LK PROPERTY FOR σ-IDEALS

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Abstract. An ideal \( J \) of subsets of a Polish space \( X \) has (LK) property if for every sequence \( (A_n) \) of analytic sets in \( X \), if \( \limsup_{n \in H} A_n \notin \mathcal{J} \) for each infinite \( H \) then \( \bigcap_{n \in G} A_n \notin \mathcal{J} \) for some infinite \( G \). In this note we present a new class of σ-ideals with (LK) property.

1. Introduction

We use standard set theoretical notation (see [Ke] or [S]). Laczkovich in [L] proved that, for every sequence \( (A_n) \) of Borel subsets of a Polish space, if \( \limsup_{n \in H} A_n \) is uncountable for each \( H \in [\mathbb{N}]^\omega \) then \( \bigcap_{n \in G} A_n \) is uncountable for some \( G \in [\mathbb{N}]^\omega \). This result was then generalized by Komjáth [Ko, Thm 1] to the case when the sets \( A_n \) are analytic. Komjáth also proved that the result of Laczkovich cannot be generalized within ZFC to the case of coanalytic sets. Namely, if \( V = L \), there is a sequence \( (A_n) \) of coanalytic sets such that \( |\limsup_{n \in H} A_n| > \omega \) and \( |\bigcap_{n \in H} A_n| \leq \omega \) for all \( H \in [\mathbb{N}]^\omega \); see [Ko, Thm. 4].

In connection with the above quoted theorem of Komjáth about analytic sets, Balcerzak and Głąb in [BG] introduce the Laczkovich-Komjáth property of ideal \( \mathcal{J} \) of subsets of Polish space \( X \). The ideal \( \mathcal{J} \) is said to have property (LK) whenever for every sequence \( (A_n) \) of analytic subsets of \( X \), if \( \limsup_{n \in H} A_n \notin \mathcal{J} \) for each \( H \in [\mathbb{N}]^\omega \) then \( \bigcap_{n \in G} A_n \notin \mathcal{J} \) for some \( G \in [\mathbb{N}]^\omega \). In particular, the Komjáth theorem states that the ideal \( [X]^{\leq \omega} \) has property (LK). Halmos [H] proved that the σ-ideal of null sets does not have (LK) property. We can reformulate the (LK) property in the following nice way. A σ-ideal \( \mathcal{I} \) has (LK) property if for any sequence \( (A_n) \) of analytic sets either there is \( H \in [\mathbb{N}]^\omega \) with \( \limsup_{n \in H} A_n \notin \mathcal{I} \) or there is \( H \in [\mathbb{N}]^\omega \) with \( \liminf_{n \in H} B_n \notin \mathcal{I} \).

In paper [BG] it was proved that for any Polish space \( X \) and any \( F_\sigma \) relation \( E \subset X \times X \) with uncountable many equivalence classes, if \( \mathcal{J} \) consist of all subsets of \( X \) that can be covered by countably many equivalence classes, then σ-ideal \( \mathcal{J} \) has (LK) property. Note that σ-ideal \( [X]^{\leq \omega} \) is of this form (it is enough to define \( xEy \) if and only if \( x = y \), and

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observe that \( E \) is closed in \( X \times X \). This note is devoted to show that there are other natural examples of \( \sigma \)-ideals with (LK) property. It seems (but we did not establish it) that these \( \sigma \)-ideals cannot be defined by \( F_\sigma \) equivalence relation.

The (LK) property was also studied by Zapletal [Z]. He proved several properties of this notion, and he gave some new examples of \( \sigma \)-ideals with (LK) property.

2. \( \sigma \)-ideals defined by \((n,F)\)-system

Let \( X \) be a Polish space. By \( K(X) \) we denote the hyperspace of all nonempty compact subsets of \( X \), endowed with the Vietoris topology, i.e. the topology generated by sets \( \{K \in K(X) : K \cap U \neq \emptyset\} \) and \( \{K \in K(X) : K \subset U\} \) for any open sets \( U \) in \( X \). The Vietoris topology is equal to the topology generated by the Hausdorff metric

\[
\rho_H(K, L) = \max\{\max_{x \in K} \rho(x, L), \max_{x \in L} \rho(x, K)\}
\]

where \( \rho(x, K) \) is the distance from a point \( x \) to a set \( K \) with respect to the metric \( \rho \) on \( X \).

By \((X)^n\) denote the set \( \{(x_1, ..., x_n) \in X^n : \forall i, j \leq n (i \neq j \Rightarrow x_i \neq x_j)\} \). Let \( S_n \) be the set of all permutations of the set \( \{1, ..., n\} \). Let \( Y \subset (X)^n \) be a Polish space. We say that \( Y \) is invariant under permutations of coefficients if for any permutation \( \sigma \in S_n \) and any \( (x_1, ..., x_n) \in (X)^n \)

\[
(x_1, ..., x_n) \in Y \iff (x_{\sigma(1)}, ..., x_{\sigma(n)}) \in Y.
\]

From now on, we assume that \( Y \) is invariant under permutations of coefficients, and for any \( x_1 \in X \) there are \( x_2, ..., x_n \in X \) with \( (x_1, x_2, ..., x_n) \in Y \).

Let \( F : Y \to K(X) \) be a continuous mapping such that for any permutation \( \sigma \in S_n \) and any \( (x_1, ..., x_n), (y_1, ..., y_n) \) from \( Y \)

\[
(i) \quad F(x_1, ..., x_n) = F(x_{\sigma(1)}, ..., x_{\sigma(n)});
(ii) \quad \{x_1, ..., x_n\} \subset F(x_1, ..., x_n);
(iii) \quad \{y_1, ..., y_n\} \subset F(x_1, ..., x_n) \Rightarrow F(y_1, ..., y_n) = F(x_1, ..., x_n).
\]

