# DICHOTOMIES FOR LORENTZ SPACES

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Abstract. Assume that $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ are Lorentz spaces. This note is devoted to answering the question what is the size of the set

$$E := \{(f_1, ..., f_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n} : f_1 \cdots f_n \in L^{p,q}\}.$$

We prove the following dichotomy: either $E = \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ or $E$ is $\sigma$-porous in $\mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$, provided $\frac{1}{p} \neq \frac{1}{p_1} + ... + \frac{1}{p_n}$. In general case we obtain that either $E = \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ or $E$ is meager.

## 1. introduction

This article is aimed at studying a size of the set of all tuples $(f_1, ..., f_n)$ from the product of $n$ Lorentz spaces such that their product $f_1 \cdots f_n$ is in another Lorentz space. This study is originated from the paper of Balcerzak and Wachowicz [BW] where it was proved that the set

$$\{(f, g) \in \mathbf{L}^1[0, 1] \times \mathbf{L}^1[0, 1] : fg \in \mathbf{L}^1[0, 1]\}$$

is a meager subset of the product $\mathbf{L}^1[0, 1] \times \mathbf{L}^1[0, 1]$. It has been generalized by Jachymski [J] who proved the following condition are equivalent when $p \geq 1$ and $(X, \Sigma, \mu)$ is any $\sigma$-finite measure space:

(i) $\{(f, g) \in \mathbf{L}^p(X) \times \mathbf{L}^p(X) : fg \in \mathbf{L}^p(X)\}$ is meager;
(ii) $\{(f, g) \in \mathbf{L}^p(X) \times \mathbf{L}^p(X) : fg \in \mathbf{L}^p(X)\} \neq \mathbf{L}^p(X) \times \mathbf{L}^p(X)$;
(iii) $\inf\{\mu(A) : \mu(A) > 0\} = 0$.

This result has been further generalized by Głąb and Strobin [GS]. Let $(X, \Sigma, \mu)$ be any measure space, $p_1, ..., p_n, p \in (0, \infty]$ and $\min\{p_1, ..., p_n\} < \infty$ (i.e., at least one of the $p_i$'s is finite). They proved that the following conditions are equivalent (we define $\frac{1}{\infty} := 0$):

(i) $\{(f_1, ..., f_n) \in \mathbf{L}^{p_1}(X) \times ... \times \mathbf{L}^{p_n}(X) : f_1 \cdots f_n \in \mathbf{L}^p(X)\}$ is $\sigma$-$\alpha$-lower porous for some $\alpha > 0$;
(ii) $\{(f_1, ..., f_n) \in \mathbf{L}^{p_1}(X) \times ... \times \mathbf{L}^{p_n}(X) : f_1 \cdots f_n \in \mathbf{L}^p(X)\} \neq \mathbf{L}^{p_1}(X) \times ... \times \mathbf{L}^{p_n}(X)$;
(iii) One of the following conditions holds:
  * $\frac{1}{p_1} + ... + \frac{1}{p_1} > \frac{1}{p}$ and $\inf\{\mu(A) : \mu(A) > 0\} = 0$;
  * $\frac{1}{p_1} + ... + \frac{1}{p_1} < \frac{1}{p}$ and $\sup\{\mu(A) : \mu(A) < \infty\} = \infty$.

In this paper we will strengthen the above result. The main idea is that if $p \in (0, \infty]$, then $\mathbf{L}^p(X)$ is a particular example of the so called Lorentz space. Hence it is interesting if the above result

1991 *Mathematics Subject Classification.* Primary: 46E30 Secondary: 46G99.

*Key words and phrases.* Lorentz space, $\mathbf{L}^p$-space, porosity, Baire Category.

can be extended by considering Lorentz spaces instead of $\mathbf{L}^p(X)$ spaces. Our main reference will be Grafakos' monograph [G].

## 2. Notation and basic facts

Let $X$ be a metric space. $B(x,R)$ stands for the ball with a radius $R$ centered at a point $x$. Let $\alpha \in (0,1]$. We say that $M \subset X$ is *$\alpha$-lower porous* [Z1], if

$$\forall_{x\in M}\ \liminf_{R\to 0^+} \frac{\gamma(x,M,R)}{R} \geq \frac{\alpha}{2},$$

where

$$\gamma(x,M,R) = \sup\{r \geq 0 : \exists z \in X\ B(z,r) \subset B(x,R)\backslash M\}.$$

Clearly, $M$ is $\alpha$-lower porous iff

$$\forall_{x\in M}\ \forall_{\beta\in\left(0,\frac{\alpha}{2}\right)}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{z\in X}\ B(z,\beta R) \subset B(x,R)\backslash M.$$

The set is *$\sigma$-$\alpha$-lower porous* if it is a countable union of $\alpha$-lower porous sets. Note that a $\sigma$-$\alpha$-lower porous set is meager, and the notion of $\sigma$-lower porosity is essentially stronger than that of meagerness. Note that the sets investigated in this paper will be $\alpha$-porous in some stronger sense, namely,

$$\forall_{x\in X}\ \forall_{\beta\in\left(0,\frac{\alpha}{2}\right)}\ \forall_{R>0}\ \exists_{z\in X}\ B(z,\beta R) \subset B(x,R)\backslash M.$$

However, we do not want to define any new notion of porosity, so in the formulations of theorems we will deal only with $\alpha$-lower porosity. For more information on porosity, we refer the reader to survey papers [Z1] and [Z2].

Assume that $(X,\Sigma,\mu)$ is a measure space, and let $p,q \in (0,\infty]$ be such that if $p=\infty$, then also $q=\infty$. A Lorentz space $\mathbf{L}^{p,q}(X,\Sigma,\mu)$ ($\mathbf{L}^{p,q}$ in short) is the space of all measurable functions (more formally, of all equivalence classes of measurable functions equal $\mu$-a.e.) with a finite quasinorm given by

$$\| f \|_{p,q} := \begin{cases} \left(\int_0^\infty p\mu(\{x : |f(x)| > \lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda\right)^{\frac{1}{q}}, & \text{if } q<\infty; \\ \sup_{\lambda>0} \lambda\mu(\{x : |f(x)| > \lambda\})^{\frac{1}{p}}, & \text{if } p<\infty \text{ and } q=\infty; \\ \operatorname{supess}|f|, & \text{if } p=q=\infty. \end{cases}$$

Note that the presented definition of quasinorm on $\mathbf{L}^{p,q}$ is equivalent to the original one (cf. [G]) and that $\mathbf{L}^{p,q}$ is linear space, but the quasinorm on $\mathbf{L}^{p,q}$ is not usually a norm since the triangle inequality does not hold for all quasinorms $\|\cdot\|_{p,q}$. However, it is always $c_{p,q}$-subadditive for some $c_{p,q}>0$.

If $p \in (0,\infty]$, then $\|\cdot\|_p$ denotes the standard $\mathbf{L}^p$-norm:

$$\|\cdot\|_p := \begin{cases} \int_X |f|^p\, d\mu, & \text{if } p\in(0,1); \\ \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}}, & \text{if } p\in[1,\infty); \\ \operatorname{supess}|f|, & \text{if } p=\infty. \end{cases}$$

The following basic facts about Lorentz spaces are known and can be easily found in [G]. In the sequel we will use them, sometimes without emphasizing it. If $(X, \Sigma, \mu)$ is a measure space, then we define $\Sigma_+ := \{A \in \Sigma : 0 < \mu(A) < \infty\}$, and if $A \in \Sigma$, then by $\chi_A$ we denote the characteristic function of the set $A$.

