LK PROPERTY FOR σ-IDEALS

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Abstract. An ideal $\mathcal{J}$ of subsets of a Polish space X has (LK) property if for every sequence (A_n) of analytic sets in X, if $\limsup_{n\in H} A_n \notin \mathcal{J}$ for each infinite H then $\bigcap_{n\in G} \notin \mathcal{J}$ for some infinite G. In this note we present a new class of σ-ideals with (LK) property.

1. Introduction

We use standard set theoretical notation (see [Ke] or [S]). Laczkovich in [L] proved that, for every sequence (A_n) of Borel subsets of a Polish space, if $\limsup_{n\in H} A_n$ is uncountable for each $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n$ is uncountable for some $G \in [\mathbb{N}]^\omega$. This result was then generalized by Komjáth [Ko, Thm 1] to the case when the sets A_n are analytic. Komjáth also proved that the result of Laczkovich cannot be generalized within ZFC to the case of coanalytic sets. Namely, if $V = L$, there is a sequence (A_n) of coanalytic sets such that $|\limsup_{n\in H} A_n| > \omega$ and $|\bigcap_{n\in H} A_n| \leq \omega$ for all $H \in [\mathbb{N}]^\omega$; see [Ko, Thm. 4].

In connection with the above quoted theorem of Komjáth about analytic sets, Balcerzak and Głąb in [BG] introduce the Laczkovich-Komjáth property of ideal $\mathcal{J}$ of subsets of Polish space X. The ideal $\mathcal{J}$ is said to have property (LK) whenever for every sequence (A_n) of analytic subsets of X, if $\limsup_{n\in H} A_n \notin \mathcal{J}$ for each $H \in [\mathbb{N}]^\omega$ then $\bigcap_{n\in G} A_n \notin \mathcal{J}$ for some $G \in [\mathbb{N}]^\omega$. In particular, the Komjáth theorem states that the ideal $[X]^{\leq\omega}$ has property (LK). Halmos [H] proved that the σ-ideal of null sets does not have (LK) property. We can reformulate the (LK) property in the following nice way. A σ-ideal $\mathcal{I}$ has (LK) property if for any sequence (A_n) of analytic sets either there is $H \in [\mathbb{N}]^\omega$ with $\limsup_{n\in H} B_n \in \mathcal{I}$ or there is $H \in [\mathbb{N}]^\omega$ with $\liminf_{n\in H} B_n \notin \mathcal{I}$.

In paper [BG] it was proved that for any Polish space X and any F_σ relation $E \subset X \times X$ with uncountable many equivalence classes, if $\mathcal{J}$ consist of all subsets of X that can be covered by countably many equivalence classes, then σ-ideal $\mathcal{J}$ has (LK) property. Note that σ-ideal $[X]^{\leq\omega}$ is of this form (it is enough to define xEy if and only if $x = y$, and

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observe that E is closed in $X \times X$). This note is devoted to show that there are other natural examples of σ-ideals with (LK) property. It seems (but we did not establish it) that these σ-ideals cannot be defined by F_σ equivalence relation.

The (LK) property was also studied by Zapletal [Z]. He proved several properties of this notion, and he gave some new examples of σ-ideals with (LK) property.

2. σ-ideals defined by (n, F)-system

Let X be a Polish space. By $\mathcal{K}(X)$ we denote the hyperspace of all nonempty compact subsets of X, endowed with the Vietoris topology, i.e. the topology generated by sets $\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ and $\{K \in \mathcal{K}(X) : K \subset U\}$ for any open sets U in X. The Vietoris topology is equal to the topology generated by the Hausdorff metric

$$\rho_H(K, L) = \max(\max_{x \in K} \rho(x, L), \max_{x \in L} \rho(x, K))$$

where $\rho(x, K)$ is the distance from a point x to a set K with respect to the metric ρ on X.

By $(X)^n$ denote the set $\{(x_1, ..., x_n) \in X^n : \forall i, j \leq n (i \neq j \Rightarrow x_i \neq x_j)\}$. Let S_n be the set of all permutations of the set $\{1, ..., n\}$. Let $Y \subset (X)^n$ be a Polish space. We say that Y is invariant under permutations of coefficients if for any permutation $\sigma \in S_n$ and any $(x_1, ..., x_n) \in (X)^n$

$$(x_1, ..., x_n) \in Y \iff (x_{\sigma(1)}, ..., x_{\sigma(n)}) \in Y.$$

From now on, we assume that Y is invariant under permutations of coefficients, and for any $x_1 \in X$ there are $x_2, ..., x_n \in X$ with $(x_1, x_2, ..., x_n) \in Y$.

Let $F : Y \to \mathcal{K}(X)$ be a continuous mapping such that for any permutation $\sigma \in S_n$ and any $(x_1, ..., x_n), (y_1, ..., y_n)$ from Y

(i) $F(x_1, ..., x_n) = F(x_{\sigma(1)}, ..., x_{\sigma(n)})$;
(ii) $\{x_1, ..., x_n\} \subset F(x_1, ..., x_n)$;
(iii) $\{y_1, ..., y_n\} \subset F(x_1, ..., x_n) \Rightarrow F(y_1, ..., y_n) = F(x_1, ..., x_n)$.

A family $\mathcal{S} = \{F(x_1, ..., x_n) : (x_1, ..., x_n) \in Y\}$ is called (n, F)–system. We say that σ–ideal $\mathcal{J}$ of subsets of X is generated by (n, F)–system $\mathcal{S}$, if $\mathcal{J}$ consists of all subsets of X which can be covered by countably many sets from $\mathcal{S}$. If $x \in X$, then there are $x_2, ..., x_n \in X$ with $(x, x_2, ..., x_n) \in Y$ and by (ii) we obtain $\{x\} \subset \{x, x_2, ..., x_n\} \subset F(x, x_2, ..., x_n)$. Hence $\mathcal{J}$ contains all singletons. If X cannot be covered by countably many elements from $\mathcal{S}$, then $\mathcal{J}$ is a proper ideal, i.e. $X \notin \mathcal{J}$ – in this case we say that (n, F)–system $\mathcal{S}$ is proper.

