subsets of the Polish space $\{0,1\}^{\mathbb{N}}$, so they can have the Baire property, be Borel, analytic, coanalytic, and so on.

The following important fact is due to Jalali-Naini [12] and Talagrand [20]; see also [21, Theorem 1, Section 8].

Lemma 1.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent:*

- *$\mathcal{I}$ has the Baire property;*
- *$\mathcal{I}$ is meager;*
- *there is an infinite sequence $n_1 < n_2 < \dots$ of integers in $\mathbb{N}$ such that no member of $\mathcal{I}$ contains infinitely many intervals $[n_i, n_{i+1})$ in $\mathbb{N}$.*

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An ideal $\mathcal{I}$ on $\mathbb{N}$ is called *maximal* if there is no ideal on $\mathbb{N}$ which is a proper superset of $\mathcal{I}$. It is well known that $\mathcal{I}$ is maximal if and only if, for every partition $\{A, B\}$ of $\mathbb{N}$ into infinite sets, exactly one of the conditions $A \notin \mathcal{I}$, $B \notin \mathcal{I}$ is true.

If $\mathcal{I}$ is an ideal on $\mathbb{N}$ and (s_n) is a sequence of reals, we say (cf. [14]) that (s_n) is *$\mathcal{I}$-convergent* to $t \in \mathbb{R}$ (and write $\mathcal{I}\text{-}\lim_n s_n = t$) if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N}\colon |s_n - t| \geq \varepsilon\} \in \mathcal{I}$. It is easy to see that if $\mathcal{I} = \mathrm{Fin}$, we get the usual convergence of (s_n) to t. Also, let us mention the case when $\mathcal{I}$ equals $\mathcal{I}_d$, *the density ideal* which consists of sets $A \subset \mathbb{N}$ with asymptotic density zero. (Recall that the *asymptotic density* of $A \subset \mathbb{N}$ is given by $d(A) := \lim_n |A \cap [1, n]|/n$, provided the limit exists.) In this case we speak about *statistical convergence* (see [9], [11]). For several interesting applications of statistical and ideal convergence, see for instance [10],[5],[7],[1],[4].

2. Results on the Baire category

Given an ideal $\mathcal{I}$ on $\mathbb{N}$ and a divergent sequence $s = (s_n) \in \mathbb{R}^{\mathbb{N}}$, the Baire category (in the spaces T and P, respectively) of the following two sets will be investigated:

$$S_s^{\mathcal{I}} := \{x \in T\colon sx \text{ is } \mathcal{I}\text{-convergent}\}, \quad R_s^{\mathcal{I}} := \{p \in P\colon (s_{p(n)}) \text{ is } \mathcal{I}\text{-convergent}\}.$$

The motivation of such studies comes from the results by Miller and Orhan [16] who proved that for $\mathcal{I} := \mathcal{I}_d$ these sets are of the first category. From now on, sets of the first category will be called *meager*, and residual sets will be called *comeager*.

Theorem 2.1. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ with the Baire property and let $s = (s_n)$ be a divergent sequence of reals. Then the sets $S_s^{\mathcal{I}}$ and $R_s^{\mathcal{I}}$ are meager in T and P, respectively.*

Proof. Note that, if s is divergent to ∞ or to $-\infty$, then $S_s^{\mathcal{I}} = \emptyset = R_s^{\mathcal{I}}$. Indeed, if a subsequence s' of s is $\mathcal{I}$-convergent to $t \in \mathbb{R}$, then a subsequence of s' is convergent to t which is impossible. Also, if a rearrangement $s'' = (s_{p(n)})$ of s is $\mathcal{I}$-convergent to $t \in \mathbb{R}$, then a subsequence of s'' is convergent to t which is impossible. Consequently, from now on we assume that s has at least two different partial limits a, b where $-\infty \leq a < b \leq \infty$. Pick α, β such that $a < \alpha < \beta < b$.

At first we will show that $S_s^{\mathcal{I}}$ is meager. Since $\mathcal{I}$ has the Baire property, by Lemma 1.1 fix an infinite sequence $n_1 < n_2 < \dots$ of integers in $\mathbb{N}$ such that no member of $\mathcal{I}$ contains infinitely many intervals $[n_i, n_{i+1})$. For any $m \in \mathbb{N}$, define

$$A_m := \{x \in T\colon (\exists k \in \mathbb{N})(n_k > m \ \& \ (sx)\restriction_{[n_k, n_{k+1})} \leq \alpha \ \& \ (sx)\restriction_{[n_{k+1}, n_{k+2})} \geq \beta)\}.$$

(Here $(sx)\restriction_{[n_k, n_{k+1})} \leq \alpha$ means that $(sx)_i \leq \alpha$ for each $i \in [n_k, n_{k+1})$.)