A family \( S = \{F(x_1, ..., x_n) : (x_1, ..., x_n) \in Y\} \) is called \((n,F)\)-system. We say that \( \sigma \)-ideal \( J \) of subsets of \( X \) is generated by \((n,F)\)-system \( S \), if \( J \) consists of all subsets of \( X \) which can be covered by countably many sets from \( S \). If \( x \in X \), then there are \( x_2, ..., x_n \in X \) with \( (x, x_2, ..., x_n) \in Y \) and by (ii) we obtain \( \{x\} \subset \{x, x_2, ..., x_n\} \subset F(x, x_2, ..., x_n) \). Hence \( J \) contains all singletons. If \( X \) cannot be covered by countably many elements from \( S \), then \( J \) is a proper ideal, i.e. \( X \notin J \) – in this case we say that \((n,F)\)-system \( S \) is proper.
Example 1. Let $X = \mathbb{R}^2$. Let $Y \subset (X)^3$ be a set of all non-collinear triples. Then $Y$ is a Polish subspace as an open subset of $(X)^3$:

$$Y = \{ (x, y, z) \in (X)^3 : (y_2 - x_2)(z_1 - y_1) \neq (y_1 - x_1)(z_2 - y_2) \}.$$ 

For a triple $(x, y, z) \in Y$ by $F(x, y, z)$ denote the unique circle which contains the points $x, y$ and $z$. Then the $\sigma$–ideal $\mathcal{J}$ generated by this $(3, F)$–system consists of all subsets of the real plane which can be covered by countably many circles.

Example 2. Let $X = [0, 1]^2$ and $Y = (X)^2$. For any pair $(x, y) \in Y$ let $F(x, y)$ be a line containing $x$ and $y$ intersected with the unite square $X$. Then the ideal $\mathcal{J}$ generated by this $(2, F)$–system consists of those subsets of the unite square $X$ which can be covered by countably many lines. Here we restrict our attention to the unite square, since we want $F(x, y)$ to be compact.

Example 3. Let $X = [0, 1] \times \mathbb{R}$ and let

$$Y = \{ ((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) \in (X)^n : \forall i \neq j (x_i \neq x_j) \}.$$ 

For $((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) \in Y$ let $F((x_1, y_1), (x_2, y_2), ..., (x_n, y_n))$ be a graph of the unique polynomial $f$ of degree not grater than $n - 1$ with $f(x_i) = y_i$, $i = 1, ..., n$. Then the $\sigma$–ideal $\mathcal{J}$ generated by this $(n, F)$–system consists of all subsets of $X$ which can be covered by countably many graphs of polynomials of degree not greater than $n - 1$.

Example 4. Let $X$ be a Polish space, $E \subset X \times X$ be a closed equivalence relation with compact equivalence classes. Put $Y = X$ and put $F(x)$ to be an equivalence class of $x$, i.e. $F(x) = \{ y : xEy \}$. Then the $\sigma$–ideal $\mathcal{J}$ generated by this $(1, F)$–system consists of all subsets of $X$ which can be covered by countably many equivalence classes.

Conversely note that $(1, F)$-system defines an equivalence relation $E_F$ by $xE_Fy \iff x \in F(y)$. The relation $E_F$ is closed. Indeed, let $(x_n, y_n) \in E_F$ be such that $(x_n, y_n) \to (x_0, y_0)$. By the continuity of $F$ we obtain that $F(x_n) \to F(x_0)$ and $F(y_n) \to F(y_0)$ in $K(X)$. Since $x_n \in F(y_n)$, then $F(x_n) = F(y_n)$. Hence $F(x_n) = F(y_n)$ and therefore $F(x_0) = F(y_0)$, which means that $x_0E_Fy_0$.

It follows now, that being generated by a $(1, F)$-system is the same as being generated by a closed equivalence relation with compact equivalence classes. In [BG] it was shown that if an $\sigma$–ideal $\mathcal{J}$ is generated by a $F_\sigma$ equivalence relation is proper, then it has (LK) property. From this reason we will consider only $(n, F)$-systems for $n \geq 2$. 

LK PROPERTY FOR $\sigma$-IDEALS
Let $\mathcal{J}$ be a proper $\sigma$-ideal of subsets of $X$ which contain all singletons. Fix a sequence $(A_n)$ of analytic subsets of $X$ such that

$$\forall H \in [\mathbb{N}]^\omega \limsup_{n \in H} A_n \notin \mathcal{J}.$$ 

Fix $H \in [\mathbb{N}]^\omega$. We say that $Y \subset X$ is good with respect to $H$, if

$$Y \cap \limsup_{n \in G} A_n \notin \mathcal{J}$$

for any $G \subset [H]^\omega$. Observe that if $Y$ is good with respect to $H$ and $Z \subset Y$, $Z \in \mathcal{J}$, then $Y \setminus Z$ is good with respect to $H$. In particular, if $Y$ is closed and good with respect to $H$, then the perfect kernel of $Y$ (cf. [S, 2.6.2]) is good with respect to $H$ – we will use this fact several times.

**Lemma 5.** If a set $Y = \bigcup_{i \in \mathbb{N}} Y_i$ is good with respect to $H \in [\mathbb{N}]^\omega$, then there are $i \in \mathbb{N}$ and $H' \in [H]^\omega$ such that $Y_i$ is good with respect to $H'$.

The proof is analogous to that given in [Ko, Lemma 1].

**Lemma 6.** Let $P, A \subset X$ be such that $P$ are closed, and $P \cap A$ is good with respect to some $H \in [\mathbb{N}]^\omega$. Then there exist $x \in P$ and $H' \in [H]^\omega$ such that for any neighborhood $U$ of $x$ the set $(P \setminus F) \cap A \cap U$ is good with respect to $H'$.

The proof can be found in [BG].

Now, we assume that $\mathcal{J}$ is a $\sigma$-ideal generated by $(n,F)$–system $\mathcal{S}$, $n \geq 2$. Let $\mathcal{S} = \{F(x_1,\ldots,x_n) : (x_1,\ldots,x_n) \in Y\}$.

**Lemma 7.** Let $R_1,\ldots,R_n,K_1,\ldots,K_m,A \subset X$ and $H \in [\mathbb{N}]^\omega$. Assume that $R_j, K_i$ are pairwise disjoint, and $R_j \cap A$, $K_i \cap A$ are good with respect to $H$, $j = 1,\ldots,n$ and $i = 0,\ldots,m$. Then there are closed sets $R'_j \subset R_j$ and $K'_i \subset K_i$ with

$$\forall (x_1,\ldots,x_n) \in R'_1 \times \cdots \times R'_n \left( \text{dist} \left( F(x_1,\ldots,x_n), \bigcup_{i=1}^m K'_i \right) > 0 \right),$$

and there is $H' \in [H]^\mathbb{N}$ such that $R'_j \cap A$, $K'_i \cap A$ are good with respect to $H'$, for $j = 1,\ldots,n$ and $i = 0,\ldots,m$.