Example 1. Let $X = \mathbb{R}^2$. Let $Y \subset (X)^3$ be a set of all non-collinear triples. Then Y is a Polish subspace as an open subset of $(X)^3$:

$$Y = \{(x, y, z) \in (X)^3 : (y_2 - x_2)(z_1 - y_1) \neq (y_1 - x_1)(z_2 - y_2)\}.$$

For a triple $(x, y, z) \in Y$ by $F(x, y, z)$ denote the unique circle which contains the points x, y and z. Then the σ–ideal $\mathcal{J}$ generated by this $(3, F)$–system consists of all subsets of the real plane which can be covered by countably many circles.

Example 2. Let $X = [0, 1]^2$ and $Y = (X)^2$. For any pair $(x, y) \in Y$ let $F(x, y)$ be a line containing x and y intersected with the unite square X. Then the ideal $\mathcal{J}$ generated by this $(2, F)$–system consists of those subsets of the unite square X which can be covered by countably many lines. Here we restrict our attention to the unit square, since we want $F(x, y)$ to be compact.

Example 3. Let $X = [0, 1] \times \mathbb{R}$ and let

$$Y = \{((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) \in (X)^n : \forall i \neq j (x_i \neq x_j)\}.$$

For $\big((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\big) \in Y$ let $F\big((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\big)$ be a graph of the unique polynomial f of degree not grater than $n - 1$ with $f(x_i) = y_i$, $i = 1, ..., n$. Then the σ–ideal $\mathcal{J}$ generated by this (n, F)–system consists of all subsets of X which can be covered by countably many graphs of polynomials of degree not greater than $n - 1$.

Example 4. Let X be a Polish space, $E \subset X \times X$ be a closed equivalence relation with compact equivalence classes. Put $Y = X$ and put $F(x)$ to be an equivalence class of x, i.e. $F(x) = \{y : xEy\}$. Then the σ–ideal $\mathcal{J}$ generated by this $(1, F)$–system consists of all subsets of X which can be covered by countably many equivalence classes.

Conversely note that $(1, F)$-system defines an equivalence relation E_F by $xE_Fy \iff x \in F(y)$. The relation E_F is closed. Indeed, let $(x_n, y_n) \in E_F$ be such that $(x_n, y_n) \to (x_0, y_0)$. By the continuity of F we obtain that $F(x_n) \to F(x_0)$ and $F(y_n) \to F(y_0)$ in $\mathcal{K}(X)$. Since $x_n \in F(y_n)$, then $F(x_n) = F(y_n)$. Hence $F(x_n) = F(y_n)$ and therefore $F(x_0) = F(y_0)$, which means that $x_0E_Fy_0$.

It follows now, that being generated by a $(1, F)$-system is the same as being generated by a closed equivalence relation with compact equivalence classes. In [BG] it was shown that if an σ-ideal $\mathcal{J}$ is generated by a F_σ equivalence relation is proper, then it has (LK) property. From this reason we will consider only (n, F)-systems for $n \geq 2$.

Let $\mathcal{J}$ be a proper σ-ideal of subsets of X which contain all singletons. Fix a sequence (A_n) of analytic subsets of X such that

$$\forall H \in [\mathbb{N}]^\omega \quad \limsup_{n\in H} A_n \notin \mathcal{J}.$$

Fix $H \in [\mathbb{N}]^\omega$. We say that $Y \subset X$ is good with respect to H, if

$$Y \cap \limsup_{n\in G} A_n \notin \mathcal{J}$$

for any $G \in [H]^\omega$. Observe that if Y is good with respect to H and $Z \subset Y$, $Z \in \mathcal{J}$, then $Y \setminus Z$ is good with respect to H. In particular, if Y is closed and good with respect to H, then the perfect kernel of Y (cf. [S, 2.6.2]) is good with respect to H – we will use this fact several times.

Lemma 5. *If a set $Y = \bigcup_{i\in\mathbb{N}} Y_i$ is good with respect to $H \in [\mathbb{N}]^\omega$, then there are $i \in \mathbb{N}$ and $H' \in [H]^\omega$ such that Y_i is good with respect to H'.*

The proof is analogous to that given in [Ko, Lemma 1].

Lemma 6. *Let $P, A \subset X$ be such that P are closed, and $P \cap A$ is good with respect to some $H \in [\mathbb{N}]^\omega$. Then there exist $x \in P$ and $H' \in [H]^\omega$ such that for any neighborhood U of x the set $(P \setminus F) \cap A \cap U$ is good with respect to H'.*

The proof can be found in [BG].

Now, we assume that $\mathcal{J}$ is a σ-ideal generated by (n, F)–system $\mathcal{S}$, $n \geq 2$. Let $\mathcal{S} = \{F(x_1, ..., x_n) : (x_1, ..., x_n) \in Y\}$.

Lemma 7. *Let $R_1, ..., R_n, K_1, ..., K_m, A \subset X$ and $H \in [\mathbb{N}]^\omega$. Assume that R_j, K_i are pairwise disjoint, and $R_j \cap A$, $K_i \cap A$ are good with respect to H, $j = 1, ..., n$ and $i = 0, ..., m$. Then there are closed sets $R'_j \subset R_j$ and $K'_i \subset K_i$ with*

$$\forall (x_1, ..., x_n) \in R'_1 \times \cdots \times R'_n \left(dist\left(F(x_1, ..., x_n), \bigcup_{i=1}^m K'_i \right) > 0 \right),$$

and there is $H' \in [H]^{\mathbb{N}}$ such that $R'_j \cap A$, $K'_i \cap A$ are good with respect to H', for $j = 1, ..., n$ and $i = 0, ..., m$.