**Proposition 2.1.** *Assume that $(X, \Sigma, \mu)$ is a measure space. The following conditions hold:*

(i) *For any $p \in (0, \infty]$,*

$$\| \cdot \|_{p,p} = \begin{cases} \| \cdot \|_p, & \text{if } p \in [1, \infty]; \\ (\| \cdot \|_p)^{\frac{1}{p}}, & \text{if } p \in (0, 1). \end{cases}$$

(ii) *If $p \in (0, \infty)$ and $q \leq q' \leq \infty$, then $\mathbf{L}^{p,q} \subset \mathbf{L}^{p,q'}$. In particular, for every $f \in \mathbf{L}^{p,q}$ there exists $M > 0$ such that $\mu(\{x : |f(x)| > \lambda\}) \leq M\lambda^{-p}$ for every $\lambda > 0$;*

(iii) *If $p, r \in (0, \infty)$ and $q \in (0, \infty]$, then $\| \, |f|^r \, \|_{p,q} = (\| f \|_{pr,qr})^r$ for every $f \in \mathbf{L}^{p,q}$;*

(iv) *If $p \in (0, \infty)$ and $q \in (0, \infty]$, then there exists $D_{p,q} > 0$ such that for every $A \in \Sigma_+$, $\| \chi_A \|_{p,q} = D_{p,q}\mu(A)^{\frac{1}{p}}$.*

(v) *If $\mathbf{L}^{p,q}$ is any Lorentz space, then for every $f \in \mathbf{L}^{p,q}$ and measurable function $g$, if for every $x \in X$, $|g(x)| \leq |f(x)|$, then $g \in \mathbf{L}^{p,q}$ and $\| g \|_{p,q} \leq \| f \|_{p,q}$;*

(vi) *If $p < \infty$ and $\Sigma_+ = \emptyset$, then for every $q \in (0, \infty]$, $\mathbf{L}^{p,q} = \{0\}$;*

(vii) *If $\mathbf{L}^{p,q}$ is a Lorentz space, $f \in \mathbf{L}^{p,q}$ and $\alpha$ is any real, then $\| \alpha f \|_{p,q} = |\alpha| \, \| f \|_{p,q}$.*

By (i), every $\mathbf{L}^p$ space is a particular example of Lorentz space.

Assume now that $\Sigma_+ \neq \emptyset$. It is well known, that every Lorentz space $L^{p,q}$ is metrizable by some metric $d_{p,q}$, which satisfies the following condition ($K_{p,q}, r_{p,q} > 0$ are some constants which depend on $p$ and $q$):

$$(\| f - g \|_{p,q})^{r_{p,q}} \leq d_{p,q}(f, g) \leq K_{p,q} \left(\| f - g \|_{p,q}\right)^{r_{p,q}} . \tag{1}$$

Indeed, if $p = q$, then we set $d_{p,p}(f, g) := \| f - g \|_p$, and in the other cases we can make use of [G, Exercises 1.4.3 and 1.1.12].

Since each Lorentz space is quasi-Banach (i.e., every Cauchy sequence with respect to a quasinorm, is convergent), (1) implies that the metrics $d_{p,q}$ are complete. Hence we can consider $\mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ as a complete metric space with the supremum metric $d_{\max}$:

$$d_{\max}((f_1, ..., f_n), (g_1, ..., g_n)) := \max\{d_{p_1,g_1}(f_1, g_1), ..., d_{p_n,g_n}(f_n, g_n)\}.$$

In particular, if $p_i = q_i$, $i = 1, ..., n$, then we have:

$$d_{\max}((f_1, ..., f_n), (g_1, ..., g_n)) := \max\{\| f_1 - g_1 \|_{p_1}, ..., \| f_n - g_n \|_{p_n}\}.$$

## 3. Results

We will assume that we work with some fixed measure space $(X,\Sigma,\mu)$. Again, we denote $\Sigma_+ := \{A\in\Sigma : 0<\mu(A)<\infty\}$.

If $n\in\mathbb{N}$ (we allow $n$ to be 1) and $\mathbf{L}^{p,q},\mathbf{L}^{p_1,q_1},...,\mathbf{L}^{p_n,q_n}$ are Lorentz spaces, then we define the set

$$E_{p,q}^{(p_1,q_1,...,p_n,q_n)} := \{(f_1,...,f_n)\in \mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n} : f_1\cdots f_n\in\mathbf{L}^{p,q}\}.$$

We also set

$$E_p^{(p_1,...,p_n)} := E_{p,p}^{(p_1,p_1,...,p_n,p_n)}$$

We will first deal with the trivial case: when $\Sigma_+=\emptyset$ or $\min\{p_1,...,p_n\}=\infty$.

**Proposition 3.1.** *Let $n\in\mathbb{N}$ and $\mathbf{L}^{p,q},\mathbf{L}^{p_1,q_1},...,\mathbf{L}^{p_n,q_n}$ be Lorentz spaces. If $\Sigma_+=\emptyset$* **or** *$\min\{p_1,...,p_n\}=\infty$, then the following conditions are equivalent:*

(i) *$E_{p,q}^{(p_1,q_1,...,p_n,q_n)}$ is 1-lower porous subset of $\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$;*
(ii) *$E_{p,q}^{(p_1,q_1,...,p_n,q_n)}\neq\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$;*
(iii) *$\mu(X)=\infty$ and $\min\{p_1,...,p_n\}=\infty$ and $p<\infty$.*

*Proof.* We first show the implication (i)⇒(ii). Let $f_1,...,f_n\in\mathbf{L}^\infty$, $R>0$ and $\alpha\in\left(0,\frac{1}{2}\right)$. For any $i=1,...,n$, set

$$\tilde{f}_i(x) := \begin{cases} f_i(x)+\frac{1}{2}R, & \text{if } f_i(x)\geq 0;\\ f_i(x)-\frac{1}{2}R, & \text{if } f_i(x)<0.\end{cases}$$

Then $\| f-\tilde{f}_i\|_\infty=\frac{1}{2}R$ for each $i=1,...,n$. Now let $a_1,...,a_n$ be such that $\| a_i-\tilde{f}_i\|_\infty<\alpha R$, $i=1,...,n$. Then for every $i=1,...,n$ and for $\mu$-almost every $x\in X$, we have $|a_i(x)|\geq\left(\frac{1}{2}-\alpha\right)R$, so for $\mu$-almost every $x\in X$,

$$|a_1(x)\cdots a_n(x)|\geq\left(R\left(\frac{1}{2}-\alpha\right)\right)^n.$$

Hence $\| a_1\cdots a_n\|_{p,q}=\infty$.

The implication (ii)⇒(iii) is trivial.

Assume now that $\mu(X)<\infty$ or $\min\{p_1,...,p_n\}<\infty$ or $p<\infty$. Since $\Sigma_+=\emptyset$ or $\min\{p_1,...,p_n\}=\infty$, one of the following conditions holds:

(a1) $\mu(X)=0$;
(a2) $0<\mu(X)<\infty$ and $\min\{p_1,...,p_n\}=\infty$;
(a3) $\min\{p_1,...,p_n\}<\infty$ and $\Sigma_+=\emptyset$;
(a4) $p=\infty$ and $\min\{p_1,...,p_n\}=\infty$.

In each case, the equality $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}=\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ is obvious. Hence we get (iii)⇒(i). □

Now we will deal with more complicated cases.

In the first result we state the condition, under which $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}=\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$.

**Proposition 3.2.** *Let $n \in \mathbb{N}$ and $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ be Lorentz spaces, and assume that $\min\{p_1, ..., p_n\} < \infty$ and $\Sigma_+ \neq \emptyset$. If one of the following conditions holds:*

(a) $\inf\{\mu(A) : A \in \Sigma_+\} > 0$ *and* $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{p}$;

(b) $\sup\{\mu(A) : A \in \Sigma_+\} < \infty$ *and* $\frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{p}$.