Note that $\bigcap_{m \in \mathbb{N}} A_m \subset T \setminus S_s^{\mathcal{I}}$. Indeed, if $x \in \bigcap_{m \in \mathbb{N}} A_m$, each of the sets $\{n \in \mathbb{N}\colon (sx)_n \leq \alpha\}$ and $\{n \in \mathbb{N}\colon (sx)_n \geq \beta\}$ contains infinitely many intervals of the form $[n_k, n_{k+1})$, hence it does not belong to $\mathcal{I}$. Thus sx is not $\mathcal{I}$-convergent. Consequently, it suffices to show that every set A_m is comeager in T.

Fix $m \in \mathbb{N}$. We will prove that every open set U from a standard countable base of topology in T (inherited from $\{0, 1\}^{\mathbb{N}}$) contains an open subset included in A_m. This will demonstrate that A_m contains a dense G_δ set, hence it is comeager.

So, consider a basic open set

$$U := T \cap \left\{x \in \{0, 1\}^{\mathbb{N}}\colon x \text{ extends } (x_1, \dots, x_d)\right\}.$$

Without loss of generality, we may assume that the sequence $\overline{x} := (x_1, \ldots, x_d)$ contains at least m ones (in fact, we may assume that every member of the base satisfies this property). Let the number of ones in $\overline{x}$ be equal to $t \geq m$. Put $k := \min\{i \in \mathbb{N}\colon n_i > t\}$. We can extend $\overline{x}$ to a sequence $(x_1, \ldots, x_q)$ in which there are precisely $n_k - 1$ ones and $x_q := 1$. Then proceed as follows. Since a is a limit point of the sequence s, find inductively indices

$$i_{n_k} < i_{n_k+1} < \cdots < i_{n_{k+1}-1},$$

greater than d, for which the respective terms of the sequence s are $\leq \alpha$. Let x_i, for these indices i, be equal to 1, and consider the next extension of $\overline{x}$ to a sequence

$$(x_1, \ldots, x_q, \ldots x_{i_{n_k}}, \ldots, x_{i_{n_{k+1}-1}})$$

where the remaining terms are filled up by zeros.

In the same manner, since b is a limit number of s, we can find the next indices

$$i_{n_{k+1}} < i_{n_{k+1}+1} \cdots < i_{n_{k+2}-1},$$

for which the respective terms of the sequence s are $\geq \beta$. Let x_i, for these indices i, be equal to 1, and consider the final extension of $\overline{x}$ to a sequence

$$\overline{\overline{x}} := (x_1, \ldots, x_q, \ldots x_{i_{n_k}}, \ldots, x_{i_{n_{k+1}-1}}, x_{i_{n_{k+1}}} \ldots, x_{i_{n_{k+2}-1}})$$

where the remaining terms are filled up by zeros. Let

$$V := T \cap \left\{ x \in \{0,1\}^{\mathbb{N}} \colon x \text{ extends } \overline{\overline{x}} \right\}.$$

Then V is an open subset of U. Moreover, $V \subset A_m$ since, if $x \in V$, then conditions $n_k > m$, $(sx) \restriction_{[n_k, n_{k+1})} \leq \alpha$ and $(sx) \restriction_{[n_{k+1}, n_{k+2})} \geq \beta$ are fulfilled.

The proof for $R_s^{\mathcal{I}}$ is similar. For any $m \in \mathbb{N}$, let

$$A_m := \left\{ p \in P \colon (\exists k \in \mathbb{N})(n_k > m \ \ \& \ \ (s_{p(\cdot)} \restriction_{[n_k, n_{k+1})} \leq \alpha \ \ \& \ \ s_{p(\cdot)} \restriction_{[n_{k+1}, n_{k+2})} \geq \beta)\right\}.$$

We show that every set A_m is comeager in P. So, fix $m \in \mathbb{N}$ and consider a basic open set

$$U := P \cap \left\{ p \in \mathbb{N}^{\mathbb{N}} \colon p \text{ extends } \overline{x} \right\}$$

where $\overline{x} := (x_1, \ldots, x_d)$ is a sequence with distinct terms. We may assume that $d \geq m$. Pick the smallest $k \in \mathbb{N}$ such that $n_k > d$. Then we can choose distinct numbers

$$p(d+1), \ldots, p(n_k), \ldots, p(n_{k+1} - 1), p(n_{k+1}), \ldots, p(n_{k+2} - 1)$$

taken from $\mathbb{N} \setminus \{x_1, \ldots, x_d\}$ and fulfilling the following conditions:

- $s_{p(i)} \leq \alpha$ for $i = n_k, n_k + 1, \ldots, n_{k+1} - 1$;
- $s_{p(i)} \geq \beta$ for $i = n_{k+1}, n_{k+1} + 1, \ldots, n_{k+2} - 1$.