Proof. If for every $j = 1, ..., n$ and every $x_j \in R_j$

$$dist\left(F(x_1, ..., x_n), \bigcup_{i=1}^m K_i \right) > 0,$$

then putting $R'_j = R_j$ and $P'_i = P_i$, we are done. If not, then by Lemma 6 (for $P = R_1$) there exist: a point $x_1 \in R_1$ and a set $H_1 \in [H]^\omega$ such that for any neighborhood U_1 of x_1 the set $R_1 \cap A \cap U_1$ is good with respect to H_1. Using again Lemma 6 (this time for $P = R_2$) we find a point $x_2 \in R_1$ and a set $H_2 \in [H_1]^\omega$ such that for any neighborhood U_2 of x_2 the set $R_2 \cap A \cap U_2$ is good with respect to H_2. In that way we find points $x_1, x_2, ..., x_n$ and $H_n \in [H]^\mathbb{N}$ such for any $j = 1, ..., n$ and any neighborhood U_j of x_j the set $(R_j \setminus \{x_1, ..., x_n\}) \cap A \cap U_j$ is good with respect to H_n.

Now, for every $i = 1, ..., m$ we will define a set H_{n+i} and a number k_i in the following way. If $F(x_1, ..., x_n) \cap K_i = \emptyset$, then put $K'_i = K_i$, $H_{n+i} = H_{n+i-1}$ and $k_i = 0$. Otherwise let

$$Y_k = \{x \in K_i : dist(x, F(x_1, x_2, ..., x_n)) \geq \frac{1}{k+1}\}.$$

Then $\bigcup_{k\in\mathbb{N}} Y_k = K_i \setminus F(x_1, x_2, ..., x_n)$. Since K_i is good with respect to H_{n+i-1}, then by Lemma 5 we find a number k_i and a set $H_{n+i} \in [H_{n+i-1}]^\omega$ such that $Y_{k_i} \cap A$ is good with respect to H_{n+i}. Put $K'_i = Y_{k_i}$. Define

$$\varepsilon = \min\left\{\frac{1}{k_i+1} : i = 1, ..., m\right\}.$$

By the continuity of F we will find neighborhoods $V_1, ..., V_n$ of points $x_1, ..., x_n$, respectively, such that the diameter of $F\big(cl(V_1), ..., cl(V_n)\big)$ is less than $\varepsilon/2$. Define $R'_j = cl(V_j)$ and $H' = H_{n+m}$. □

Lemma 8. *Let $m \in \mathbb{N}$, $P_0, ..., P_m, A \subset X$. Assume that for $i = 0, ..., m$ the sets P_i are closed, pairwise disjoint and such that any set from $\mathcal{S}$ does not intersect more than $n+1$ sets P_i. Let $H \in [\mathbb{N}]^\omega$ and $\varepsilon > 0$. If $P_i \cap A$ is good with respect to H and $i = 0, ..., m$, then there are pairwise disjoint closed sets $P'_m, ..., P'_{m+n-1} \subset P_m$, $P'_i \subset P_i$ for $i < m$ and there is $H' \in [H]^\omega$ such that each P'_i has diameter less than ε, any set from $\mathcal{S}$ does not intersect more than $n+1$ sets P'_i, and sets $A \cap P'_i$ are good with respect to H'.*

Proof. In the same way as in Lemma 7 we find a set $H_{-1} \in [H]^\mathbb{N}$ and points $x_m, x_{m+1}, ..., x_{m+n-1}$ in P_m such that any $i = 0, ..., n-1$ and any neighborhood U_{m+i} of point x_{m+i} sets $U_{m+i} \cap P_m \cap A$ is good with respect to H_{-1}.

Inductively for $i = 0, 1, ..., m-1$ we define P'_i and H_i in the following way. If distance between $F(x_m, x_{m+1}, ..., x_{m+n-1})$ and P_i is greater than zero, then we put $P'_i = P_i$. Otherwise let

$$Y_k = \{x \in P_i : dist(x, F(x_m, x_{m+1}, ..., x_{m+n-1})) \geq \frac{1}{k+1}\}.$$

Then $\bigcup_{k\in\mathbb{N}} Y_k = P_i \setminus F(x_m, x_{m+1}, ..., x_{m+n-1})$. Since P_i is good with respect to H_{i-1}, then by Lemma 5 we find a number k and a set $H_i \in [H_{i-1}]^\omega$ such that $Y_k \cap A$ is good with respect to H_i. We may assume that $diam(Y_k) < \varepsilon$. Put $P'_i = Y_k$.

Now, let $\delta > 0$ be such that for any $i \in \{0, 1, ..., m-1\}$ the distance from P'_i to the set $F(x_m, x_{m+1}, ..., x_{m+n-1})$ is greater than δ. By continuity of F we find neighborhoods U_m, U_{m+1},..., U_{m+n-1} of points x_m, x_{m+1},..., x_{m+n-1}, respectively, such that

$$diam(F(U_m, U_{m+1}, ..., U_{m+n-1})) < \delta.$$

Put $P'_m = cl(U_m)$,..., $P'_{m+n-1} = cl(U_{m+n-1})$.

Now, any set from $\mathcal{S}$ which intersect $P'_m, ..., P'_{m+n-1}$ does not intersect any of $P'_0, ..., P'_{m-1}$. But our choice guarantees that any set from $\mathcal{S}$ does not intersect more than $n+1$ sets $P'_0, ..., P'_{m+n-1}$ only in the case if $n = 2$. If $n > 2$, we will shrink each P'_i and H' finitely many times in $n-2$ steps.