*then* $E_{p,q}^{(p_1,q_1,...,p_n,q_n)} = \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$.

Before we start the proof of Proposition 3.2, we need some lemmas. The first one is an easy consequence of [G, Exercise 1.1.15] and we skip its proof.

**Lemma 3.3.** *Let $n \in \mathbb{N}$ and $p, p_1, ..., p_n \in (0, \infty]$ be with $\frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{p}$. Then $E_{p,\infty}^{(p_1,\infty,...,p_n,\infty)} = \mathbf{L}^{p_1,\infty} \times ... \times \mathbf{L}^{p_n,\infty}$.*

**Lemma 3.4.** *Assume that $\Sigma_+ \neq \emptyset$. The following conditions hold:*

(I) *If $\inf\{\mu(A) : A \in \Sigma_+\} > 0$, then every element of any Lorentz space is $\mu$-a.e. bounded;*

(II) *If $\sup\{\mu(A) : A \in \Sigma_+\} < \infty$, then there exists $A \in \Sigma_+$ such that for every $p \in (0, \infty)$ and $q \in (0, \infty]$, the projection $f \to f_{|A}$ is an isometry between Lorentz spaces $\mathbf{L}^{p,q}(X, \Sigma, \mu)$ and $\mathbf{L}^{p,q}\left(A, \Sigma_{|A}, \mu_{|A}\right)$.*

*Proof.* We first show (I). Let $\mathbf{L}^{p,q}$ be a Lorentz space. If $p = q = \infty$, the thesis holds by the definition of $\|\cdot\|_\infty$. Hence assume that $p < \infty$ and let $f \in L^{p,q}$. By Proposition 2.1 (ii), there exists $M > 0$ such that for every $\lambda > 0$, we have

$$\mu(\{x : |f(x)| > \lambda\}) \leq M\lambda^{-p}.$$

Hence $\lim_{\lambda\to\infty} \mu(\{x : |f(x)| > \lambda\}) = 0$, so for some $\lambda_0 > 0$, we get $\mu(\{x : |f(x)| > \lambda_0\}) = 0$, which proves (I).

Now we show (II). Set $K := \sup\{\mu(A) : A \in \Sigma_+\} < \infty$. For every $n \in \mathbb{N}$, there is $A_n \in \Sigma_+$ with $K \geq \mu(A_n) \geq K - \frac{1}{n}$. Set $A = \bigcup_{n\in\mathbb{N}} A_n$. Then $\mu(A) = K$ and for any measurable $D \subset X \setminus A$, we have that either $\mu(D) = 0$ or $\mu(D) = \infty$. Hence if $f$ is an element of any Lorentz space $\mathbf{L}^{p,q}(X, \Sigma, \mu)$, then $\mu(\{x \in X \setminus A : |f(x)| > 0\}) = 0$. This easily gives (II). □

The following lemma seems to be known, but we will give a proof.

**Lemma 3.5.** *Assume that $\Sigma_+ \neq \emptyset$ and $p, p', q \in (0, \infty]$. If one of the following conditions holds:*

(i) $\inf\{\mu(A) : A \in \Sigma_+\} > 0$ *and* $\frac{1}{p} < \frac{1}{p'}$;

(ii) $\sup\{\mu(A) : A \in \Sigma_+\} < \infty$ *and* $0 < \frac{1}{p'} < \frac{1}{p}$,

*then* $\mathbf{L}^{p',\infty} \subset \mathbf{L}^{p,q}$.

*Proof.* Assume that (i) holds and let $f \in \mathbf{L}^{p',\infty}$. By Lemma 3.4 (I), there exists $S < \infty$ such that $|f(x)| < S$ for $\mu$-almost $x \in X$. Hence if $\mathbf{L}^{p,q} = \mathbf{L}^\infty$, then obviously $f \in \mathbf{L}^{p,q}$. Thus assume that

$p < \infty$. By Proposition 2.1 (ii), we may also assume that $q < \infty$.
Since $f \in \mathbf{L}^{p',\infty}$, there exists $M > 0$ such that for every $\lambda > 0$,

$$\mu(\{x : |f(x)| > \lambda\}) \leq M\lambda^{-p'},$$

and since $|f(x)| < S$ for $\mu$-almost $x \in X$, for any $\lambda \geq S$ we have $\mu(\{x : |f(x)| > \lambda\}) = 0$. Hence

$$\int_0^\infty \mu\left(\{x : |f(x)| > \lambda\}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda = \int_0^S \mu\left(\{x : |f(x)| > \lambda\}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda \leq$$

$$\int_0^S M^{\frac{q}{p}} \lambda^{\frac{-p'q}{p}} \lambda^{q-1} d\mu = \int_0^S M^{\frac{q}{p}} \lambda^{q\left(1-\frac{p'}{p}\right)-1} d\mu < \infty,$$

so $f \in \mathbf{L}^{p,q}$ and the result follows.
Assume now that condition (ii) holds. By Proposition 2.1 (ii), we may assume that $q < \infty$. By Lemma 3.4 (II), we may assume that $K := \mu(X) < \infty$. Now let $f \in \mathbf{L}^{p',\infty}$. Then there exists $M > 0$ such that for every $\lambda > 0$,

$$\mu(\{x : |f(x)| > \lambda\}) \leq M\lambda^{-p'}.$$

Hence we have

$$\int_0^\infty \mu\left(\{x : |f(x)| > \lambda\}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda =$$

$$\int_0^1 \mu\left(\{x : |f(x)| > \lambda\}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda + \int_1^\infty \mu\left(\{x : |f(x)| > \lambda\}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda \leq$$

$$\int_0^1 K^{\frac{q}{p}} \lambda^{q-1} d\lambda + \int_1^\infty M^{\frac{q}{p}} \lambda^{\frac{-p'q}{p}} \lambda^{q-1} d\mu =$$

$$\int_0^1 K^{\frac{q}{p}} \lambda^{q-1} d\lambda + \int_1^\infty M^{\frac{q}{p}} \lambda^{q\left(1-\frac{p'}{p}\right)-1} d\mu < \infty,$$

so $f \in \mathbf{L}^{p,q}$.

□

We are ready to give a proof of Proposition 3.2

*Proof.* (of Proposition 3.2) Let $p' \in (0, \infty)$ be such that $\frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{p'}$, and let $(f_1, ..., f_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$. By Proposition 2.1 (ii), $(f_1, ..., f_n) \in \mathbf{L}^{p_1,\infty} \times ... \times \mathbf{L}^{p_n,\infty}$, and by Lemma 3.3, we get $f_1 \cdots f_n \in \mathbf{L}^{p',\infty}$. Hence and by Lemma 3.5, $f_1 \cdots f_n \in \mathbf{L}^{p,q}$, so the result follows. □

The next theorem deals with the case when $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}$ is small. Recall that we consider the product of Lorentz spaces as a metric space with a metric $d_{\max}$ defined in the previous section, and in the case of $\mathbf{L}^{p_1} \times ... \times \mathbf{L}^{p_n}$, we have $d_{\max}((f_1, ..., f_n), (g_1, ..., g_n)) = \max\{\| f_1 - g_1 \|_{p_1}, ..., \| f_n - g_n \|_{p_n}\}$

**Theorem 3.6.** *Let $n \in \mathbb{N}$ and $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ be Lorentz spaces. Assume that* $\min\{p_1, ..., p_n\} < \infty$ *and $\Sigma_+ \neq \emptyset$. If one of the following conditions holds*

(i) $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{p}$ *and* $\inf\{\mu(A) : A \in \Sigma_+\} = 0$*;*
(ii) $\frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{p}$ *and* $\sup\{\mu(A) : A \in \Sigma_+\} = \infty$*,*

*then the set* $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}$ *is* $\sigma$*-*$\alpha$*-lower porous in* $\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ *for some* $\alpha>0$.
*If additionally* $q=p$ *and* $q_i=p_i$ *for* $i=1,...,n$, *then we can take* $\alpha=\frac{2}{m+1}$, *where* $m$ *is the number of* $i's$ *for which* $p_i<\infty$.