**Proof.** If for every $j = 1,\ldots,n$ and every $x_j \in R_j$

$$\text{dist} \left( F(x_1,\ldots,x_n), \bigcup_{i=1}^m K_i \right) > 0,$$
then putting $R'_j = R_j$ and $P'_i = P_i$, we are done. If not, then by Lemma 6 (for $P = R_1$) there exist: a point $x_1 \in R_1$ and a set $H_1 \in [H]^{\omega}$ such that for any neighborhood $U_1$ of $x_1$ the set $R_1 \cap A \cap U_1$ is good with respect to $H_1$. Using again Lemma 6 (this time for $P = R_2$) we find a point $x_2 \in R_1$ and a set $H_2 \in [H_1]^{\omega}$ such that for any neighborhood $U_2$ of $x_2$ the set $R_2 \cap A \cap U_2$ is good with respect to $H_2$. In that way we find points $x_1, x_2, ..., x_n$ and $H_n \in [H]^{\omega}$ such for any $j = 1, ..., n$ and any neighborhood $U_j$ of $x_j$ the set $(R_j \setminus \{x_1, ..., x_n\}) \cap A \cap U_j$ is good with respect to $H_n$.

Now, for every $i = 1, ..., m$ we will define a set $H_{n+i}$ and a number $k_i$ in the following way. If $F(x_1, ..., x_n) \cap K_i = \emptyset$, then put $K'_i = K_i$, $H_{n+i} = H_{n+i-1}$ and $k_i = 0$. Otherwise let

$$Y_k = \{x \in K_i : \text{dist}(x, F(x_1, x_2, ..., x_n)) \geq \frac{1}{k+1}\}.$$  

Then $\bigcup_{k \in \mathbb{N}} Y_k = K_i \setminus F(x_1, x_2, ..., x_n)$. Since $K_i$ is good with respect to $H_{n+i-1}$, then by Lemma 5 we find a number $k_i$ and a set $H_{n+i} \in [H_{n+i-1}]^{\omega}$ such that $Y_{k_i} \cap A$ is good with respect to $H_{n+i}$. Put $K'_i = Y_{k_i}$. Define

$$\varepsilon = \min \left\{ \frac{1}{k_i + 1} : i = 1, ..., m \right\}.$$  

By the continuity of $F$ we will find neighborhoods $V_1, ..., V_n$ of points $x_1, ..., x_n$, respectively, such that the diameter of $F(cl(V_1), ..., cl(V_n))$ is less than $\varepsilon/2$. Define $R'_j = cl(V_j)$ and $H' = H_{n+m}$.

**Lemma 8.** Let $m \in \mathbb{N}$, $P_0, ..., P_m, A \subset X$. Assume that for $i = 0, ..., m$ the sets $P_i$ are closed, pairwise disjoint and such that any set from $\mathcal{S}$ does not intersect more than $n + 1$ sets $P_i$. Let $H \in [\mathbb{N}]^{\omega}$ and $\varepsilon > 0$. If $P_i \cap A$ is good with respect to $H$ and $i = 0, ..., m$, then there are pairwise disjoint closed sets $P'_m, ..., P'_{m+n-1} \subset P_m$, $P'_i \subset P_i$ for $i < m$ and there is $H' \in [H]^{\omega}$ such that each $P'_i$ has diameter less than $\varepsilon$, any set from $\mathcal{S}$ does not intersect more than $n + 1$ sets $P'_i$, and sets $A \cap P'_i$ are good with respect to $H'$.

**Proof.** In the same way as in Lemma 7 we find a set $H_1 \in [H]^{\omega}$ and points $x_m, x_{m+1}, ..., x_{m+n-1}$ in $P_m$ such that any $i = 0, ..., n - 1$ and any neighborhood $U_{m+i}$ of point $x_{m+i}$ sets $U_{m+i} \cap P_m \cap A$ is good with respect to $H_1$.

Inductively for $i = 0, 1, ..., m - 1$ we define $P'_i$ and $H_i$ in the following way. If distance between $F(x_m, x_{m+1}, ..., x_{m+n-1})$ and $P_i$ is greater than zero, then we put $P'_i = P_i$. Otherwise let

$$Y_k = \{x \in P_i : \text{dist}(x, F(x_m, x_{m+1}, ..., x_{m+n-1})) \geq \frac{1}{k+1}\}.$$
Then $\bigcup_{k \in \mathbb{N}} Y_k = P_i \setminus F(x_m, x_{m+1}, \ldots, x_{m+n-1})$. Since $P_i$ is good with respect to $H_{i-1}$, then by Lemma 5 we find a number $k$ and a set $H_i \in [H_{i-1}]^\omega$ such that $Y_k \cap A$ is good with respect to $H_i$. We may assume that $\text{diam}(Y_k) < \varepsilon$. Put $P_i' = Y_k$.

Now, let $\delta > 0$ be such that for any $i \in \{0, 1, \ldots, m-1\}$ the distance from $P_i'$ to the set $F(x_m, x_{m+1}, \ldots, x_{m+n-1})$ is greater than $\delta$. By continuity of $F$ we find neighborhoods $U_m, U_{m+1}, \ldots, U_{m+n-1}$ of points $x_m, x_{m+1}, \ldots, x_{m+n-1}$, respectively, such that

$$\text{diam}(F(U_m, U_{m+1}, \ldots, U_{m+n-1})) < \delta.$$ 

Put $P'_m = cl(U_m), \ldots, P'_{m+n-1} = cl(U_{m+n-1})$.

Now, any set from $\mathcal{S}$ which intersect $P'_m, \ldots, P'_{m+n-1}$ does not intersect any of $P'_0, \ldots, P'_{m-1}$. But our choice guarantees that any set from $\mathcal{S}$ does not intersect more than $n + 1$ sets $P'_0, \ldots, P'_{m+n-1}$ only in the case if $n = 2$. If $n > 2$, we will shrink each $P_i'$ and $H'$ finitely many times in $n - 2$ steps.