In the first step we use Lemma 7 for $H = H'$, $R_1 = P'_m, ..., R_{n-1} = P'_{m+n-2}, R_n = P'_{m-1}$ and $K_1 = P_1, ..., K_{m-1} = P_{m-1}, K_m = P'_{m+n-1}$ to find $H'' \in [H']^{\mathbb{N}}$ and closed sets $R'_1, ..., R'_{n-1}, R'_n$ and $K'_1, ..., K'_{m-1}, K'_m$ such that $R'_j \subset R_j$, $K'_i \subset K_i$,

$$\forall (x_1, ..., x_n) \in R'_1 \times \cdots \times R'_n \left(dist\left(F(x_1, ..., x_n), \bigcup_{i=1}^{m} K'_i \right) > 0 \right),$$

and such that $R'_j \cap A$, $K'_i \cap A$ are good with respect to H''. Let H'' be our new choice for H', R'_1 be a new choice for P'_m etc. Now, we use Lemma 7 for H' and each combination $R_1, ..., R_{n-1}, R_n$ such that $R_1, ..., R_{n-1}$ is collection of $n-1$ sets from $P'_m, ..., P'_{m+n-1}$ and R_n is a set from $P'_0, ..., P'_{m-1}$, and $K_1, ..., K_m$ the remaining sets from $P'_0, ..., P'_{m+n-1}$. After this we obtain that any set which intersect $n-1$ sets from $P'_m, ..., P'_{m+n-1}$ do intersect at least one set from $P'_0, ..., P'_{m-1}$.

In the second step we use Lemma 7 for H' and each combination $R_1, ..., R_{n-1}, R_n$ such that $R_1, ..., R_{n-2}$ is collection of $n-2$ sets from $P'_m, ..., P'_{m+n-1}$ and R_{n-1}, R_n are sets from $P'_0, ..., P'_{m-1}$, and $K_1, ..., K_m$ the remaining sets from $P'_0, ..., P'_{m+n-1}$. After this we obtain that any set which intersect $n-2$ sets from $P'_m, ..., P'_{m+n-1}$ do intersect at least two sets from $P'_0, ..., P'_{m-1}$.

In the last $n-2$th step we use Lemma 7 for H' and each combination $R_1, ..., R_{n-1}, R_n$ such that R_1, R_2 is collection of 2 sets from $P'_m, ..., P'_{m+n-1}$ and $R_3, ..., R_n$ are sets from $P'_0, ..., P'_{m-1}$, and $K_1, ..., K_m$ the remaining sets from $P'_0, ..., P'_{m+n-1}$. After this we obtain that any set which intersect 2 sets from $P'_m, ..., P'_{m+n-1}$ do intersect at least $n-2$ sets from $P'_0, ..., P'_{m-1}$.

Suppose that there is $D \in S$ which intersects $n+1$ sets from $P'_1, ..., P'_{m+n-1}$. Then D cannot intersect more than one set from $P'_m, ..., P'_{m+n-1}$. Hence D intersects $n+1$ sets from $P_1, ..., P_m$ which contradicts our assumptions. $\square$

The next theorem shows that σ-ideals genereted by (n, F)-systems have (LK) property. Its proof is quite similar to that in [Ko]. The main difference lays in Lemma 8.

Theorem 9. *Let $\mathcal{J}$ be a σ–ideal generated by a proper (n, F)–system $\mathcal{S}$. Then for any sequence (A^j) of analytic sets such that*

$$\forall H \in [\mathbb{N}]^\omega \quad \limsup_{j \in H} A^j \notin \mathcal{J}$$

there exist: a set $G \in [\mathbb{N}]^\omega$ and a homeomorph P of the Cantor set 2^ω such that any $n+1$ distinct point of P are not the member of the same set from family $\mathcal{S}$ and such that $P \subset \bigcap_{j \in G} A^j$. In particular, a σ–ideal $\mathcal{J}$ has (KL) property.

Proof. We may assume that X is a perfect set (if not, then removing countably many points from X we obtain a perfect set). Additionally we may assume that $diam(X) < 1$. Let A^j be a sequence of analytic sets with

$$\forall H \in [\mathbb{N}]^\omega \quad \limsup_{j \in H} A^j \notin \mathcal{J}.$$

We may write A^j using a Suslin operation (cf. [Ke, 25.7]):

$$A^j = \bigcup_{z \in \mathbb{N}^\mathbb{N}} \bigcap_{m \in \mathbb{N}} C^j_{z|m},$$

where $C^j_{z|m}$ are closed with $diam(C^j_{z|m}) < \frac{1}{m+1}$ and

$$\forall k, m \in \mathbb{N}(k > m \Rightarrow C^j_{z|k} \subset C^j_{z|m}).$$

For $s \in \mathbb{N}^m$ put $A^j_s = \bigcup_{z \in \mathbb{N}^\mathbb{N}, z|m=s} \bigcap_{k \in \mathbb{N}} C^j_{z|k}$.

Without loss of generality we may assume that $A^0 = X$. Our construction will be inductive. In the m-th step we choose a number $j_m \in \mathbb{N}$, perfect sets P_s $(s \in \{1, ..., n\}^m)$, finite sequences $t(k, s) \in \mathbb{N}^m$ $(k \leq m,\ s \in \{1, ..., n\}^m)$ and a set $H_m \in [\mathbb{N}]^\omega$ fulfilling the following conditions

(W1) $j_m > j_{m-1}$, $H_m \in [H_{m-1}]^\omega$, $j_m \in H_{m-1}$;

(W2) $P_{s\hat{\ }i} \subset P_s$ for $i \in \{1, ..., n\}$, $P_{s\hat{\ }i}$ are pairwise disjoint for $s \in \{1, ..., n\}^{m-1}$, and any set from $\mathcal{S}$ does not intersect $n+1$ or more sets from $\{P_s : s \in \{1, ..., n\}^m\}$;

(W3) $diam(P_s) < \frac{1}{m+1}$ for $s \in \{1, ..., n\}^m$;

(W4) $P_s \cap A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)}$ is good with respect to H_m, if $s \in \{1, ..., n\}^m$;

(W5) $P_s \subset C^{j_0}_{t(0,s)} \cap ... \cap C^{j_m}_{t(m,s)}$ for $s \in \{1, ..., n\}^m$;

(W6) $t(k, s) \subset t(k, s\hat{\ }i)$, for $i \in \{1, ..., n\}$, $s \in \{1, ..., n\}^{m-1}$ and $k \leq m-1$.