Note that the last statement of the above result, together with part (i) of Proposition 3.1, give the thesis of the main result of [GS] ([GS, Theorem 6]). Before we prove the result, we will present some lemmas. Note that they are refinements of [GS, Lemmas 4 and 5].

**Lemma 3.7.** *Let* $n\in\mathbb{N}$, $A,A_1,...,A_n$ *be measurable sets and* $s_1,...,s_n\geq 0$ *be such that* $\sum_{i=1}^n s_i\leq 1$. *If* $A_i\subset A$ *and* $\mu(A_i)>(1-s_i)\mu(A)$ *for any* $i=1,...,n$, *then*

$$\mu\left(\bigcap_{i=1}^n A_i\right)>0.$$

*Proof.* Using the induction principle, it is easy to show that

$$\mu\left(\bigcap_{i=1}^k A_i\right)>\left(1-\sum_{i=1}^k s_i\right)\mu(A)\quad\text{for any } k=1,...,n.$$

In particular, for $k=n$, we get that $\mu(\bigcap_{i=1}^n A_i)>0$. □

Recall (cf. Proposition 2.1 (iv)) that if $q\in(0,\infty]$ and $p<\infty$, then for some $D_{p,q}>0$, $\parallel\chi_A\parallel_{p,q}=D_{p,q}\mu(A)^{\frac{1}{p}}$ for every $A\in\Sigma_+$.

**Lemma 3.8.** *Assume that* $n\in\mathbb{N}$, $p_1,...,p_n\in(0,\infty)$, $q_1,...,q_n\in(0,\infty]$ *and let* $A\in\Sigma_+$. *If* $f_1,...,f_n,g_1,...,g_n$ *are such that* $|g_i(x)|\geq 1$ *for* $x\in A$ *and* $i=1,...,n$, *and* $\parallel(f_i-g_i)\chi_A\parallel_{p_i,q_i}\leq s_i$, $i=1,...,n$ *for some* $s_1,...,s_n$, *then*

$$\parallel f_1\cdots f_n\chi_A\parallel_{p,q}\quad\geq D_{p,q}C^n\left(\mu(A)-\sum_{i=1}^n\left(\frac{s_i}{D_{p_i,q_i}(1-C)}\right)^{p_i}\right)^{\frac{1}{q}}$$

*for any* $C\in(0,1)$, $p\in(0,\infty)$ *and* $q\in(0,\infty]$, *provided that* $\mu(A)\geq\sum_{i=1}^n\left(\frac{s_i}{D_{p_i,q_i}(1-C)}\right)^{p_i}$.

*Proof.* For simplicity, let $D,D_1,...,D_n$ stand for $D_{p,q},D_{p_1,q_1},...,D_{p_n,q_n}$, respectively. For $i=1,...,n$, we define the sets $A_i:=\{x\in A:|f_i(x)|<C|g_i(x)|\}$. Now let $i=1,...n$. Then for every $x\in A_i$,

$$|f_i(x)-g_i(x)|\geq|g_i(x)|-|f_i(x)|\geq(1-C)|g_i(x)|\geq 1-C.$$

Hence

$$s_i\geq\parallel(f_i-g_i)\chi_A\parallel_{p_i,q_i}\geq\parallel(1-C)\chi_{A_i}\parallel_{p_i,q_i}=D_i(1-C)\mu(A_i)^{\frac{1}{p_i}},$$

and therefore

$$\mu(A_i)\leq\left(\frac{s_i}{D_i(1-C)}\right)^{p_i}.$$

By combining this with the fact that for any $x\in A\setminus A_i$, $|f_i(x)|\geq C$, we obtain

$$\parallel f_1\cdots f_n\chi_A\parallel_{p,q}\geq\parallel f_1\cdots f_n\chi_{A\setminus\bigcup_{1=1}^n A_i}\parallel_{p,q}\geq\parallel C^n\chi_{A\setminus\bigcup_{1=1}^n A_i}\parallel_{p,q}\geq$$

$$DC^n\left(\mu(A)-\sum_{i=1}^n\mu(A_i)\right)^{\frac{1}{p}}\geq DC^n\left(\mu(A)-\sum_{i=1}^n\left(\frac{s_i}{D_i(1-C)}\right)^{p_i}\right)^{\frac{1}{p}}.$$

□

The following lemma is crucial for the proof of Theorem 3.6. If $u>0$, then we set

$$E_u:=\{(f_1,...,f_n)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}:\| f_1\cdots f_n\|_{p,q}\leq u\}. \tag{2}$$

If $f\in\mathbf{L}^{p,q}$ and $r>0$, then we denote $B_{p,q}(f,r):=\{g\in\mathbf{L}^{p,q}:\| f-g\|_{p,q}<r\}$.

**Lemma 3.9.** *Let $n\in\mathbb{N}$ and $\mathbf{L}^{p,q},\mathbf{L}^{p_1,q_1},...,\mathbf{L}^{p_n,q_n}$ be Lorentz spaces and let the assumptions of Theorem 3.6 be satisfied. Assume that for some $m\geq 1$ and $k\geq 0$ with $m+k=n$, we have $p_1,...,p_m<\infty$ and $p_{m+1}=...=p_{m+k}=\infty$. Let $r_i\in(0,\infty)$ and $\delta_i\in\left(0,\frac{1}{2}\right)$, $i=1,...,n$, be such that*

$$\sum_{i=1}^m\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{p_i}{r_i}}<1\quad\textit{and}\quad r_{m+1}=...=r_n=1.$$

*For every $u>0$, $R>0$ and $(f_1,...,f_n)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$, there exists $\left(\tilde f_1,...,\tilde f_n\right)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ such that*

$$\left(\| f_i-\tilde f_i\|_{p_i,q_i}\right)^{r_i}=(1-\delta_i)R\quad\textit{for every } i=1,...,n,$$

*and*

$$B_{p_1,q_1}\left(\tilde f_1,(\delta_iR)^{\frac{1}{r_1}}\right)\times...\times B_{p_n,q_n}\left(\tilde f_n,(\delta_nR)^{\frac{1}{r_n}}\right)\cap E_u=\emptyset.$$

*Proof.* For simplicity, we will write $\|\cdot\|_i$ and $D_i$ instead of $\|\cdot\|_{p_i,q_i}$ and $D_{p_i,q_i}$ for $i=1,...,n$, respectively, and $\|\cdot\|$ and $D$ instead of $\|\cdot\|_{p,q}$ and $D_{p,q}$, respectively.

Let $u>0$. As we assumed, $p_i<\infty$ for $i=1,...,m$ and $p_i=\infty$ for $i=m+1,...,m+k$, for some $m\geq 1$ and $k\geq 0$ with $m+k=n$.