In the first step we use Lemma 7 for $H = H'$, $R_1 = P'_m, \ldots, R_{n-1} = P'_{m+n-2}, R_n = P'_{m-1}$ and $K_1 = P_1, \ldots, K_{m-1} = P_{m-1}, K_m = P'_{m+n-1}$ to find $H'' \in [H'][\mathbb{N}]$ and closed sets $R'_1, \ldots, R'_{n-1}, R'_n$ and $K'_1, \ldots, K'_{m-1}, K'_m$ such that $R'_j \subset R_j$, $K'_i \subset K_i$,

$$\forall (x_1, \ldots, x_n) \in R'_1 \times \cdots \times R'_n \left( \text{dist} \left( F(x_1, \ldots, x_n), \bigcup_{i=1}^{m} K_i \right) > 0 \right),$$

and such that $R'_j \cap A$, $K'_i \cap A$ are good with respect to $H''$. Let $H''$ be our new choice for $H'$, $R'_1$ be a new choice for $P'_m$ etc. Now, we use Lemma 7 for $H'$ and each combination $R_1, \ldots, R_{n-1}, R_n$ such that $R_1, \ldots, R_{n-1}$ is collection of $n - 1$ sets from $P'_m, \ldots, P'_{m+n-1}$ and $R_n$ is a set from $P'_0, \ldots, P'_{m-1}$, and $K_1, \ldots, K_m$ the remaining sets from $P'_0, \ldots, P'_{m+n-1}$. After this we obtain that any set which intersect $n - 1$ sets from $P'_m, \ldots, P'_{m+n-1}$ do intersect at least one set from $P'_0, \ldots, P'_{m-1}$.

In the second step we use Lemma 7 for $H'$ and each combination $R_1, \ldots, R_{n-1}, R_n$ such that $R_1, \ldots, R_{n-2}$ is collection of $n - 2$ sets from $P'_m, \ldots, P'_{m+n-1}$ and $R_{n-1}, R_n$ are sets from $P'_0, \ldots, P'_{m-1}$, and $K_1, \ldots, K_m$ the remaining sets from $P'_0, \ldots, P'_{m+n-1}$. After this we obtain that any set which intersect $n - 2$ sets from $P'_m, \ldots, P'_{m+n-1}$ do intersect at least two sets from $P'_0, \ldots, P'_{m-1}$.

In the last $n - 2$th step we use Lemma 7 for $H'$ and each combination $R_1, \ldots, R_{n-1}, R_n$ such that $R_1, R_2$ is collection of 2 sets from $P'_m, \ldots, P'_{m+n-1}$ and $R_3, \ldots, R_n$ are sets from $P'_0, \ldots, P'_{m-1}$, and $K_1, \ldots, K_m$ the remaining sets from $P'_0, \ldots, P'_{m+n-1}$. After this we obtain
that any set which intersect 2 sets from $P'_m, \ldots, P'_{m+n-1}$ do intersect at least $n-2$ sets from $P'_0, \ldots, P'_{m-1}$.

Suppose that there is $D \in S$ which intersects $n + 1$ sets from $P'_1, \ldots, P'_{n+n-1}$. Then $D$ cannot intersect more than one set from $P'_m, \ldots, P'_{m+n-1}$. Hence $D$ intersects $n + 1$ sets from $P_1, \ldots, P_m$ which contradicts our assumptions. \hfill \Box

The next theorem shows that $\sigma$-ideals generated by $(n, F)$-systems have (LK) property. Its proof is quite similar to that in [Ko]. The main difference lays inLemma 8.

**Theorem 9.** Let $\mathcal{J}$ be a $\sigma$–ideal generated by a proper $(n, F)$–system $S$. Then for any sequence $(A^j)$ of analytic sets such that

$$\forall H \in [\mathbb{N}]^\omega \limsup_{j \in H} A^j \notin \mathcal{J}$$

there exist: a set $G \in [\mathbb{N}]^\omega$ and a homeomorphic $P$ of the Cantor set $2^\omega$ such that any $n + 1$ distinct point of $P$ are not the member of the same set from family $S$ and such that $P \subset \bigcap_{j \in G} A^j$. In particular, a $\sigma$–ideal $\mathcal{J}$ has (KL) property.

**Proof.** We may assume that $X$ is a perfect set (if not, then removing countably many points from $X$ we obtain a perfect set). Additionally we may assume that $\text{diam}(X) < 1$. Let $A^j$ be a sequence of analytic sets with

$$\forall H \in [\mathbb{N}]^\omega \limsup_{j \in H} A^j \notin \mathcal{J}.$$   

We may write $A^j$ using a Suslin operation (cf. [Ke, 25.7]):

$$A^j = \bigcup_{z \in \mathbb{N}^m} \bigcap_{m \in \mathbb{N}} C^j_{z|m},$$

where $C^j_{z|m}$ are closed with $\text{diam}(C^j_{z|m}) < \frac{1}{m+1}$ and

$$\forall k, m \in \mathbb{N} (k > m \Rightarrow C^j_{z|m} \subset C^j_{z|k}).$$

For $s \in \mathbb{N}^m$ put $A^j_s = \bigcup_{z \in \mathbb{N}^m, z|m=s} \bigcap_{k \in \mathbb{N}} C^j_{z|k}$.

Without loss of generality we may assume that $A^0 = X$. Our construction will be inductive. In the $m$-th step we choose a number $j_m \in \mathbb{N}$, perfect sets $P_s (s \in \{1, \ldots, n\}^m)$, finite sequences $t(k, s) \in \mathbb{N}^m (k \leq m, s \in \{1, \ldots, n\}^m)$ and a set $H_m \in [\mathbb{N}]^\omega$ fulfilling the following conditions

(W1) $j_m > j_{m-1}$, $H_m \in [H_{m-1}]^\omega$, $j_m \in H_{m-1}$;
(W2) \( P_{s; i} \subset P_s \) for \( i \in \{1, ..., n\} \), \( P_{s; i} \) are pairwise disjoint for \( s \in \{1, ..., n\}^{m-1} \), and any set from \( S \) does not intersect \( n + 1 \) or more sets from \( \{P_s : s \in \{1, ..., n\}^m\} \);

(W3) \( \text{diam}(P_s) < \frac{1}{m+1} \) for \( s \in \{1, ..., n\}^m \);

(W4) \( P_s \cap A^m_{t(0,s)} \cap ... \cap A^m_{t(m,s)} \) is good with respect to \( H_m \), if \( s \in \{1, ..., n\}^m \);

(W5) \( P_s \subset C^m_{t(0,s)} \cap ... \cap C^m_{t(m,s)} \) for \( s \in \{1, ..., n\}^m \);

(W6) \( t(k, s) \subset t(k, s' j) \), for \( i \in \{1, ..., n\} \), \( s \in \{1, ..., n\}^{m-1} \) and \( k \leq m - 1 \).