Conditions (2) and (3) guarantee that the set

$$P = \bigcap_{m\in\mathbb{N}} \bigcup_{s\in\{1,...,n\}^m} P_s$$

is perfect and that any set from $\mathcal{S}$ does not contain $n+1$ or more points from P. Hence $P \notin \mathcal{J}$. If $x \in P$, then from (2) it follows that for any $m \in \mathbb{N}$ there is an unique sequence s_m with $x \in P_{s_m}$. Moreover $s_0 \subset s_1 \subset s_2 \subset ....$ Fix $i \in \mathbb{N}$. From (5) for $m \geq i$ we obtain that $x \in C^{j_i}_{t(i,s_m)}$, and by (6) we get $t(i, s_i) \subset t(i, s_{i+1}) \subset t(i, s_{i+2}) \subset ....$ Hence $x \in A^{j_i}_{t(i,s_i)} \subset A^{j_i}$. Finally $P \subset \bigcap_{i\in\mathbb{N}} A^{j_i}$, and putting $G = \{j_0, j_1, ...\}$ we obtain the assertion.

It suffices to define the fulfilling (1)–(6). We will construct them by induction on m. Put $j_0 = 0$, $P_\emptyset = X$, $H_0 = \mathbb{N}$. Clearly, X is good with respect to $\mathbb{N}$. Putting $t(0, \emptyset) = \emptyset$, we define objects fulfilling (1)–(6) for the first step.

Assume that for $m \in \mathbb{N}$ we already choose j_k (for $k \leq m$), P_s (for $s \in \{1, ..., n\}^k$, $k \leq m$), $t(k, s)$ (for $k \leq l \leq m$, $s \in \{1, ..., n\}^l$) and H_k (for $k \leq m$).

At first we show that there exist a number $j \in H_m$, $j > j_m$, and a set $H'_m \in [H_m]^\omega$ such that

(7) $\forall s \in \{1, ..., n\}^m (P_s \cap A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)} \cap A^j)$ is good with respect to H'_m).

Assume to the contrary that for any $j \in H_m$, $j > j_m$, and for any $H \in [H_m]^\omega$ we have

$$\exists G \in [H]^\omega\ \exists s \in \{1, ..., n\}^m\ (P_s \cap A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)} \cap A^j \cap \limsup_{r\in G} A^r \in \mathcal{J}).$$

Proceeding inductively, we find numbers $k_0 < k_1 < ...$ and sets $H_m = G_0 \supset G_1 \supset ...$ such that $k_r \in G_r \in [\mathbb{N}]^\omega$ and

$$\forall r \in \mathbb{N}\ \exists s_r \in \{1, ..., n\}^m\ (P_{s_r} \cap A^{j_0}_{t(0,s_r)} \cap ... \cap A^{j_m}_{t(m,s_r)} \cap A^{k_r} \cap \limsup_{p\in G_{r+1}} A^p \in \mathcal{J}).$$

Since there is only n^m possibilities of choosing s_r, there is a sequence $s \in \{1, ..., n\}^m$ such that a set $\Gamma = \{k_r : s_r = s\}$ is infinite. Then Γ is almost contained in G_r, for every $r \in \mathbb{N}$.

So we obtain

$$P_s \cap A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)} \cap (\bigcup_{r\in\Gamma} A^r) \cap \limsup_{p\in\Gamma} A^p \in \mathcal{J}.$$

But this is impossible, since $\limsup_{p\in\Gamma} A^p \subset \bigcup_{r\in\Gamma} A^r$ and (4). Hence there is a number $j > j_m$, $j \in H_m$, fulfilling (7). It is our choice for j_{m+1}.

Using n^m many times (7) and Lemma 8 to the sets $\{P_s : s \in \{1,...,n\}^m\}$, and considering perfect kernels of appropriate closed sets we will find pairwise disjoint perfect sets $\overline{P}_{s\hat{}i}$, for $i \in \{1,...,n\}$ with $diam(\overline{P}_{s\hat{}i}) < \frac{1}{m+1}$, and such that any set from $\mathcal{S}$ have no common point with $n+1$ or more sets from $\{P_s : s \in \{1,...,n\}^{m+1}\}$, and a set $H''_m \in [H'_m]^\omega$ such that for any $s \in \{1,...,n\}^m$ and $i = 1,...,n$ we have

$$\overline{P}_{s\hat{}i} \cap A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)} \cap A^{j_{m+1}} \text{ is good with respect to } H''_m.$$

The set $A^{j_0}_{t(0,s)} \cap ... \cap A^{j_m}_{t(m,s)} \cap A^{j_{m+1}}$ is contained in the following union

$$\bigcup_{z_0\in\mathbb{N}^{m+1}, z_0\supset t(0,s)} \cdots \bigcup_{z_m\in\mathbb{N}^{m+1}, z_m\supset t(m,s)} \bigcup_{z_{m+1}\in\mathbb{N}^{m+1}} (A^{j_0}_{z_0} \cap ... \cap A^{j_{m+1}}_{z_{m+1}}).$$

By Lemma 5 it follows that some element of this union is good with respect to H''_m. Using n^{m+1} times Lemma 5, we will find $\overline{H}_m$ with that property for all $s \in \{1,...,n\}^m$ and all $i = 1,...,n$. We define sequences $t(0, s\hat{}i)$,..., $t(m+1, s\hat{}i)$ as z_0,...,z_{m+1} corresponding to $s\hat{}i$. We finally need only to "repair" sets $\overline{P}_{s\hat{}i}$ to fulfill (5). To do this put

$$Q_{s\hat{}i} = \overline{P}_{s\hat{}i} \cap C^{j_0}_{t(0,s\hat{}i)} \cap ... \cap C^{j_{m+1}}_{t(m+1,s\hat{}i)}.$$