Put

$$\overline{\overline{x}} := (x_1, \ldots, x_d, \ldots p(n_k), \ldots, p(n_{k+1} - 1), p(n_{k+1}) \ldots, p(n_{k+2} - 1)$$

and

$$V := P \cap \left\{ \ p \in \mathbb{N}^{\mathbb{N}} \colon x \text{ extends } \overline{\overline{x}} \right\}.$$

Then V is open and included in $A_m \cap U$. This implies that A_m is comeager in P, and so is $A := \bigcap_{m \in \mathbb{N}} A_m$. Hence $R_s^{\mathcal{I}}$ included in $P \setminus A$ is meager. $\square$

Theorem 2.1 generalizes results of [16, Theorems 3.1, 5.1(b)] dealing with a special case when $\mathcal{I} := \mathcal{I}_d$. Note that, in [16], a different (but equivalent) approach was proposed to code subsequences (instead of $x \in \{0,1\}^{\mathbb{N}}$, binary expansions of reals $x \in (0,1]$ are used).

Let us show some consequences of Theorem 2.1. We will formulate several equivalent conditions for the usual convergence of a sequence in the language of the Baire category of ideal convergent subsequences and rearrangements.

Corollary 2.2. *Let $s = (s_n)$ be a sequence of reals and let $\mathcal{I}$ be an ideal on $\mathbb{N}$ with the Baire property. The following conditions are equivalent:*

(i) *s is convergent;*

(ii) *$S_s^{\mathcal{I}} = T$;*

(iii) *$S_s^{\mathcal{I}}$ is comeager in T;*

(iv) *$S_s^{\mathcal{I}}$ is nonmeager in T.*

Proof. To show (i)⇒(iii) recall that, if (i) holds, say s is convergent to $t \in \mathbb{R}$, then all subsequences of s are convergent to t, and also $\mathcal{I}$-convergent to t. Hence $S_s^{\mathcal{I}} = T$ and so, (ii) is true. Implications (ii)⇒(iii) and (iii)⇒(iv) are trivial, and Theorem 2.1 yields implication (iv)⇒(i). □

Analogously, we obtain

Corollary 2.3. *Let $s = (s_n)$ be a sequence of reals and let $\mathcal{I}$ be an ideal on $\mathbb{N}$ with the Baire property. The following conditions are equivalent:*

(i) *s is convergent;*

(ii) *$R_s^{\mathcal{I}} = P$;*

(iii) *$R_s^{\mathcal{I}}$ is comeager in P;*

(iv) *$R_s^{\mathcal{I}}$ is nonmeager in P.*

Observe that, if $\mathcal{I}$ is a maximal ideal on $\mathbb{N}$, the assertion of Theorem 2.1 is false. Indeed, it is known that every bounded sequence of reals is $\mathcal{I}$-convergent (cf. [14, Lemma 5.2]). Hence it suffices to consider a bounded divergent sequence s and then $S_s^{\mathcal{I}} = T$, $R_s^{\mathcal{I}} = P$. Note that maximal ideals do not have the Baire property (cf. [13, 8.50]).

3. Some results in the measure case

It is natural to ask whether a measure counterpart of Theorem 2.1 is true in the case of the set $S_s^{\mathcal{I}}$. Namely, consider the uniform probability measure on $\{0,1\}$ and let μ denote the respective product measure on $\{0,1\}^{\mathbb{N}}$ which sometimes is called *Lebesgue measure on* $\{0,1\}^{\mathbb{N}}$ (cf. [19, Example 3.4.10]). In fact, μ is strictly associated with linear Lebesgue measure on $[0,1]$ (when one uses the Cantor continuous function from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$). Then, by measurable subsets of $\{0,1\}^{N}$ we mean sets belonging to the μ-completion of the respective product σ-algebra on $\{0,1\}^{\mathbb{N}}$. Of course, T as a cocountable subset of $\{0,1\}$ is of full μ-measure (that is $\mu(T) = 1$). If we treat $\{0,1\}$ as the group $\mathbb{Z}_2$, the space $\{0,1\}^{\mathbb{N}}$ can be treated as the compact metric group $(\mathbb{Z}_2)^{\mathbb{N}}$, and then μ is the respective Haar measure.

