DICHOTOMIES FOR L^p SPACES

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ABSTRACT. Assume that (X, Σ, μ) is a measure space and $p_1, ..., p_n, r > 0$. We prove that $\{(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n} : f_1 \cdot ... \cdot f_n \in L^r\}$ is either $L^{p_1} \times ... \times L^{p_n}$ or a σ-porous subset of $L^{p_1} \times ... \times L^{p_n}$. This dichotomy depends on properties of μ and the sign of the number $\frac{1}{r} - \frac{1}{p_1} - ... - \frac{1}{p_n}$.

1. INTRODUCTION

Among linear topological spaces there are spaces X consisting of sequences or functions such that a natural multiplication is defined on pairs $(x_1, x_2) \in X^2$, however, its result need not necessarily belong to X. It is an interesting question about the size of the set of such "bad" pairs in a various sense. Such a kind of studies was initiated in [BW] and [J]. Balcerzak and Wachowicz proved in [BW] that $\{(f, g) \in L^1[0, 1] \times L^1[0, 1] : f \cdot g \in L^1[0, 1]\}$ is a meager subset of $L^1[0, 1] \times L^1[0, 1]$. They also proved that

$$\left\{(x, y) \in c_0 \times c_0 : \left(\sum_{i=1}^{n} x(i)y(i)\right)_{n=1}^{\infty} \text{ is bounded}\right\}$$

is a meager subset of $c_0 \times c_0$. These meagerness results were generalized by Jachymski in the following extension of the classical Banach–Steinhaus theorem. Recall that a function $\varphi : X \to \mathbb{R}_+$ is L-subadditive for some $L \geq 1$, if $\varphi(x + y) \leq L(\varphi(x) + \varphi(y))$ for any $x, y \in X$.

Theorem 1 (Jachymski [J]). *Given $k \in \mathbb{N}$, let $X_1, ..., X_k$ be Banach spaces, $X = X_1$ if $k = 1$, and $X = X_1 \times ... \times X_k$ if $k > 1$. Assume that $L \geq 1$, $F_n : X \to \mathbb{R}_+$ ($n \in \mathbb{N}$) are lower semicontinuous and such that all functions $x_i \mapsto F_n(x_1, ..., x_k)$ ($i = 1, ..., k$) are L-subadditive and even. Let $E = \{x \in X : (F_n(x))_{n=1}^{\infty}$ is bounded$\}$. Then the following statements are equivalent:*

(i) *E is meager;*

(ii) *$E \neq X$;*

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(iii) $\sup\{F_n(x) : n \in \mathbb{N},\ ||x|| \leq 1\} = \infty$.

At first, we were interested in a further generalization of this theorem changing meagerness by σ-porosity. It turns out that this is not possible. To see it, consider the following set:

$$E = \left\{ x \in \mathbb{R} : \left(\sum_{k=1}^{n} \frac{|\sin(k!\pi x)|}{k} \right)_{n=1}^{\infty} \text{ is bounded} \right\}.$$

Using Theorem 1 for $F_n(x) = \sum_{k=1}^n |\sin(k!\pi x)|/k$ (clearly, each F_n is subadditive) we obtain that this set is meager ($E \neq \mathbb{R}$ since it is of measure zero) and is not σ-upper porous ([Z1], p. 341). Hence we could not generalize Jachymski's theorem in this manner.

Assume that (X, Σ, μ) is a measure space. In our paper we answer the question about a size of the set (in the following we will write L^p instead of $L^p(X, \Sigma, \mu)$):

$$\{(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n} : f_1 \cdot ... \cdot f_n \in L^r\}.$$

We do not restrict our attention only to Banach L^p spaces for $p \in [1, \infty]$, but we consider all linear metric L^p spaces for $p \in (0, \infty]$. It appears that this set is either $L^{p_1} \times ... \times L^{p_n}$ or a σ-c-lower porous (for some $c > 0$) subset of $L^{p_1} \times ... \times L^{p_n}$. So, it is either the whole space or a very small set. We determine this dichotomy for every type of a measure space (X, Σ, μ). Surprisingly it depends on the following parameters (in the sequel the symbol $\frac{1}{\infty}$ means 0):

- the sign of the number $\frac{1}{r} - \frac{1}{p_1} - ... - \frac{1}{p_n}$;
- $\inf\{\mu(A) : \mu(A) > 0\}$ (it is important whether it is equal or greater than zero);
- $\sup\{\mu(A) : \mu(A) < \infty\}$ (it is important whether it is finite or infinite).