Since for every s we have $A^j_s \subset C^j_s$, then the sets $Q_{s\hat{}i}$ and $\overline{P}_{s\hat{}i}$ have the same intersection with

$$A^{j_0}_{t(0,s\hat{}i)} \cap ... \cap A^{j_{m+1}}_{t(m+1,s\hat{}i)}.$$

Hence (4) valid. Removing from each closed set $Q_{s\hat{}i}$ at most countably many point we obtain its perfect kernel $P_{s\hat{}i}$. It is still good with respect to $\overline{H}_m$, which will be our choice for H_{m+1}. Therefore conditions (1)–(6) are fulfilled. □

In [EKM] it was proved that if an analytic set on the real plane cannot be covered by coutably many lines then it contains a perfect set which also cannot be covered by countably many lines. We can generalized this in the following.

Corollary 10. *Let A an analytic subset of the plane and let $\mathcal{J}$ be a σ-ideal generated by a proper (n,F)-system $\mathcal{S}$. If $A \notin \mathcal{J}$, then there is $P \subset A$ a homeomorph of the Cantor set such that any $n+1$ points of P are not contained in the same set from family $\mathcal{S}$.*

Proof. It is enough to put $A_m = A$ for any $m \in \mathbb{N}$. □

Assume that $\mathcal{J} \subset \mathcal{P}([0,1]\times\mathbb{R})$ consist of those subsets of $[0,1]\times\mathbb{R}$ which can be covered by countably many graphs of polynomials. This σ-ideal is not of the form we considered in the previous section. But it is still very similar. This led us to the following definition. Let $\{(n_i,F_i)\}_{i\in\mathbb{N}}$ be a sequence of (n_i,F_i) systems. Let $\mathcal{S}_i = \{F_i(x_1,...,x_{n_i}) : x_1,...,x_{n_i} \in Y_i\}$. We say that $\mathcal{J}$ is generated by $\{(n_i,F_i)\}_{i\in\mathbb{N}}$ if $\mathcal{J}$ consists of those sets which can be covered by countably many sets from $\mathcal{S} = \bigcup_{i\in\mathbb{N}} \mathcal{S}_i$. Then the proof that $\mathcal{J}$ has (LK) property goes in an analogous way as the proof of Theorem 9. In the proof we need only to change condition (W2) to

(W2') $P_{s\hat{\ }i} \subset P_s$ for $i \in \{1,...,n\}$, $P_{s\hat{\ }i}$ are pairwise disjoint for $s \in \{1,...,n\}^{m-1}$, and any set from $\mathcal{S}_1,...,\mathcal{S}_m$ does not intersect more than $n_1,...,n_m$ sets from $\{P_s : s \in \{1,...,n\}^m\}$, respectively.

Proving the existence of such $P_{s\hat{\ }i}$ we use Lemma 8 for $\mathcal{S}_1$, then for $\mathcal{S}_2$, etc.

Using this one can get the following interesting colloraly. Let $A \subset \mathbb{R}^2$ be analytic. Suppose that A cannot be covered by countably many graphs of polynomials. Then there is a perfect set $P \subset A$ such that any n points of P cannot be covered by the graph of polymonial of degree less than n.

3. Parametric Laczkovich-Komjáth property

By the Mazurkiewicz-Sierpiński theorem [Ke, 29.19], if X, Z are Polish spaces then for each analytic set $A \subset X \times Z$, the set $\{x \in X : |A(x)| > \omega\}$ is also analytic. We say that an ideal $\mathcal{J} \subset \mathcal{P}(Z)$ has the Mazurkiewicz-Sierpiński property if for any Polish space X and analytic set $A \subset X \times Z$, the set $\{x \in X : A(x) \notin \mathcal{J}\}$ is analytic. This property holds true, besides the ideal of countable sets, the ideal of meager sets in Z and the ideal of Lebesgue null sets in $\mathbb{R}$. Ideal which has Mazurkiewicz-Sierpiński property is also called Π^1_1-on-Σ^1_1.

We say that an ideal $\mathcal{J}$ of subsets of Z has parametric property (LK), whenever for every uncountable Polish space X and every sequence (A_n) of analytic subsets of $X \times Z$, if $\limsup_{n\in H} A_n(x) \notin \mathcal{J}$ for all $x \in X$ and $H \in [\mathbb{N}]^\omega$ then there are a perfect set $P \subset X$ and

$G \in [\mathbb{N}]^\omega$ such that $\bigcap_{n\in G} A_n(x) \notin \mathcal{J}$ for each $x \in P$. In [G], it was proved that the ideal $[Z]^{\leq\omega}$ of all countable subsets of Y has parametric property (LK). In [BG], it was proved that the σ-ideal generated by F_σ equivalence relation has parametric property (LK). The proof in [BG] was based on the fact that σ-ideal generated by F_σ equivalence relation has Mazurkiewicz-Sierpiński property and following fact:

Proposition 11 ([BG]). *Let Z be an uncountable Polish space and let $\mathcal{J} \subset \mathcal{P}(Z)$ be a σ-ideal with property (LK) and with Mazurkiewicz-Sierpiński property. Then $\mathcal{J}$ has parametric property (LK).*

Now, we will prove that σ-ideals generated by (n, F)-systems have Mazurkiewicz-Sierpiński property. As a corollary we will obtain that σ-ideals generated by (n, F)-systems have parametric property (LK). We say that P is a perfect partial transversal (in short ppt) for (n, F)-system $\mathcal{S}$ if P is perfect and $x_{n+1} \notin F(x_1, ..., x_n)$ for any $x_1, ..., x_n, x_{n+1} \in P$; it is the same as saying that no $n+1$ points of P are contained in the same member of family $\mathcal{S}$.