Let $(f_1,...,f_m,g_1,...,g_k)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$, $R>0$. We can choose $C\in(0,1)$ such that

$$\sum_{i=1}^m\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i}<1. \tag{3}$$

Now set:

$$G:=\frac{\prod_{i=1}^m(R(1-\delta_i))^{\frac{1}{r_i}}}{\prod_{i=1}^m D_i}, \tag{4}$$

$$H:=\left(1-\sum_{i=1}^m\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i}\right)^{\frac{1}{p}}. \tag{5}$$

By our assumptions ((i) or (ii)), there is a set $A\in\Sigma_+$ such that
if $p<\infty$, then

$$\left(\prod_{i=m+1}^n R\,(1-2\delta_i)\right)GDC^mH\mu(A)^{\left(\frac{1}{p}-\frac{1}{p_1}-...-\frac{1}{p_m}\right)}>u \tag{6}$$

and if $p = \infty$, then

$$\left(\prod_{i=m+1}^{n} R\left(1-2\delta_i\right)\right) GC^m \mu(A)^{-\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m}\right)} > u. \tag{7}$$

Note that in the case when $\frac{1}{p}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}<0$, we take the set $A$ with an appropriate small positive measure and in the case when $\frac{1}{p}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}>0$, we take the set $A$ with an appropriate large finite measure.

Next, let $M_1, \ldots, M_m$ be such that for $i = 1, \ldots, m$,

$$(\| M_i \chi_A \|_i)^{r_i} = (1-\delta_i)R. \tag{8}$$

Then for every $i = 1, \ldots, m$,

$$\mu(A) = \left(\frac{((1-\delta_i)R)^{\frac{1}{r_i}}}{M_i D_i}\right)^{p_i}. \tag{9}$$

Now, let us define $\tilde{f}_1, \ldots, \tilde{f}_m, \tilde{g}_1, \ldots, \tilde{g}_k$ as follows.

For $i = 1, \ldots, m$, we set

$$\tilde{f}_i(x) := \begin{cases} f_i(x) + M_i, & x \in A \text{ and } f_i(x) \geq 0; \\ f_i(x) - M_i, & x \in A \text{ and } f_i(x) < 0; \\ f_i(x), & x \notin A, \end{cases}$$

and for $i = 1, \ldots, k$, we set

$$\tilde{g}_i(x) := \begin{cases} g_i(x) + (1-\delta_{m+i})R, & \text{if } g_i(x) \geq 0; \\ g_i(x) - (1-\delta_{m+i})R, & \text{if } g_i(x) < 0. \end{cases}$$

Using (8), we obtain for every $i = 1, \ldots, m$:

$$\left(\| \tilde{f}_i - f_i \|_i\right)^{r_i} = (\| M_i \chi_A \|_i)^{r_i} \overset{(8)}{=} (1-\delta_i)R$$

and similarly for every $i = 1, \ldots, k$ (recall that $r_i = 1$ for $i = m+1, \ldots, n$),

$$||\tilde{g}_i - g_i||_i = (1-\delta_{m+i})R.$$

Now let

$$(a_1, \ldots, a_m, b_1, \ldots, b_k) \in B_{p_1,q_1}\left(\tilde{f}_1, (\delta_1 R)^{\frac{1}{r_1}}\right) \times \ldots \times B_{p_n,q_n}\left(\tilde{g}_k, (\delta_n R)^{\frac{1}{r_n}}\right).$$

Clearly, since for every $i = 1, \ldots, k$ and $x \in X$, $|\tilde{g}_i(x)| \geq (1-\delta_{m+i})R$, then for $\mu$-almost every $x \in X$,

$$|b_i(x)| \geq (1-2\delta_{m+i})R. \tag{10}$$

Consider two cases:

*Case 1.* $p < \infty$. For any $i = 1, \ldots, m$, we have

$$(\delta_i R)^{\frac{1}{r_i}} \geq \| a_i - \tilde{f}_i \|_i \geq \left|\left|\left(a_i - \tilde{f}_i\right) \chi_A\right|\right|_i = M_i \left|\left|\left(\frac{a_i}{M_i} - \frac{\tilde{f}_i}{M_i}\right) \chi_A\right|\right|_i \tag{11}$$

and also, by (9),

$$\left(\frac{(R\delta_i)^{\frac{1}{r_i}}}{M_iD_i(1-C)}\right)^{p_i} = \mu(A)\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i} \tag{12}$$

and

$$M_i\mu(A)^{\frac{1}{p_i}}D_i = ((1-\delta_i)R)^{\frac{1}{r_i}}. \tag{13}$$

Hence, by (4) – (6), (10) – (13) and Lemma 3.8 (used for $g_i := \frac{\tilde{f}_i}{M_i}$, $f_i := \frac{a_i}{M_i}$ and $s_i := \frac{(\delta_i R)^{\frac{1}{r_i}}}{M_i}$), we obtain the following

$$\| a_1\cdots a_m\cdot b_1\cdots b_k \| \geq \| a_1\cdots a_m\cdot b_1\cdots b_k\chi_A \| \overset{(10)}{\geq}$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right)\| a_1\cdots a_m\chi_A \| =$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right) M_1\cdots M_m\left\|\frac{a_1}{M_1}\cdots\frac{a_m}{M_m}\chi_A\right\| \overset{L\ 3.8,(11)}{\geq}$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right) M_1\cdots M_m DC^m\left(\mu(A)-\sum_{i=1}^{m}\left(\frac{(\delta_iR)^{\frac{1}{r_i}}}{M_iD_i(1-C)}\right)^{p_i}\right)^{\frac{1}{p}} \overset{(12)}{=}$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right) M_1\cdots M_m DC^m\left(\mu(A)-\sum_{i=1}^{m}\mu(A)\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i}\right)^{\frac{1}{p}} \overset{(5)}{=}$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right)\frac{\prod_{i=1}^m M_i\mu(A)^{\frac{1}{p_i}}D_i}{\prod_{i=1}^m D_i}DC^m\mu(A)^{\left(\frac{1}{p}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}\right)}H \overset{(4),(13)}{=}$$
$$\left(\prod_{i=m+1}^{n} R\,(1-2\delta_i)\right) GDC^m\mu(A)^{\left(\frac{1}{p}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}\right)}H \overset{(6)}{>} u.$$

Hence $(a_1,...,b_k)\notin E_u$.

*Case 2.* $p=\infty$. This case is possible only if $\inf\{\mu(E): E\in\Sigma_+\}=0$. For any $i=1,...,m$, we define:

$$A_i^1 := \{x\in A: |a_i(x)|\geq CM_i\},\quad A_i^2 := A\setminus A_i^1.$$

Then for every $i=1,...,m$, we have

$$(\delta_iR)^{\frac{1}{r_i}} > \| a_i-\tilde{f}_i \|_i \geq \left\|\left(a_i-\tilde{f}_i\right)\chi_{A_i^2}\right\|_i \geq \| (M_i(1-C))\chi_{A_i^2} \|_i = D_iM_i(1-C)\mu(A_i^2)^{\frac{1}{p_i}}.$$

Hence by (12) (which works in this case), for every $i=1,...,m$,

$$\mu(A_i^2) < \left(\frac{(\delta_iR)^{\frac{1}{r_i}}}{D_iM_i(1-C)}\right)^{p_i} \overset{(12)}{=} \mu(A)\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i}.$$

Then for each $i=1,...,m$,

$$\mu(A_i^1) = \mu(A)-\mu(A_i^2) > \mu(A)\left(1-\left(\left(\frac{\delta_i}{1-\delta_i}\right)^{\frac{1}{r_i}}\frac{1}{(1-C)}\right)^{p_i}\right).$$

By Lemma 3.7 and (3), we obtain that $\mu\left(A_1^1\cap...\cap A_m^1\right)>0$. Also, for $\mu$-almost every $x\in A_1^1\cap...\cap A_m^1$, we make use of (4), (7), (10) and (13) (which, clearly, works in this case) to obtain

$$|a_1(x)\cdots a_m(x)\cdot b_1(x)\cdots b_k(x)| \overset{(10)}{\geq} C^m M_1\cdots M_m\left(\prod_{i=m+1}^{n} R\left(1-2\delta_i\right)\right) \overset{(4),(13)}{=}$$

$$=\left(\prod_{i=m+1}^{n} R\left(1-2\delta_i\right)\right) C^m G\mu(A)^{-\left(\frac{1}{p_1}+...+\frac{1}{p_m}\right)} \overset{(7)}{>} u.$$

Hence

$$||a_1\cdots a_m\cdot b_1\cdots b_k||>u.$$

This ends the proof. □

We are ready to give a proof of Theorem 3.6

*Proof.* (of Theorem 3.6) For simplicity, we will write $\|\cdot\|_i$ and $d_i$, $i=1,...,n$ instead of $\|\cdot\|_{p_i,q_i}$ and $d_{p_i,q_i}$, $i=1,...,n$, respectively.