Conditions (2) and (3) guarantee that the set

\[
P = \bigcap_{m \in \mathbb{N}} \bigcup_{s \in \{1, ..., n\}^m} P_s
\]

is perfect and that any set from \( S \) does not contain \( n + 1 \) or more points from \( P \). Hence \( P \notin J \). If \( x \in P \), then from (2) it follows that for any \( m \in \mathbb{N} \) there is an unique sequence \( s_m \) with \( x \in P_{s_m} \). Moreover \( s_0 \subset s_1 \subset s_2 \subset ... \). Fix \( i \in \mathbb{N} \). From (5) for \( m \geq i \) we obtain that \( x \in C^i_{t(i, s_m)} \), and by (6) we get \( t(i, s_i) \subset t(i, s_{i+1}) \subset t(i, s_{i+2}) \subset ... \). Hence \( x \in A^i_{t(i, s_i)} \subset A^i \).

Finally \( P \subset \bigcap_{i \in \mathbb{N}} A^i \), and putting \( G = \{j_0, j_1, ...\} \) we obtain the assertion.

It suffices to define the fulfilling (1)–(6). We will construct them by induction on \( m \). Put \( j_0 = 0 \), \( P_0 = X \), \( H_0 = \mathbb{N} \). Clearly, \( X \) is good with respect to \( \mathbb{N} \). Putting \( t(0, \emptyset) = \emptyset \), we define objects fulfilling (1)–(6) for the first step.

Assume that for \( m \in \mathbb{N} \) we already choose \( j_k \) (for \( k \leq m \)), \( P_s \) (for \( s \in \{1, ..., n\}^k \), \( k \leq m \), \( t(k, s) \) (for \( k \leq l \leq m \), \( s \in \{1, ..., n\}^l \) and \( H_k \) (for \( k \leq m \)).

At first we show that there exist a number \( j \in H_m \), \( j > j_m \), and a set \( H'_m \in [H_m]^{\omega} \) such that

\[
\forall s \in \{1, ..., n\}^m (P_s \cap A^m_{t(0,s)} \cap ... \cap A^m_{t(m,s)} \cap A^j) \text{ is good with respect to } H'_m.
\]

Assume to the contrary that for any \( j \in H_m \), \( j > j_m \), and for any \( H \in [H_m]^{\omega} \) we have

\[
\exists G \in [H]^{\omega} \exists s \in \{1, ..., n\}^m (P_s \cap A^m_{t(0,s)} \cap ... \cap A^m_{t(m,s)} \cap A^j \cap \limsup_{r \in G} A^r \in J).
\]

Proceeding inductively, we find numbers \( k_0 < k_1 < ... \) and sets \( H_m = G_0 \supset G_1 \supset ... \) such that \( k_r \in G_r \in [\mathbb{N}]^{\omega} \) and

\[
\forall r \in \mathbb{N} \exists s_r \in \{1, ..., n\}^m (P_{s_r} \cap A^m_{t(0,s_r)} \cap ... \cap A^m_{t(m,s_r)} \cap A^{k_r} \cap \limsup_{p \in G_{r+1}} A^p \in J).
\]

Since there is only \( n^m \) possibilities of choosing \( s_r \), there is a sequence \( s \in \{1, ..., n\}^m \) such that a set \( \Gamma = \{k_r : s_r = s\} \) is infinite. Then \( \Gamma \) is almost contained in \( G_r \), for every \( r \in \mathbb{N} \).
So we obtain
\[ P_s \cap A_{t(0,s)}^{j_0} \cap ... \cap A_{t(m,s)}^{j_m} \cap \left( \bigcup_{r \in \Gamma} A^r \right) \cap \limsup_{p \in \Gamma} A^p \in \mathcal{J}. \]

But this is impossible, since \( \limsup_{p \in \Gamma} A^p \subset \bigcup_{r \in \Gamma} A^r \) and (4). Hence there is a number \( j > j_m, j \in H_m \), fulfilling (7). It is our choice for \( j_{m+1} \).

Using \( n^m \) many times (7) and Lemma 8 to the sets \( \{ P_s : s \in \{ 1, ..., n \}^m \} \), and considering perfect kernels of appropriate closed sets we will find pairwise disjoint perfect sets \( P_s^\hat{i} \), for \( i \in \{ 1, ..., n \} \) with \( \text{diam}(P_s^\hat{i}) < \frac{1}{m+1} \), and such that any set from \( S \) have no common point with \( n+1 \) or more sets from \( \{ P_s : s \in \{ 1, ..., n \}^m \} \), and a set \( H_m'' \subset [H_m']^\omega \) such that for any \( s \in \{ 1, ..., n \}^m \) and \( i = 1, ..., n \) we have
\[
\mathcal{P}_{s^\hat{i}} \cap A_{t(0,s)}^{j_0} \cap ... \cap A_{t(m,s)}^{j_m} \cap A_{t(m+1,s)}^{j_{m+1}} \text{ is good with respect to } H_m''.
\]

The set \( A_{t(0,s)}^{j_0} \cap ... \cap A_{t(m,s)}^{j_m} \cap A_{t(m+1,s)}^{j_{m+1}} \) is contained in the following union
\[
\bigcup_{z_0 \in \mathbb{N}^{m+1}, z_0 \supset t(0,s)} \bigcup_{z_m \in \mathbb{N}^{m+1}, z_m \supset t(m,s)} \bigcup_{z_{m+1} \in \mathbb{N}^{m+1}} (A_{z_0}^{j_0} \cap ... \cap A_{z_{m+1}}^{j_{m+1}}).
\]

By Lemma 5 it follows that some element of this union is good with respect to \( H_m'' \). Using \( n^{m+1} \) times Lemma 5, we will find \( H_m'' \) with that property for all \( s \in \{ 1, ..., n \}^m \) and all \( i = 1, ..., n \). We define sequences \( t(0,s^\hat{i}), ..., t(m+1,s^\hat{i}) \) as \( z_0, ..., z_{m+1} \) corresponding to \( s^\hat{i} \).