Lemma 12. *Let X be an uncoutable Polish space and consider (n, F)-system defined on X. Then the family of all sets $L \in \mathcal{K}(X)$ containing a perfect partial transversal for (n, F)-system is analytic.*

Proof. Fix a contable base (U_n) for X. For $L \in \mathcal{K}(X)$ we have the following equivalence

$$L \text{ contains a ppt for } (n, F)\text{-system} \iff \exists K \in \mathcal{K}(L) \forall m \in \mathbb{N} \forall i_1, ..., i_{n+1} \in \mathbb{N}$$

$$(\forall k = 1, ..., n+1 \; U_{i_k} \cap K \neq \emptyset \Rightarrow \exists j_1, ..., j_{n+1} \in \mathbb{N} \forall k = 1, ..., n+1 \; clU_{j_k} \subset U_{i_k},$$
$$diam U_{j_k} < \frac{1}{m+1}, U_{j_k} \cap K \neq \emptyset, F(U_{j_1}, ..., U_{j_n}) \cap U_{j_{n+1}} = \emptyset.$$

Hence, in a standard way (cf. [Ke, 4.29], [S, 2.4.11]) we show that the family of all sets $L \in \mathcal{K}(Y)$ containing an (n, F)-ppt is analytic. Thus to finish the proof it suffices to show that the equivalence does hold.

If $L \in \mathcal{K}(Y)$ contains an (n, F)-ppt K, we easily conclude that K satisfies the right hand side of the equivalence. Conversely, if $K \in \mathcal{K}(L)$ satisfies the right hand side of the equivalence, we can define by recursion a family $\{V_s \colon s \in \{1, ..., n\}^{<\mathbb{N}}\} \subset \{U_i \colon i \in \mathbb{N}\}$ such that for each $s \in \{1, ..., n\}^{<\mathbb{N}}$ the following conditions hold:

(i) $V_s \cap K \neq \emptyset$;

(ii) $clV_{s\hat{}1} \cup ... \cup clV_{s\hat{}n} \subset V_s$, $clV_{s\hat{}1}, ..., clV_{s\hat{}n}$ are pairwise disjoint;

(iii) $diamV_s < 1/(|s|+1)$;

and additionally,

(iv) $F(V_{s_1}, ..., V_{s_n}) \cap V_{s_{n+1}} = \emptyset$ for all $m \in \mathbb{N}$ and distinct $s_1, ..., s_{n+1} \in \{1, ..., n\}^m$.

The construction is similar to that given in the proof of Theorem 9 (cf. conditions (W1)–(W3)). Then $\bigcap_{m\in\mathbb{N}} \bigcup_{s\in\{1,\dots,n\}^m} (K \cap clV_s)$ is an (n, F)-ppt contained in L.

□

Theorem 13. *Let X be an uncoutable Polish space and consider (n, F)-system $\mathcal{S}$ defined on X. Then the σ-ideal $\mathcal{J}$ generated by (n, F)-system $\mathcal{S}$ has the Mazurkiewicz-Sierpiński property.*

Proof. Set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. For an analytic set $B \subset X$ pick a closed set $C \subset X \times \mathcal{N}$ such that $\mathrm{pr}_X(C) = B$ where pr_X stands for the projection from $X \times \mathcal{N}$ to X. Observe that

$$B \notin \mathcal{J} \iff (\exists K \in \mathcal{K}(X \times \mathcal{N}))(K \subset C \text{ and } \mathrm{pr}_X(K) \text{ contains a } (n, F)\text{-ppt}).$$

Indeed, to show "⇒" assume that $B \notin \mathcal{J}$. By Corollary 10, B contains an (n, F)-ppt P. Note that $P = \mathrm{pr}_X((P \times \mathcal{N}) \cap C)$. By [Ke, 29.20] there is a set $K \subset (P \times \mathcal{N}) \cap C$ such that the both K and $\mathrm{pr}_X(K)$ are homeomorphic with $\{0,1\}^{\mathbb{N}}$. Since $\mathrm{pr}_X(K) \subset P$ so $\mathrm{pr}_X(K)$ is an (n, F)-ppt with $K \subset C$. Implication"⇐" is obvious.

Now, let Z be a Polish space and let $A \subset X \times Z$ be an analytic set. Pick a closed set $C \subset X \times Z \times \mathcal{N}$ such that $\mathrm{pr}_{X\times Z}(C) = A$. Then $A(x) = \mathrm{pr}_Y(C(x))$ and $C(x) \subset Z \times \mathcal{N}$ is closed for each $x \in X$. For each $x \in X$ we have

$$A(x) \notin \mathcal{J} \iff (\exists K \in \mathcal{K}(Z \times \mathcal{N}))(K \subset C(x) \text{ and } \mathrm{pr}_Z(K) \text{ contains an } (n, F)\text{-ppt}).$$

Observe that the set $\{(x, K) \in X \times \mathcal{K}(Z \times \mathcal{N}) \colon K \subset C(x)\}$ is closed and note that the mapping $K \mapsto \mathrm{pr}_Z(K)$ from $\mathcal{K}(Z \times \mathcal{N})$ to $\mathcal{K}(Z)$ is continuous [Ke, 4.29(vi)]. Hence by Lemma 13 the assertion follows. □

4. Closing remarks and open questions

We say that ideals $\mathcal{J}$ and $\mathcal{I}$ of subsets of a set X are orthogonal, if there are sets $A \in \mathcal{J}$ and $B \in \mathcal{I}$ with $A \cup B = X$.