By (1), for each $i=1,...,n$, there exist $K_i,r_i>0$ such that for every $f,g\in\mathbf{L}^{p_i,q_i}$,

$$(\| f-g\|_i)^{r_i}\leq d_i(f,g)\leq K_i\left(\| f-g\|_i\right)^{r_i}, \tag{14}$$

and if $p_i=\infty$, then $K_i=r_i=1$.

Since $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}=\bigcup_{u\in\mathbb{N}}E_u$ (where each $E_u$ is defined as in (2)), we have to show that there exists $\alpha>0$ such that for each $u>0$, the set $E_u$ is $\alpha$-lower porous.

Let $u>0$. Without loss of generality, we assume that $p_i<\infty$ for $i=1,...,m$ and $p_i=\infty$ for $i=m+1,...,m+k$, where $m+k=n$, $m\geq 1$ and $k\geq 0$.

Now let $K:=\max\{K_1,...,K_n\}$ and $\lambda>0$ be such that

$$\sum_{i=1}^{m}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{p_i}{r_i}}=1.$$

It is easy to see that $\lambda\leq\frac{1}{2}$. Take $\delta\in(0,\lambda)$. Then

$$\sum_{i=1}^{m}\left(\frac{\delta}{1-\delta}\right)^{\frac{p_i}{r_i}}<1.$$

Now take $(f_1,...,f_n)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ and $R>0$. Let $\left(\tilde{f}_1,...,\tilde{f}_n\right)\in\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ be as in Lemma 3.9, chosen for $r_1,...,r_n$, $u$, $(f_1,...,f_n)$, $R':=\frac{R}{K}$ and $\delta_i:=\delta$, $i=1,...,n$.

By (14) and Lemma 3.9, for every $i=1,...,n$, we have that

$$d_i\left(f_i,\tilde{f}_i\right)\overset{(14)}{\leq} K\left(\| f_i-\tilde{f}_i\|_i\right)^{r_i}\overset{L\,3.9}{=}K(1-\delta)R'=(1-\delta)R,$$

so

$$B_d\left(\left(\tilde{f}_1,...,\tilde{f}_n\right),\frac{\delta R}{K}\right)\subset B_d\left(\left(\tilde{f}_1,...,\tilde{f}_n\right),\delta R\right)\subset B_d\left((f_1,...,f_n),R\right), \tag{15}$$

where $B_d((\cdot,...,\cdot),\cdot)$ denotes an open ball in $(\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n},d_{\max})$.
On the other hand, by (14), for any $i=1,...,n$ and any $a_i\in\mathbf{L}^{p_i,q_i}$, if $d_i\left(\tilde{f}_i,a_i\right)<\frac{\delta R}{K}$, then also $\left(\|\tilde{f}_i-a_i\|_i\right)^{r_i}<\frac{\delta R}{K}=\delta R'$. Hence and by Lemma 3.9, $(a_1,...,a_n)\notin E_u$, which shows that

$$B_d\left(\left(\tilde{f}_1,...,\tilde{f}_n\right),\frac{\delta R}{K}\right)\cap E_u=\emptyset. \tag{16}$$

By (15) and (16), $E_u$ is $\frac{2\lambda}{K}$-lower porous.
Now we will prove the last statement of the thesis. We may assume that $p_1,...,p_j<1$, $p_{j+1},...,p_m\in[1,\infty)$ and $p_{m+1}=...=p_n=\infty$. For $i=1,...,j$, set $r_i:=p_i$, and for $i=j+1,...,n$, set $r_i:=1$. Now if $\lambda:=\frac{1}{m+1}$, then $\frac{\lambda}{1-\lambda}=\frac{1}{m}$, so

$$\sum_{i=1}^m\left(\frac{\lambda}{1-\lambda}\right)^{\frac{p_i}{r_i}}\leq\sum_{i=1}^m\frac{1}{m}=1.$$

Proceeding similarly as above we get that each $E_u$ is $\frac{2\frac{1}{m}}{1}$-lower porous (note that here $K=1$). □

Finally, Propositions 3.1, 3.2 and Theorem 3.6 imply the following partial dichotomy:

**Corollary 3.10.** *Let $n\in\mathbb{N}$ and $\mathbf{L}^{p,q},\mathbf{L}^{p_1,q_1},...,\mathbf{L}^{p_n,q_n}$ be Lorentz spaces such that if $p<\infty$, then $\frac{1}{p}\neq\frac{1}{p_1}+...+\frac{1}{p_n}$. Then the following conditions are equivalent:*

(a) *the set $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}$ is $\sigma$-$\alpha$-lower porous in $\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ for some $\alpha>0$;*
(b) *$E_{p,q}^{(p_1,q_1,...,p_n,q_n)}\neq\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$;*
(c) *one of the conditions holds:*
  (i) *$\Sigma_+\neq\emptyset$ and $\inf\{\mu(A):A\in\Sigma_+\}=0$ and $\frac{1}{p_1}+...+\frac{1}{p_n}>\frac{1}{p}$;*
  (ii) *$\Sigma_+\neq\emptyset$ and $\sup\{\mu(A):A\in\Sigma_+\}=\infty$ and $\frac{1}{p_1}+...+\frac{1}{p_n}<\frac{1}{p}$;*
  (iii) *$\mu(X)=\infty$ and $\min\{p_1,...,p_n\}=\infty$ and $p<\infty$.*

*Proof.* Implication (a)⇒(b) is trivial. Implication (c)⇒(a) follows from Proposition 3.1 and Theorem 3.6. We will prove implication (b)⇒(c). Assume that (i), (ii) and (iii) do not hold. Consider two cases:
*Case 1.* $\Sigma_+=\emptyset$ or $\min\{p_1,...,p_n\}=\infty$.
Then the equality $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}=\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ follows from negation of (iii) and Proposition 3.1.
*Case 2.* $\Sigma_+\neq\emptyset$ and $\min\{p_1,...,p_n\}<\infty$.
Then the negation of (i) and (ii) easily imply the assumptions of Proposition 3.2.
The result follows. □

Now we will deal with the case when $\frac{1}{p_1}+...+\frac{1}{p_n}=\frac{1}{p}$ and $\min\{p_1,...,p_n\}<\infty$.

**Proposition 3.11.** *Assume that $\frac{1}{p_1}+...+\frac{1}{p_n}=\frac{1}{p}$. Then the following conditions hold:*

(i) *for $q_1,...,q_n\in(0,\infty]$, $E_{p,\infty}^{(p_1,q_1,...,p_n,q_n)}=\mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$;*

(ii) *for every* $q_i \in (0, p_i]$, $i = 1, ..., n$ *and* $q \in [p, \infty]$, $E_{p,q}^{(p_1,q_1,...,p_n,q_n)} = \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$.