We finally need only to "repair" sets \( \mathcal{P}_{s^\hat{i}} \) to fulfill (5). To do this put
\[
Q_{s^\hat{i}} = \mathcal{P}_{s^\hat{i}} \cap C_{t(0,s^\hat{i})}^{j_0} \cap ... \cap C_{t(m+1,s^\hat{i})}^{j_{m+1}}.
\]

Since for every \( s \) we have \( A_s^j \subset C_s^j \), then the sets \( Q_{s^\hat{i}} \) and \( \mathcal{P}_{s^\hat{i}} \) have the same intersection with
\[
A_{t(0,s^\hat{i})}^{j_0} \cap ... \cap A_{t(m+1,s^\hat{i})}^{j_{m+1}}.
\]

Hence (4) valid. Removing from each closed set \( Q_{s^\hat{i}} \) at most countably many point we obtain its perfect kernel \( P_{s^\hat{i}} \). It is still good with respect to \( H_m'' \), which will be our choice for \( H_{m+1} \). Therefore conditions (1)–(6) are fulfilled. \( \square \)

In [EKM] it was proved that if an analytic set on the real plane cannot be covered by countably many lines then it contains a perfect set which also cannot be covered by countably many lines. We can generalized this in the following.
Corollary 10. Let $A$ an analytic subset of the plane and let $J$ be a $\sigma$-ideal generated by a proper $(n, F)$-system $\mathcal{S}$. If $A \notin J$, then there is $P \subset A$ a homeomorph of the Cantor set such that any $n+1$ points of $P$ are not contained in the same set from family $\mathcal{S}$.

Proof. It is enough to put $A_m = A$ for any $m \in \mathbb{N}$. $\square$

Assume that $J \subset \mathcal{P}([0,1] \times \mathbb{R})$ consist of those subsets of $[0,1] \times \mathbb{R}$ which can be covered by countably many graphs of polynomials. This $\sigma$-ideal is not of the form we considered in the previous section. But it is still very similar. This led us to the following definition. Let $\{(n_i, F_i)\}_{i \in \mathbb{N}}$ be a sequence of $(n_i, F_i)$ systems. Let $S_i = \{F_i(x_1, ..., x_{n_i}) : x_1, ..., x_{n_i} \in Y_i\}$. We say that $J$ is generated by $\{(n_i, F_i)\}_{i \in \mathbb{N}}$ if $J$ consists of those sets which can be covered by countably many sets from $S = \bigcup_{i \in \mathbb{N}} S_i$. Then the proof that $J$ has (LK) property goes in an analogous way as the proof of Theorem 9. In the proof we need only to change condition (W2) to

(W2') $P_{s;i} \subset P_s$ for $i \in \{1, ..., n\}$, $P_{s;i}$ are pairwise disjoint for $s \in \{1, ..., n\}^{m-1}$, and any set from $S_1, ..., S_m$ does not intersect more than $n_1, ..., n_m$ sets from $\{P_s : s \in \{1, ..., n\}^m\}$, respectively.

Proving the existence of such $P_{s;i}$ we use Lemma 8 for $S_1$, then for $S_2$, etc.

Using this one can get the following interesting corollary. Let $A \subset \mathbb{R}^2$ be analytic. Suppose that $A$ cannot be covered by countably many graphs of polynomials. Then there is a perfect set $P \subset A$ such that any $n$ points of $P$ cannot be covered by the graph of polynomial of degree less than $n$.

3. Parametric Laczkovich-Komjáth property

By the Mazurkiewicz-Sierpiński theorem [Ke, 29.19], if $X, Z$ are Polish spaces then for each analytic set $A \subset X \times Z$, the set $\{x \in X : |A(x)| > \omega\}$ is also analytic. We say that an ideal $J \subset \mathcal{P}(Z)$ has the Mazurkiewicz-Sierpiński property if for any Polish space $X$ and analytic set $A \subset X \times Z$, the set $\{x \in X : A(x) \notin J\}$ is analytic. This property holds true, besides the ideal of countable sets, the ideal of meager sets in $Z$ and the ideal of Lebesgue null sets in $\mathbb{R}$. Ideal which has Mazurkiewicz-Sierpiński property is also called $\Pi^1_1$-on-$\Sigma^1_1$.

We say that an ideal $J$ of subsets of $Z$ has parametric property (LK), whenever for every uncountable Polish space $X$ and every sequence $(A_n)$ of analytic subsets of $X \times Z$, if $\limsup_{n \in H} A_n(x) \notin J$ for all $x \in X$ and $H \in [\mathbb{N}]^\omega$ then there are a perfect set $P \subset X$ and
$G \in [\mathbb{N}]^\omega$ such that $\bigcap_{n \in G} A_n(x) \notin J$ for each $x \in P$. In [G], it was proved that the ideal $[Z]^\omega$ of all countable subsets of $Y$ has parametric property (LK). In [BG], it was proved that the $\sigma$-ideal generated by $F_\sigma$ equivalence relation has parametric property (LK). The proof in [BG] was based on the fact that $\sigma$-ideal generated by $F_\sigma$ equivalence relation has Mazurkiewicz-Sierpiński property and following fact:

**Proposition 11 ([BG]).** Let $Z$ be an uncountable Polish space and let $J \subset P(Z)$ be a $\sigma$-ideal with property (LK) and with Mazurkiewicz-Sierpiński property. Then $J$ has parametric property (LK).

Now, we will prove that $\sigma$-ideals generated by $(n, F)$-systems have Mazurkiewicz-Sierpiński property. As a corollary we will obtain that $\sigma$-ideals generated by $(n, F)$-systems have parametric property (LK). We say that $P$ is a perfect partial transversal (in short ppt) for $(n, F)$-system $S$ if $P$ is perfect and $x_{n+1} \notin F(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n, x_{n+1} \in P$; it is the same as saying that no $n + 1$ points of $P$ are contained in the same member of family $S$.