We propose a preliminary observation (Proposition 3.1) which shows a dichotomy for $S_s^{\mathcal{I}}$, provided that $\mathcal{I}$ is an analytic or a coanalytic ideal on $\mathbb{N}$ (thus it is a special measurable subset of $\{0,1\}^{\mathbb{N}}$).

We need one more notion. Given an ideal $\mathcal{I}$ on $\mathbb{N}$, we say that a sequence (s_n) of reals satisfies *$\mathcal{I}$-Cauchy condition* (cf. [6]) if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\{n \in \mathbb{N} \colon |s_n - s_N| \geq \varepsilon\} \in \mathcal{I}$. In a similar way, one can define $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy condition in a metric space.

Recall that (cf. [6]), in a complete metric space, the classes of $\mathcal{I}$-convergent and of $\mathcal{I}$-Cauchy sequences are equal. So, the sets $S_s^{\mathcal{I}}$ can be expressed in the form

$$S_s^{\mathcal{I}} = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \{x \in T \colon \{i \in \mathbb{N} \colon |(sx)_i - (sx)_N| \geq \varepsilon\} \in \mathcal{I}\}; \tag{1}$$

Proposition 3.1. *Let $s \in \mathbb{R}^{\mathbb{N}}$ and let $\mathcal{I}$ be an ideal on $\mathbb{N}$ which is an analytic or a coanalytic subset of $\{0,1\}^{\mathbb{N}}$. Then $S_s^{\mathcal{I}}$ is analytic or coanalytic in T, and either of measure 0 or 1.*

Proof. A set $E \subset \{0,1\}^{\mathbb{N}}$ (see [17]) is called a *tail set* if, whenever $x \in E$ and $y \in \{0,1\}^{\mathbb{N}}$ differs from x in a finite number of coordinates, then $y \in E$. By [17, Theorem 21.3] if a tail set is measurable then it is either of measure 0 or 1. Observe that $S_s^{\mathcal{I}}$, treated as a subset of $\{0,1\}^{\mathbb{N}}$, is a tail set. If we show that $S_s^{\mathcal{I}}$ is measurable, we will get the second assertion (note that $\{0,1\}^{\mathbb{N}} \setminus T$ is countable).

We consider the expression (1) with ε taken from the set $\mathbb{Q}_+$ of positive rationals. For $\varepsilon \in \mathbb{Q}_+$ and $N \in \mathbb{N}$ define $f_{\varepsilon,N} \colon T \to \{0,1\}^{\mathbb{N}}$ as the sequence of characteristic functions

$$f_{\varepsilon,N}(x) := (\chi_{\{i\in\mathbb{N}\colon |(sx)_i-(sx)_N|\geq\varepsilon\}}(j))_{j\in\mathbb{N}}, \quad x \in T.$$

Then $S_s^{\mathcal{I}} = \bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcup_{N\in\mathbb{N}} f_{\varepsilon,N}^{-1}[\mathcal{I}]$. To finish the proof we need to show that every function $f_{\varepsilon,N}$ is continuous since then (by the respective properties of analytic and coanalytic sets; see [19]) the set $S_s^{\mathcal{I}}$ will be analytic or coanalytic, hence it will be measurable.

Let $f := f_{\varepsilon,N}$. It suffices to prove that every coordinate $f_j := \chi_{\{i\in\mathbb{N}\colon |(sx)_i-(sx)_N|\geq\varepsilon\}}(j)$ of f is continuous. Fix $j \in \mathbb{N}$. Let $x_n \in T$ for $n \in \mathbb{N}$ and assume that $x_n \to x \in T$. We will show that $f_j(x_n) \to f_j(x)$. Let for example $f_j(x) = 1$ (the second case is analogous). Hence $|(sx)_j - (sx)_N| \geq \varepsilon$. Denote by k the maximum of the Nth and the jth positions of ones in the sequence x. Since $x_n \to x$, there is $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, the first k terms of x_n and x are equal. That is why $|(sx_n)_j - (sx_n)_N| \geq \varepsilon$, and finally $f_j(x_n) = f_j(x) = 1$ for any $n \geq n_0$. □

Remark 1. In an analogous way, one can prove that, if $\mathcal{I}$ is analytic or coanalytic, then for any sequence $s \in \mathbb{R}^{\mathbb{N}}$ and any $t \in \mathbb{R}$, the set

$$\left\{x \in T \colon \mathcal{I}\text{-}\lim_i (sx)_i = t\right\} = \bigcap_{\varepsilon\in\mathbb{Q}^+} \{x \in T \colon \{i \in \mathbb{N} \colon |(sx)_i - t| \geq \varepsilon\} \in \mathcal{I}\}$$

is analytic or coanalytic, and either of measure 0 or 1 (observe that this is a tail set).