Theorem 14. *Let $\mathcal{J}$ be a σ-ideal of subsets of an uncountable Polish space X, which is not orthogonal to the σ-ideal of meager subsets of X. If $\mathcal{J}$ has KL property, then there is a family of continuum many pairwise disjoint G_δ sets which do not belong to $\mathcal{J}$.*

Proof. Observe that if X and Y are uncountable Polish spaces and a σ-ideal $\mathcal{J} \subset \mathcal{P}(X)$ has property (LK) then, for every Borel isomorphism $\varphi : X \to Y$, the σ-ideal $\{\varphi(A) : A \in \mathcal{J}\} \subset \mathcal{P}(Y)$ has property (LK). Note that between any two perfect Polish spaces there is a Borel isomorphism preserving the Baire category (see e.g. [CKW, 3.15]). Hence we may assume that $X = \{0,1\}^{\mathbb{N}}$.

Let $\alpha \in \{0,1\}^{\mathbb{N}}$. Put $A_n^\alpha = \{x \in \{0,1\}^{\mathbb{N}} : x(n) = \alpha(n)\}$. If $H \in [\mathbb{N}]^{\mathbb{N}}$, then $\limsup_{n\in H} A_n^\alpha$ is a dense G_δ. By the assumption

$$\forall H \in [\mathbb{N}]^{\mathbb{N}} \quad \limsup_{n\in H} A_n^\alpha \notin \mathcal{J}.$$

Let $\{G_\beta : \beta < 2^{\aleph_0}\}$ be a family of almost disjoint sets from $[\mathbb{N}]^{\mathbb{N}}$. Then for any $\beta < 2^{\aleph_0}$ and any $\alpha = \chi_{G_\beta}$ (where χ_{G_β} is a characteristic function of G_β, i.e. $\alpha(k) = 1 \iff k \in G_\beta$) we have

$$\forall H \in [G_\beta]^{\mathbb{N}} \quad \limsup_{n\in H} A_n^\alpha \notin \mathcal{J}.$$

Since $\mathcal{J}$ has a KL property, then there is $H_\beta \in [G_\beta]^{\mathbb{N}}$ with

$$\bigcap_{n\in H_\beta} A_n^\alpha \notin \mathcal{J}.$$

Since for distinct β and β', the set H_β and $H_{\beta'}$ are almost disjoint, the family

$$\left\{ \bigcap_{n\in H_\beta} A_n^\alpha : \beta < 2^{\aleph_0} \right\}$$

consist of pairwise-disjoint sets of type G_δ which do not belong to $\mathcal{J}$. □

Corollary 15. *Let (X,τ) be a Polish space. Assume that $\mathcal{J}$ is a σ-ideal of subsets of X, which is not orthogonal to all σ-ideals of meager subsets of (X,τ'), where τ' is a Polish topology which gives the same Borel sets as (X,τ). If $\mathcal{J}$ has KL property, then there is a family of continuum many pairwise disjoint Borel sets which do not belong to $\mathcal{J}$.*

The next example shows that the assumptions in Corollary 15 are not always fullfiled.

Theorem 16. *(CH) Let (X,τ) be a Polish space. There exist non-trivial σ-ideal of subsets of X with a Borel base, which is orthogonal to every σ-ideals of meager subsets of (X,τ') for any Polish topology giving the same Borel sets as τ.*

Proof. Note that any Polish topology (X,τ') which gives the same Borel σ-algebra is Borel isomorphic to (X,τ). Any Borel isomorphism f is uniquely determined by preimages of sets U_n, where (U_n) is fixed base for (X,τ). Hence there are $|\mathcal{B}(X)|^\omega = \omega_1$ such Borel isomorphisms. Let $\{\tau_\alpha : \alpha < \omega_1\}$ be a family of all Polish topologies on X giving the same Borel sets as (X,τ).

Let B_0 be a dense G_δ in (X,τ_0) such that $X\setminus B_0$ is uncountable. Suppose that we have already defined pairwise disjoint sets $\{B_\beta : \beta<\alpha\}$ for some $\alpha<\omega_1$. We will define B_α. If $\bigcup_{\beta<\alpha} B_\beta$ contain a dense G_δ in τ_α, then put $B_\alpha=\emptyset$. Otherwise we find a dense G_δ set A in τ_α such that $X\setminus(\bigcup_{\beta<\alpha} B_\beta\cup A)$ is uncountable. Put $B_\alpha = A\setminus\bigcup_{\beta<\alpha} B_\beta$.

Let $\mathcal{J}$ be σ-ideal generated by all singletons and family $\{B_\alpha : \alpha<\omega_1\}$. Clearly $\mathcal{J}$ is a proper σ-ideal with Borel base which is orthogonal to each σ-ideal of meager sets in topologies on X giving the same Borel σ-algebra. □

We end the paper with some open questions:

1. For uncountable Polish spaces X,Y and for σ-ideals $\mathcal{I}\subset\mathcal{P}(X)$, $\mathcal{J}\subset\mathcal{P}(Y)$, put

$$\mathcal{I}\otimes\mathcal{J} = \{A\subset X\times Y : \{x\in X : A(x)\notin\mathcal{J}\}\in\mathcal{I}\}.$$

Then $\mathcal{I}\otimes\mathcal{J}$ forms a σ-ideal. Suppose that $\mathcal{I}$ and $\mathcal{J}$ have (LK) property. Does it follow that $\mathcal{I}\otimes\mathcal{J}$ has (LK) property?

2. We will say that a σ-ideal $\mathcal{J}$ of subsets of X has *property (M)* if there is a Borel function $f\colon X\to[0,1]$ such that $f^{-1}(x)\notin\mathcal{J}$ for every $x\in[0,1]$. Is it true that any σ-ideal $\mathcal{J}$ with property (LK) has property (M)?

3. Let $\mathcal{J}$ be a σ-ideal generated by a (n,F)-system or by an equivalence relation of type F_σ. Then $\mathcal{J}$ has property (LK) and

($\star$) there is a perfect set $P\notin\mathcal{J}$ such that $P'\notin\mathcal{J}$ for any perfect subset $P'\subset P$.

Is there any relation between property (LK) and property ($\star$)?

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