*Proof.* Part (i) follows from Lemma 3.3 and Proposition 2.1 (ii).
Part (ii) follows from a general version of the Hölder inequality [G, Exercise 1.1.2] and Proposition 2.1 (ii). □

Now we show that we can also have $E_{p,q}^{(p_1,q_1,...,p_n,q_n)} \neq \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$.

**Proposition 3.12.** *Let* $X = \mathbb{R}^k$ *and* $\mu$ *be the Lebesgue measure on* $X$, $p' < \infty$, $p = \frac{p'}{n}$, $q' \in (0, \infty]$ *and* $q < \frac{q'}{n}$. *Then* $E_{p,q}^{(p',q',...,p',q')} \neq \mathbf{L}^{p',q'} \times ... \times \mathbf{L}^{p',q'}$.

*Proof.* Let $t = \frac{q'}{n}$. By [G, Exercise 1.4.8], there is $f \in \mathbf{L}^{p,t} \setminus \mathbf{L}^{p,q}$. Clearly, we may assume that $f \geq 0$. By Proposition 2.1, $f^{\frac{1}{n}} \in \mathbf{L}^{pn,tn} = \mathbf{L}^{p',q'}$. Hence $(f^{\frac{1}{n}}, ..., f^{\frac{1}{n}}) \in \mathbf{L}^{p',q'} \times ... \times \mathbf{L}^{p',q'}$ and $f^{\frac{1}{n}} \cdots f^{\frac{1}{n}} = f \notin \mathbf{L}^{p,q}$. □

Now we show that the following dichotomy holds:

**Theorem 3.13.** *Let* $n \in \mathbb{N}$ *and* $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ *be Lorentz spaces with* $\frac{1}{p_1} + ... + \frac{1}{p_n} = \frac{1}{p}$. *Then the following conditions are equivalent:*

(i) $E_{p,q}^{(p_1,q_1,...,p_n,q_n)} \neq \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$;
(ii) $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}$ *is a meager subset of* $\mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$.

To prove the above fact, we need the following lemma. If $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ are Lorentz spaces such that $p, q < \infty$, then for every reals $v, u > 0$, we put

$$E_u^v := \left\{ (f_1, ..., f_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n} : \int_0^\infty \mu(\{x : |f_1(x) \cdots f_n(x)| > v\lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda < u \right\}. \tag{17}$$

**Lemma 3.14.** *Let* $n \in \mathbb{N}$ *and* $\mathbf{L}^{p,q}, \mathbf{L}^{p_1,q_1}, ..., \mathbf{L}^{p_n,q_n}$ *be Lorentz spaces such that* $q, p, p_1, ..., p_n \in (0, \infty)$. *If* $(h_1, ..., h_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ *is such that* $h_1 \cdots h_n \notin \mathbf{L}^{p,q}$, *then for every* $u, v > 0$, *there exists* $r > 0$ *such that*

$$B_{p_1,q_1}(h_1, r) \times ... \times B_{p_n,q_n}(h_n, r) \cap E_u^v = \emptyset.$$

*Proof.* Let $(h_1, ..., h_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ be such that $h_1 \cdots h_n \notin \mathbf{L}^{p,q}$. In particular, $\frac{h_1 \cdots h_n}{v 2^n} \notin \mathbf{L}^{p,q}$, so

$$\int_0^\infty \mu(\{x : |h_1(x) \cdots h_n(x)| > v 2^n \lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda = \infty. \tag{18}$$

Now for every $k \in \mathbb{N}$, set

$$A_k := \left\{ x \in X : k > |h_i(x)| > \frac{1}{k},\ i = 1, ..., n \right\}.$$

Since for every $i = 1, ..., n$, $p_i < \infty$, we have that $\mu(A_k) < \infty$ for every $k \in \mathbb{N}$. Now let $A := \bigcup_{k\in\mathbb{N}} A_k$. Then $A = \{x \in X : \infty > |h_1(x) \cdots h_n(x)| > 0\}$, so by (18) and a fact that for each $i = 1, ..., n$, $\mu\{x \in X : |h_i(x)| = \infty\} = 0$, we get

$$\int_0^\infty \mu(\{x \in A : |h_1(x) \cdots h_n(x)| > v2^n\lambda\})^{\frac{q}{p}} \lambda^{q-1}\, d\lambda = \infty.$$

Hence by the Lebesgue monotone convergence theorem, there exists $k > 0$ such that

$$\int_0^\infty \mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\})^{\frac{q}{p}} \lambda^{q-1}\, d\lambda > u. \tag{19}$$

Define

$$s_0 := \inf\{\lambda > 0 : \mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\}) = 0\}.$$

By (19), we get $s_0 > 0$, and since for $\lambda > \frac{k^n}{v2^n}$ we have $\mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\}) = 0$, we also have $s_0 < \infty$. Moreover, again by (19), we obtain

$$\int_0^{s_0} \mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda > u,$$

therefore for some $s \in (0, s_0)$,

$$\int_0^{s} \mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda > u.$$

By the definition of $s_0$, we have $\mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n s\}) > 0$. Hence and by the Lebesgue monotone convergence theorem, there exists $m > 0$ such that for every $\lambda \in (0, s]$, $\mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\}) > \frac{1}{m}$ and

$$\int_0^{s} \left(\mu(\{x \in A_k : |h_1(x) \cdots h_n(x)| > v2^n\lambda\}) - \frac{1}{m}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda > u. \tag{20}$$

Now set $r > 0$ such that

$$\sum_{i=1}^n \left(\frac{2rk}{D_i}\right)^{p_i} < \frac{1}{m}, \tag{21}$$

where $D_i := D_{p_i,q_i}$, $i = 1, ..., n$. Now let $(a_1, ..., a_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ be such that $\| h_i - a_i \|_{p_i,q_i} < r$ for every $i = 1, ..., n$. For every $i = 1, ..., n$, put

$$A^i := \left\{x \in A_k : |a_i(x)| \le \frac{1}{2}|h_i(x)|\right\}. \tag{22}$$

Then for every $i = 1, ..., n$, we have

$$r > \| h_i - a_i \|_{p_i,q_i} \ \ge\ \left\|\frac{1}{2}h_i\chi_{A^i}\right\|_{p_i,q_i} \ \ge\ \frac{1}{2k} D_i \mu(A^i)^{\frac{1}{p_i}},$$

so for every $i = 1, ..., n$, we have

$$\mu(A^i) < \left(\frac{2rk}{D_i}\right)^{p_i}. \tag{23}$$

Hence, by (21) – (23), for every $\lambda \geq 0$, we get

$$\mu(\{x \in X : |a_1(x)\cdots a_n(x)| > v\lambda\}) \geq \mu\left(\left\{x \in A_k \setminus \bigcup_{i=1}^{n} A^i : |a_1(x)\cdots a_n(x)| > v\lambda\right\}\right) \overset{(22)}{\geq}$$

$$\mu\left(\left\{x \in A_k \setminus \bigcup_{i=1}^{n} A^i : |h_1(x)\cdots h_n(x)| > v2^n\lambda\right\}\right) \overset{(21),(23)}{\geq} \mu(\{x \in A_k : |h_1(x)\cdots h_n(x)| > v2^n\lambda\}) - \frac{1}{m}.$$

Therefore by (20),

$$\int_0^\infty \mu(\{x \in X : |a_1(x)\cdots a_n(x)| > v\lambda\})^{\frac{q}{p}} \lambda^{q-1} d\lambda \geq$$

$$\int_0^s \left(\mu(\{x \in A_k : |h_1(x)\cdots h_n(x)| > v2^n\lambda\}) - \frac{1}{m}\right)^{\frac{q}{p}} \lambda^{q-1} d\lambda \overset{(20)}{>} u,$$

so $(a_1, ..., a_n) \notin E_u^v$. □

*Proof.* (Theorem 3.13) We only have to prove implication (i)⇒(ii). Hence assume (i). By Proposition 3.11 (i), we can assume that

$$\min\{p_1, ..., p_n\} < \infty \quad \text{and} \quad p, q < \infty.$$

Moreover, without loss of generality, we can assume that $p_1, ..., p_m < \infty$ and $p_{m+1} = ... = p_{m+k} = \infty$ for some $m \geq 1$ and $k \geq 0$ with $m + k = n$.