In [15], [16], the measure of $S_s^{\mathcal{I}}$ was investigated, in the case $\mathcal{I} := \mathcal{I}_d$. Note that $\mathcal{I}_d$ is an $F_{\sigma\delta}$ subset of $\{0,1\}^{\mathbb{N}}$ (cf. [8]). We summarize the respective results of [15] and [16] in the following theorem.

Theorem 3.2. ([15],[16]) For $\mathcal{I} := \mathcal{I}_d$ and a sequence $s = (s_n)$ of reals, the following conditions are equivalent:

(i) s is $\mathcal{I}$-convergent;
(ii) $\mu(\{x \in T \colon \mathcal{I}\text{-}\lim_n (sx)_n = t\}) = 1$ for some $t \in \mathbb{R}$;
(iii) $\mu(S_s^{\mathcal{I}}) = 1$.

Proof. The equivalence (i)$\Leftrightarrow$(ii) was shown in [15, Theorem 3], with $\mathcal{I}\text{-}\lim_n s_n = t$ in (i). Implication (ii)$\Rightarrow$(iii) is obvious. By [16, Theorem 3.5], if s is not $\mathcal{I}$-convergent, then $\mu(T \setminus S_s^{\mathcal{I}}) = 1$. Consequently, if $\mu(S_s^{\mathcal{I}}) = 1$, then s is $\mathcal{I}$-convergent. This yields (iii)$\Rightarrow$(i). □

Theorem 3.2 shows that the measure analogue of Theorem 2.1 is false. Indeed, it suffices to consider a divergent sequence s which is $\mathcal{I}_d$-convergent and then $\mu(S_s^{\mathcal{I}_d}) = 1$ by Theorem 3.2.

Now, we are going to extend Theorem 3.2 to a wider class of ideals. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A function $f\colon \mathbb{N} \to \mathbb{N}$ is called *bi-$\mathcal{I}$-invariant* if $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{I}$ for every $A \subset \mathbb{N}$ (if "$\Rightarrow$" is true, we say that f is $\mathcal{I}$-invariant). We will need the following fact (cf. [3, Proposition 24]).

Proposition 3.3 ([3]). *Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be bi-$\mathcal{I}$-invariant injections such that $f[\mathbb{N}] \cap g[\mathbb{N}] = \emptyset$ and $f[\mathbb{N}] \cup g[\mathbb{N}] = \mathbb{N}$. Then, for any sequence (s_n) of reals and a point $t \in \mathbb{R}$, we have*

$$\mathcal{I}\text{-}\lim_n s_n = t \Leftrightarrow (\mathcal{I}\text{-}\lim_n s_{f(n)} = t \text{ and } \mathcal{I}\text{-}\lim_n s_{g(n)} = t).$$

For a 0-1 sequence $x \in T$ let $\{n_1 < n_2 < \dots\} := \{k \in \mathbb{N}\colon x_k = 1\}$. Define $f_x\colon \mathbb{N} \to \mathbb{N}$ by $f_x(k) = n_k$, $k \in \mathbb{N}$, and let $T_{\mathcal{I}} := \{x \in T\colon f_x \text{ is bi-}\mathcal{I}\text{-invariant}\}$. We will say that an ideal $\mathcal{I}$ on $\mathbb{N}$ *has property* (G) if $\mu(T_{\mathcal{I}}) = 1$.

Theorem 3.4. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. For a sequence $s = (s_n)$ of reals, consider the following conditions:*

(i) *s is $\mathcal{I}$-convergent;*
(ii) *$\mu(\{x \in T\colon \mathcal{I}\text{-}\lim_n (sx)_n = t\}) = 1$ for some $t \in \mathbb{R}$;*
(iii) *$\mu(S_s^{\mathcal{I}}) = 1$.*

Then (i) *and* (ii) *are equivalent, provided that $\mathcal{I}$ has property (G). Implication* (ii)$\Rightarrow$(iii) *is always true. Implication* (iii)$\Rightarrow$(i) *holds provided that $\mathcal{I}$ is analytic or coanalytic and $\mathcal{I}$ has property (G). Consequently, under these two assumptions on $\mathcal{I}$, conditions* (i),(ii),(iii) *are equivalent.*