**Lemma 12.** Let $X$ be an uncoutable Polish space and consider $(n, F)$-system defined on $X$. Then the family of all sets $L \in K(X)$ containing a perfect partial transversal for $(n, F)$-system is analytic.

**Proof.** Fix a contable base $(U_n)$ for $X$. For $L \in K(X)$ we have the following equivalence

$L$ contains a ppt for $(n, F)$-system $\iff \exists K \in K(L) \forall m \in \mathbb{N} \forall i_1, \ldots, i_{n+1} \in \mathbb{N}

(\forall k = 1, \ldots, n + 1 U_{i_k} \cap K \neq \emptyset \Rightarrow \exists j_1, \ldots, j_{n+1} \in \mathbb{N} \forall k = 1, \ldots, n + 1 \text{ cl} U_{j_k} \subset U_{i_k},

diam U_{j_k} < \frac{1}{m+1}, U_{j_k} \cap K \neq \emptyset, F(U_{j_1}, \ldots, U_{j_n}) \cap U_{j_{n+1}} = \emptyset.$

Hence, in a standard way (cf. [Ke, 4.29], [S, 2.4.11]) we show that the family of all sets $L \in K(Y)$ containing an $(n, F)$-ppt is analytic. Thus to finish the proof it suffices to show that the equivalence does hold.

If $L \in K(Y)$ contains an $(n, F)$-ppt $K$, we easily conclude that $K$ satisfies the right hand side of the equivalence. Conversely, if $K \in K(L)$ satisfies the right hand side of the equivalence, we can define by recursion a family $\{V_s : s \in \{1, \ldots, n\}^{<\mathbb{N}}\} \subset \{U_i : i \in \mathbb{N}\}$ such that for each $s \in \{1, \ldots, n\}^{<\mathbb{N}}$ the following conditions hold:

(i) $V_s \cap K \neq \emptyset$;

(ii) $cl V_{s-1} \cup \ldots \cup cl V_{s-n} \subset V_s$, $cl V_{s-1}, \ldots, cl V_{s-n}$ are pairwise disjoint;
(iii) $\text{diam} V_s < 1/(|s| + 1)$;
and additionally,

(iv) $F(V_{s_1}, \ldots, V_{s_n}) \cap V_{s_{n+1}} = \emptyset$ for all $m \in \mathbb{N}$ and distinct $s_1, \ldots, s_{n+1} \in \{1, \ldots, n\}^m$.

The construction is similar to that given in the proof of Theorem 9 (cf. conditions (W1)–(W3)). Then $\bigcap_{m \in \mathbb{N}} \bigcup_{s \in \{1, \ldots, n\}} m(K \cap cl V_s)$ is an $(n, F)$-ppt contained in $L$.

\[\square\]

**Theorem 13.** Let $X$ be an uncountable Polish space and consider $(n, F)$-system $S$ defined on $X$. Then the $\sigma$-ideal $J$ generated by $(n, F)$-system $S$ has the Mazurkiewicz-Sierpiński property.

**Proof.** Set $\mathcal{N} = \mathbb{N}^\mathbb{N}$. For an analytic set $B \subset X$ pick a closed set $C \subset X \times \mathcal{N}$ such that $\text{pr}_X(C) = B$ where $\text{pr}_X$ stands for the projection from $X \times \mathcal{N}$ to $X$. Observe that

$B \notin J \iff (\exists K \in \mathcal{K}(X \times \mathcal{N}))(K \subset C$ and $\text{pr}_X(K)$ contains a $(n, F)$-ppt).

Indeed, to show “$\Rightarrow$” assume that $B \notin J$. By Corollary 10, $B$ contains an $(n, F)$-ppt $P$. Note that $P = \text{pr}_X((P \times \mathcal{N}) \cap C)$. By [Ke, 29.20] there is a set $K \subset (P \times \mathcal{N}) \cap C$ such that the both $K$ and $\text{pr}_X(K)$ are homeomorphic with $\{0, 1\}^\mathbb{N}$. Since $\text{pr}_X(K) \subset P$ so $\text{pr}_X(K)$ is an $(n, F)$-ppt with $K \subset C$. Implication “$\Leftarrow$” is obvious.

Now, let $Z$ be a Polish space and let $A \subset X \times Z$ be an analytic set. Pick a closed set $C \subset X \times Z \times \mathcal{N}$ such that $\text{pr}_{X \times Z}(C) = A$. Then $A(x) = \text{pr}_Y(C(x))$ and $C(x) \subset Z \times \mathcal{N}$ is closed for each $x \in X$. For each $x \in X$ we have

$A(x) \notin J \iff (\exists K \in \mathcal{K}(Z \times \mathcal{N}))(K \subset C(x)$ and $\text{pr}_Z(K)$ contains a $(n, F)$-ppt).

Observe that the set $\{(x, K) \in X \times \mathcal{K}(Z \times \mathcal{N}): K \subset C(x)\}$ is closed and note that the mapping $K \mapsto \text{pr}_Z(K)$ from $\mathcal{K}(Z \times \mathcal{N})$ to $\mathcal{K}(Z)$ is continuous [Ke, 4.29(vi)]. Hence by Lemma 13 the assertion follows. \[\square\]

4. CLOSING REMARKS AND OPEN QUESTIONS

We say that ideals $J$ and $\mathcal{I}$ of subsets of a set $X$ are orthogonal, if there are sets $A \in J$ and $B \in \mathcal{I}$ with $A \cup B = X$.

**Theorem 14.** Let $J$ be a $\sigma$-ideal of subsets of an uncountable Polish space $X$, which is not orthogonal to the $\sigma$-ideal of meager subsets of $X$. If $J$ has KL property, then there is a family of continuum many pairwise disjoint $G_\delta$ sets which do not belong to $J$. 
Proof. Observe that if \( X \) and \( Y \) are uncountable Polish spaces and a \( \sigma \)-ideal \( \mathcal{J} \subset \mathcal{P}(X) \) has property (LK) then, for every Borel isomorphism \( \varphi : X \rightarrow Y \), the \( \sigma \)-ideal \( \{ \varphi(A) : A \in \mathcal{J} \} \subset \mathcal{P}(Y) \) has property (LK). Note that between any two perfect Polish spaces there is a Borel isomorphism preserving the Baire category (see e.g. [CKW, 3.15]). Hence we may assume that \( X = \{0,1\}^\mathbb{N} \).