Now take $(h_1, ..., h_n) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_n,q_n}$ with $h_1 \cdots h_n \notin \mathbf{L}^{p,q}$. Since $h_{m+1}, ..., h_n \in \mathbf{L}^\infty$, we get $h_1 \cdots h_m \notin \mathbf{L}^{p,q}$.

For any $v, u > 0$, let $E_u^v$ be defined as in (17). Clearly, we have $E_{p,q}^{(p_1,q_1...,p_n,q_n)} = \bigcup_{u\in\mathbb{N}} E_u^1$, so we only have to show that for every $u > 0$, the set $E_u^1$ is nowhere dense.

Let $d_i, K_i, r_i$, $i = 1, ..., n$ be as in (14). Set $u > 0$ and let $(f_1, ..., f_m, g_1, ..., g_k) \in \mathbf{L}^{p_1,q_1} \times ... \times \mathbf{L}^{p_m,q_m} \times \mathbf{L}^\infty \times ... \times \mathbf{L}^\infty$ and $R > 0$. Now take $r_0 > 0$ such that for every $i = 1, ..., m$,

$$K_i r_0^{r_i} \leq \frac{1}{2}R. \tag{24}$$

Define for every $i = 1, ..., m$,

$$\tilde{f}_i(x) := \begin{cases} f_i(x) + \frac{r_0}{\|h_i\|_{p_i,q_i}}|h_i(x)|, & f_i(x) \geq 0; \\ f_i(x) - \frac{r_0}{\|h_i\|_{p_i,q_i}}|h_i(x)|, & f_i(x) < 0, \end{cases}$$

and for $i = 1, ..., k$,

$$\tilde{g}_i(x) := \begin{cases} g_i(x) + \frac{R}{2}, & g_i(x) \geq 0; \\ g_i(x) - \frac{R}{2}, & g_i(x) < 0. \end{cases}$$

Then, clearly, for each $i = 1, ..., m$, $\| \tilde{f}_i - f_i \|_{p_i,q_i} = r_0$, and for each $i = 1, ..., k$, $\| \tilde{g}_i - g_i \|_{p_i,q_i} = \frac{R}{2}$. In particular, by (14) and (24), for every $i = 1, ..., m$, $d_i\left(\tilde{f}_i, f_i\right) \leq \frac{1}{2}R$, and for each $i = 1, ..., k$, $d_{m+i}(\tilde{g}_i, g_i) \leq \frac{1}{2}R$. Hence

$$B_d\left(\left(\tilde{f}_1, ..., \tilde{f}_m, \tilde{g}_1, ..., \tilde{g}_k\right), \frac{1}{2}R\right) \subset B_d\left((f_1, ..., f_m, g_1, ..., g_k), R\right). \tag{25}$$

Since for every $i=1,...,m$ and every $x\in X$, $|\tilde{f}_i(x)|\geq \frac{r_0}{\|h_i\|_{p_i,q_i}}|h_i(x)|$, we get that $\tilde{f}_1\cdots\tilde{f}_m\notin L^{p,q}$. Now let $l>0$ be as in the thesis of Lemma 3.14, chosen for $\left(\tilde{f}_1,...,\tilde{f}_m\right)$, $u$ and $v=\left(\frac{4}{R}\right)^k$. Clearly, we may assume that

$$K_i l^{r_i}\leq \frac{1}{2}R,\ i=1,...,n. \tag{26}$$

Let

$$(a_1,...,a_m,b_1,...,b_k)\in B_{p_1,q_1}\left(\tilde{f}_1,l\right)\times...\times B_{p_n,q_n}(\tilde{g}_k,l).$$

Then for $\mu$-almost all $x\in X$,

$$|b_1(x)\cdots b_k(x)|\geq\left(\frac{1}{2}R-l\right)^k\geq\left(\frac{R}{4}\right)^k.$$

Hence and by Lemma 3.14,

$$\int_0^\infty \mu(\{x:|a_1(x)\cdots a_m(x)b_1(x)\cdots b_k(x)|>\lambda\})^{\frac{q}{p}}\lambda^{q-1}\,d\lambda\geq$$

$$\int_0^\infty \mu\left(\left\{x:|a_1(x)\cdots a_m(x)|>\left(\frac{4}{R}\right)^k\lambda\right\}\right)^{\frac{q}{p}}\lambda^{q-1}\,d\lambda\overset{L\ 3.14}{>}u,$$

so $(a_1,...,a_m,b_1,...,b_k)\notin E_1^u$. Hence

$$B_{p_1,q_1}\left(\tilde{f}_1,l\right)\times...\times B_{p_n,q_n}(\tilde{g}_k,l)\cap E_u^1=\emptyset. \tag{27}$$

Moreover, by (14) and (26), for every $i=1,...,m$, $d_i\left(a_i,\tilde{f}_i\right)<\frac{1}{2}R$, and for every $i=1,...,k$, $d_i\left(b_i,\tilde{g}_i\right)<\frac{1}{2}R$. Hence

$$B\left(\tilde{f}_1,l\right)\times...\times B\left(\tilde{f}_m,l\right)\times B(\tilde{g}_1,l)\times...\times B(\tilde{g}_k,l)\subset B_d\left(\left(\tilde{f}_1,...,\tilde{f}_m,\tilde{g}_1,...,\tilde{g}_k\right),\frac{1}{2}R\right)$$

Since each open "ball" with respect to quasinorm has a nonempty interior (this follows from (14)), the above together with (25) and (27), show that $E_u^1$ is nowhere dense. □

**Problem 3.15.** It would be interesting to find the necessary and sufficient condition, under which $E_{p,q}^{(p_1,q_1,...,p_n,q_n)}\neq \mathbf{L}^{p_1,q_1}\times...\times\mathbf{L}^{p_n,q_n}$ in the case when $\frac{1}{p_1}+...+\frac{1}{p_n}=\frac{1}{p}$.

The first author has been supported by the Polish Ministry of Science and Higher Education Grant No. N N201 414939.

## References

[BW] Balcerzak, M.; Wachowicz, A.: Some examples of meager sets in Banach spaces. Real Anal. Exchange 26 (2000/01), no. 2, 877–884.

[G] L. Grafakos, *Classical Fourier analysis. Second edition*, Graduate Texts in Mathematics, **249** Springer, New York, 2008.

[GS] Głąb, S.; Strobin, F.: Dichotomies for $L^p$ spaces, J. Math. Anal. Appl., 368 (2010) 382-390.

[J] Jachymski, J.: A nonlinear Banach-Steinhaus theorem and some meager sets in Banach spaces. Studia Math. 170 (2005), no. 3, 303–320.

[Z1] Zajíček, L.: Porosity and $\sigma$-porosity, Real Anal. Exchange 13 (1987/1988), 314–350.

[Z2] Zajíček, L.: On $\sigma$-porous sets in abstract spaces, Abstr. Appl. Anal. 5 (2005), 509–534.

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