Proof. Assume that $\mathcal{I}$ has property (G). To show (i)$\Rightarrow$(ii), assume that $\mathcal{I}\text{-}\lim_n s_n = t \in \mathbb{R}$. Let $x \in T_{\mathcal{I}}$. Hence f_x is bi-$\mathcal{I}$-invariant. Let $\{n_1 < n_2 < \dots\} := \{k \in \mathbb{N}\colon x_k = 1\}$. Fix $\varepsilon > 0$. Then

$$\{i \in \mathbb{N}\colon |(sx)_i - t| \geq \varepsilon\} = \{i \in \mathbb{N}\colon |s_{n_i} - t| \geq \varepsilon\} = f_x^{-1}[\{i \in \mathbb{N}\colon |s_i - t| \geq \varepsilon\}].$$

We have $\{i \in \mathbb{N}\colon |s_i - t| \geq \varepsilon\} \in \mathcal{I}$. Thus $f_x^{-1}[\{i \in \mathbb{N}\colon |s_i - t| \geq \varepsilon\}] \in \mathcal{I}$ by the bi-$\mathcal{I}$-invariance of f_x. Consequently, sx is $\mathcal{I}$-convergent to t. By property (G) and the choice of x, we obtain (ii).

To show (ii)$\Rightarrow$(i), let $t \in \mathbb{R}$ be such that $\mu(B) = 1$ where $B := \{x \in T\colon \mathcal{I}\text{-}\lim_n (sx)_n = t\}$. If $x \in T$, denote by $x^c := \mathbf{1} - x$, the converse of x in the group $(\mathbb{Z}_2)^{\mathbb{N}}$, where $\mathbf{1} := (1, 1, \dots)$. By property (G) we have $\mu(T_{\mathcal{I}}) = 1$, hence $\mu(\mathbf{1} - T_{\mathcal{I}}) = 1$ since μ is the Haar measure. Pick $x \in B \cap T_{\mathcal{I}} \cap (\mathbf{1} - T_{\mathcal{I}})$. Let $\{n_1 < n_2 < \dots\} := \{k \in \mathbb{N}\colon x_k = 1\} = f_x[\mathbb{N}]$ and $\{m_1 < m_2 < \dots\} := \{k \in \mathbb{N}\colon x_k^c = 1\} = f_{x^c}[\mathbb{N}]$. Then $f_x[\mathbb{N}] \cap f_{x^c}[\mathbb{N}] = \emptyset$, $f_x[\mathbb{N}] \cup f_{x^c}[\mathbb{N}] = \mathbb{N}$ and

$$\mathcal{I}\text{-}\lim_n (sx)_n = t \Leftrightarrow \mathcal{I}\text{-}\lim_n s_{n_i} = t \Leftrightarrow \mathcal{I}\text{-}\lim_n s_{f_x(n)} = t,$$

$$\mathcal{I}\text{-}\lim_n (sx^c)_n = t \Leftrightarrow \mathcal{I}\text{-}\lim_n s_{m_i} = t \Leftrightarrow \mathcal{I}\text{-}\lim_n s_{f_{x^c}(n)} = t.$$

Hence, using the bi-invariance of f_x and f_{x^c}, we obtain $\mathcal{I}\text{-}\lim_n s_n = t$ by Proposition 3.3.

Implication (ii)$\Rightarrow$(iii) is obvious (no extra assumption on $\mathcal{I}$ is needed).

Now, let $\mathcal{I}$ be analytic or coanalytic, with property (G). To prove (iii)$\Rightarrow$(i), we follow some ideas from [16, Theorem 3.5]. Assume that $\mu(S_s^{\mathcal{I}})=1$. Pick x_0 from the set $H := T_{\mathcal{I}}\cap(\mathbf{1}-T_{\mathcal{I}})\cap S_s^{\mathcal{I}}\cap(\mathbf{1}-S_s^{\mathcal{I}})$ which is of μ-measure 1. Then f_{x_0} and $f_{x_0^c}$ are bi-$\mathcal{I}$-invariant, and the subsequences sx_0 and sx_0^c are $\mathcal{I}$-convergent to t and t', respectively. If $t=t'$, then (i) follows from Proposition 3.3. Assume that $t\neq t'$. Fix any $x\in H$ and suppose that sx is $\mathcal{I}$-convergent to some $u\notin\{t,t'\}$. Take $\varepsilon>0$ such that the sets $(t-\varepsilon,t+\varepsilon)$, $(t'-\varepsilon,t'+\varepsilon)$ and $(u-\varepsilon,u+\varepsilon)$ are pairwise disjoint. Then $\{n\colon |(sx_0)_n-t|\geq\varepsilon\}\in\mathcal{I}$ and since f_{x_0} is $\mathcal{I}$-invariant, we have $\{f_{x_0}(n)\colon |(sx_0)_n-t|\geq\varepsilon\}\in\mathcal{I}$. Thus $\{f_{x_0}(n)\colon |(sx_0)_n-u|<\varepsilon\}\in\mathcal{I}$. Similarly $\{f_{x_0^c}(n)\colon |(sx_0^c)_n-u|<\varepsilon\}\in\mathcal{I}$. Since $\{n\colon |(sx)_n-u|\geq\varepsilon\}\in\mathcal{I}$, then $\{f_x(n)\colon |(sx)_n-u|\geq\varepsilon\}\in\mathcal{I}$. Moreover, $\{f_x(n)\colon |(sx)_n-u|<\varepsilon\}\subset\{f_{x_0}(n)\colon |(sx_0)_n-u|<\varepsilon\}\cup\{f_{x_0^c}(n)\colon |(sx_0^c)_n-u|<\varepsilon\}$. Therefore f_x maps $\mathbb{N}$ onto a set from $\mathcal{I}$ which contradicts the bi-$\mathcal{I}$-invariance of f_x. Hence, for μ-almost every $x\in T$, the sequence sx is $\mathcal{I}$-convergent either to t or to t'.