Let \( \alpha \in \{0,1\}^\mathbb{N} \). Put \( A^\alpha_n = \{ x \in \{0,1\}^\mathbb{N} : x(n) = \alpha(n) \} \). If \( H \in [\mathbb{N}]^\mathbb{N} \), then \( \limsup_{n \in H} A^\alpha_n \) is a dense \( G_δ \). By the assumption

\[
\forall H \in [\mathbb{N}]^\mathbb{N} \quad \limsup_{n \in H} A^\alpha_n \notin \mathcal{J}.
\]

Let \( \{G_\beta : \beta < 2^{\aleph_0}\} \) be a family of almost disjoint sets from \([\mathbb{N}]^\mathbb{N}\). Then for any \( \beta < 2^{\aleph_0} \) and any \( \alpha = \chi_{G_\beta} \) (where \( \chi_{G_\beta} \) is a characteristic function of \( G_\beta \), i.e. \( \alpha(k) = 1 \iff k \in G_\beta \)) we have

\[
\forall H \in [G_\beta]^\mathbb{N} \quad \limsup_{n \in H} A^\alpha_n \notin \mathcal{J}.
\]

Since \( \mathcal{J} \) has a KL property, then there is \( H_\beta \in [G_\beta]^\mathbb{N} \) with

\[
\bigcap_{n \in H_\beta} A^\alpha_n \notin \mathcal{J}.
\]

Since for distinct \( \beta \) and \( \beta' \), the set \( H_\beta \) and \( H_{\beta'} \) are almost disjoint, the family

\[
\left\{ \bigcap_{n \in H_\beta} A^\alpha_n : \beta < 2^{\aleph_0} \right\}
\]

consist of pairwise-disjoint sets of type \( G_δ \) which do not belong to \( \mathcal{J} \). □

Corollary 15. Let \((X, \tau)\) be a Polish space. Assume that \( \mathcal{J} \) is a \( \sigma \)-ideal of subsets of \( X \), which is not orthogonal to all \( \sigma \)-ideals of meager subsets of \((X, \tau')\), where \( \tau' \) is a Polish topology which gives the same Borel sets as \((X, \tau)\). If \( \mathcal{J} \) has KL property, then there is a family of continuum many pairwise disjoint Borel sets which do not belong to \( \mathcal{J} \).

The next example shows that the assumptions in Corollary 15 are not always fullfiled.

Theorem 16. (CH) Let \((X, \tau)\) be a Polish space. There exist non-trivial \( \sigma \)-ideal of subsets of \( X \) with a Borel base, which is orthogonal to every \( \sigma \)-ideals of meager subsets of \((X, \tau')\) for any Polish topology giving the same Borel sets as \( \tau \).
Proof. Note that any Polish topology \((X, \tau')\) which gives the same Borel \(\sigma\)-algebra is Borel isomorphic to \((X, \tau)\). Any Borel isomorphism \(f\) is uniquely determined by preimages of sets \(U_n\), where \((U_n)\) is fixed base for \((X, \tau)\). Hence there are \(|B(X)|^\omega = \omega_1\) such Borel isomorphisms. Let \(\{\tau_\alpha : \alpha < \omega_1\}\) be a family of all Polish topologies on \(X\) giving the same Borel isomorphisms. Let \(\{\tau_\alpha : \alpha < \omega_1\}\) be a family of all Polish topologies on \(X\) giving the same Borel \(\sigma\)-algebra.

Let \(B_0\) be a dense \(G_\delta\) in \((X, \tau_0)\) such that \(X \setminus B_0\) is uncountable. Suppose that we have already defined pairwise disjoint sets \(\{B_\beta : \beta < \alpha\}\) for some \(\alpha < \omega_1\). We will define \(B_\alpha\). If \(\bigcup_{\beta<\alpha} B_\beta\) contain a dense \(G_\delta\) in \(\tau_\alpha\), then put \(B_\alpha = \emptyset\). Otherwise we find a dense \(G_\delta\) set \(A\) in \(\tau_\alpha\) such that \(X \setminus (\bigcup_{\beta<\alpha} B_\beta \cup A)\) is uncountable. Put \(B_\alpha = A \setminus \bigcup_{\beta<\alpha} B_\beta\).

Let \(J\) be \(\sigma\)-ideal generated by all singletons and family \(\{B_\alpha : \alpha < \omega_1\}\). Clearly \(J\) is a proper \(\sigma\)-ideal with Borel base which is orthogonal to each \(\sigma\)-ideal of meager sets in topologies on \(X\) giving the same Borel \(\sigma\)-algebra. \(\square\)

We end the paper with some open questions:

1. For uncountable Polish spaces \(X, Y\) and for \(\sigma\)-ideals \(\mathcal{I} \subset \mathcal{P}(X), \mathcal{J} \subset \mathcal{P}(Y)\), put

\[
\mathcal{I} \otimes \mathcal{J} = \{A \subset X \times Y : \{x \in X : A(x) \notin \mathcal{J}\} \in \mathcal{I}\}.
\]

Then \(\mathcal{I} \otimes \mathcal{J}\) forms a \(\sigma\)-ideal. Suppose that \(\mathcal{I}\) and \(\mathcal{J}\) have (LK) property. Does it follow that \(\mathcal{I} \otimes \mathcal{J}\) has (LK) property?

2. We will say that a \(\sigma\)-ideal \(\mathcal{J}\) of subsets of \(X\) has property \((M)\) if there is a Borel function \(f : X \to [0, 1]\) such that \(f^{-1}(x) \notin \mathcal{J}\) for every \(x \in [0, 1]\). Is it true that any \(\sigma\)-ideal \(\mathcal{J}\) with property (LK) has property (M)?

3. Let \(\mathcal{J}\) be a \(\sigma\)-ideal generated by a \((n,F)\)-system or by an equivalence relation of type \(F_\sigma\). Then \(\mathcal{J}\) has property (LK) and

\[(*)\] there is a perfect set \(P \notin \mathcal{J}\) such that \(P' \notin \mathcal{J}\) for any perfect subset \(P' \subset P\).

Is there any relation between property (LK) and property (\(\ast\))?}

**References**


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