Since $\mathcal{I}$ is analytic or coanalytic, by Remark 1 we infer that $\{x\in T\colon (sx)$ is $\mathcal{I}$-convergent to $t\}$ and $\{x\in T\colon (sx)$ is $\mathcal{I}$-convergent to $t'\}$ are 0-1 sets with respect to μ. However, their union is of full μ-measure. Hence one of them has full μ-measure. Now, we can use (ii)$\Rightarrow$(i). □

Let $NR := \{x\in T\colon d(\{n\in\mathbb{N}\colon x_n=1\})=1/2\}$. Recall that $\mu(NR)=1$, by the Borel theorem on normal numbers. If $\mathcal{I}=\mathcal{I}_d$, the set $\{x\in T\colon f_x$ is bi-$\mathcal{I}$-invariant$\}$ contains NR and therefore it is of full μ-measure. (This was used in the proofs of [15, Theorem 3] and [16, Theorem 3.5].) We will present two classes of ideals $\mathcal{I}$ that fulfil the inclusion $NR\subset T_{\mathcal{I}}$. This will witness that the class of ideals with property (G) is quite rich.

Let (a_n) be a sequence of nonnegative real numbers with $\sum_{n\in\mathbb{N}}a_n=\infty$. By $\mathcal{I}_{(a_n)}$ denote the ideal of all sets $A\subset\mathbb{N}$ with $\sum_{n\in A}a_n<\infty$. This is called the *summable ideal* associated with (a_n); cf. [8]. Another class consists of density-like ideals considered in [2]. For any $\alpha\in(0,1]$ let

$$\mathcal{I}\langle\alpha\rangle := \left\{A\subset\mathbb{N} : \limsup_{n\to\infty}\frac{|A\cap[1,n]|}{n^\alpha}=0\right\}.$$

Note that $\mathcal{I}\langle 1\rangle=\mathcal{I}_d$.

Proposition 3.5. (I) *Let (a_n) be a nonincreasing sequence of positive reals with $\sum_{n\in\mathbb{N}}a_n=\infty$. Assume that there is $C>0$ such that $a_n/a_{2n}\leq C$ for each $n\in\mathbb{N}$. Then $\mathcal{I}_{(a_n)}$ has property (G).*

(II) *The ideal $\mathcal{I}\langle\alpha\rangle$ has property (G) for each $\alpha\in(0,1]$.*

Proof. Fix $x\in NR$. We claim that $f_x(n)\leq 4n$ for all but finitely many n's. Suppose not. Then there are infinitely many n's such that $f_x(n)>4n$. Fix $n_0\in\mathbb{N}$ such that

$$\frac{|\{i\leq n\colon x_i=1\}|}{n}>\frac{1}{3}$$

for every $n\geq n_0$. Take $n_1\geq n_0$ with $f_x(n_1)>4n_1$. Since f_x is increasing, $f_x(n)>4n_1$ for every $n\geq n_1$. Thus

$$\frac{|\{i\leq 4n_1\colon x_i=1\}|}{4n_1}\leq\frac{1}{4},$$

a contradiction. From now on, we fix $m_0\in\mathbb{N}$ such that $f_x(n)\leq 4n$ for all $n\geq m_0$. We will prove that f_x is bi-$\mathcal{I}$-invariant for ideals $\mathcal{I}$ considered in statements (I) and (II).

(I) Let $A \subset \mathbb{N}$. Since (a_n) is nonincreasing, $A \in \mathcal{I}_{(a_n)}$ implies $f_x[A] \in \mathcal{I}_{(a_n)}$. Assume that $f_x[A] \in \mathcal{I}_{(a_n)}$. Then

$$\sum_{n\in A, n\geq m_0} a_n \leq C^2 \sum_{n\in A, n\geq m_0} a_{4n} \leq \sum_{n\in A, n\geq m_0} a_{f_x(n)} < \infty$$

which shows that $A \in \mathcal{I}_{(a_n)}$.

(II) Note that each increasing injection from $\mathbb{N}$ to $\mathbb{N}$ is $\mathcal{I}\langle\alpha\rangle$-invariant. Let $A \subset \mathbb{N}$ and assume that $f_x[A] \in \mathcal{I}\langle\alpha\rangle$. Fix $\varepsilon > 0$ and find $k_0 \geq m_0$ such that for all $k \geq k_0$ we have

$$\frac{|\{f_x(n) \leq k \colon n \in A\}|}{k^\alpha} < \frac{\varepsilon}{4^\alpha}.$$

Then for all $k \geq k_0$ we have

$$\frac{|\{n \leq k \colon n \in A\}|}{k^\alpha} \leq 4^\alpha \frac{|\{f_x(n) \leq 4k \colon n \in A\}|}{(4k)^\alpha} \leq \varepsilon$$

which shows that $A \in \mathcal{I}\langle\alpha\rangle$. □

Now, we present an example of summable ideal and a sequence of reals for which implication (i)⇒(iii) in Theorem 3.4 is false.

Example 2. Define $\mathcal{I}$ as follows: for $A \subset \mathbb{N}$ let $A \in \mathcal{I} \Leftrightarrow A \cap (2\mathbb{N}+1) \in \mathrm{Fin}$. Note that $\mathcal{I}$ is a summable ideal; namely $\mathcal{I} = \mathcal{I}_{(a_n)}$ where $a_{2n} := 0$ and $a_{2n+1} := 1$ for all $n \in \mathbb{N}$. By the Borel-Cantelli lemma, a sequence $x \in T$ contains infinitely many blocks $(1,0,1,1,0,1)$ with probability 1 (i.e. with μ-measure 1). Define

$$E := \{x \in T \colon x_k = x_{k+2} = x_{k+3} = x_{k+5} = 1, x_{k+1} = x_{k+4} = 0, \text{ for infinitely many } k \in \mathbb{N}\}.$$

Let $s := (a_n)$. Clearly, s is $\mathcal{I}$-convergent to 1. Observe that every $\mathcal{I}$-convergent subsequence of s with odd indices is convergent in the usual way. We will show that sx is not $\mathcal{I}$-convergent for every $x \in E$.

Fix $x \in E$. By K denote the infinite set of indices k such that $(x_k, \dots, x_{k+5}) = (1,0,1,1,0,1)$. Let $\{k \in \mathbb{N} \colon x_k = 1\} = \{n_1 < n_2 < n_3 < \dots\}$. To prove that sx is not $\mathcal{I}$-convergent, we need to show that the sequence $(a_{n_{2k+1}})_{k\in\mathbb{N}}$ contains infinitely many zeros and infinitely many ones. This holds since, for infinitely many even k's, the indices n_k are odd, and for infinitely many odd k's, the indices n_k are also odd. Ideed, fix $k \in K$. Consider the following four cases.

Case 1. k is even and n_k is even. Then $k+2$ is even and n_{k+2} is odd, and $k+5$ is odd and n_{k+5} is odd.

Case 2. k is even and n_k is odd. Then $k+3$ is odd and n_{k+3} is odd.

Case 3. k is odd and n_k is even. Then $k+2$ is odd and n_{k+2} is odd, and $k+5$ is even and n_{k+5} is odd.

Case 4. k is odd and n_k is odd. Then $k+3$ is even and n_{k+2} is odd.

Since sx is not $\mathcal{I}$-convergent for every $x \in E$, we have $\mu(S_c^{\mathcal{I}}) = 0$.

Note that summable ideals and ideals of the form $\mathcal{I}\langle\alpha\rangle$, $\alpha \in (0,1]$, are analytic P-ideals. An ideal $\mathcal{I}$ on $\mathbb{N}$ is called a *P-ideal* if for any sequence (A_n) of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for every $n \in \mathbb{N}$. We leave unsolved the problem how to describe the class of analytic P-ideals for which the equivalence of conditions (i),(ii),(iii) in Theorem 3.2 is true. Recall an important theorem of Solecki [18]: $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ if and only if $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ on $\mathbb{N}$ (where $\mathrm{Exh}(\varphi) := \{A \subset \mathbb{N}\colon \lim_n \varphi(A \setminus [1, n]) = 0\}$).

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