The dichotomy is stated in Proposition 2 and Theorems 9, 10.

Let X be a metric space. $B(x, R)$ stands for the ball with a radius R centered at a point x. Let $c \in (0, 1]$. We say that $M \subset X$ is *c-lower porous* [Z2], if

$$\forall x \in M \ \liminf_{R \to 0^+} \frac{\gamma(x, M, R)}{R} \geq \frac{c}{2},$$

where

$$\gamma(x, M, R) = \sup\{r \geq 0 : \exists z \in X \ B(z, r) \subset B(x, R) \setminus M\}.$$

Clearly, M is c-lower porous iff

$$\forall x \in M \ \forall \beta \in (0, c/2) \ \exists R_0 > 0 \ \forall R \in (0, R_0) \ \exists z \in X \ B(z, \beta R) \subset B(x, R) \setminus M.$$

The set is σ-c-lower porous if it is a countable union of c-lower porous sets. Note that a σ-c-lower porous set is meager, and the notion of σ-porosity is essentially stronger than that of meagerness.

Note that the sets investigated in this paper will be c-porous in some stronger sense, namely,

$$\forall x \in X \ \forall \beta \in (0, c/2) \ \ \forall R > 0 \ \exists z \in X \ B(z, \beta R) \subset B(x, R) \backslash M.$$

However, we do not want to define any new notion of porosity, so in the formulations of theorems we will deal only with c-lower porosity.

2. Algebraic product of functions from $L^{p_1} \times ... \times L^{p_n}$

Throughout the paper, (X, Σ, μ) is a measure space. If $p \in (0, 1)$, then we consider L^p as a metric linear space with the metric

$$d(f, g) = \int_X |f - g|^p d\mu.$$

Additionally we put

$$||f||_p = d(f, 0) = \int_X |f|^p d\mu.$$

If $p \in [1, \infty)$, then we consider L^p as a normed linear space with the norm

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

Finally, if $p = \infty$, then we consider L^p as a normed linear space with the norm $||f||_\infty = supess|f|$. Note that in all cases L^p is a complete space.

For every $n \in \mathbb{N}$ and any $p_1, ..., p_n, r \in (0, \infty]$, we define the set (we allow n to be 1):

$$E_r^{(p_1,...,p_n)} = \{(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n} : f_1 \cdot ... \cdot f_n \in L^r\}.$$

In this paper we consider $L^{p_1} \times ... \times L^{p_n}$ as a space with the metric defined as the maximum of distances on all coordinates in $L^{p_1}, ..., L^{p_n}$.

Using the general Hölder inequality ([G], p. 10) we obtain that:

Proposition 2. *Let $p_1, ..., p_n, r \in (0, \infty]$ be such that*

$$\frac{1}{r} = \frac{1}{p_1} + ... + \frac{1}{p_n}.$$

Then $E_r^{(p_1,...,p_n)} = L^{p_1} \times ... \times L^{p_n}$.

Now we will give some helpful lemmas.

Lemma 3. *Let $h \geq 0$, $h \in L^1$, $\varepsilon > 0$. Then*

(i) *if* $\inf\{\mu(A) : A \in \Sigma, \mu(A) > 0\} = 0$, *there is* $A \in \Sigma$ *with* $0 < \mu(A) \leq \varepsilon$ *and* $\int_A h\, d\mu \leq \varepsilon$;

(ii) *if* $\sup\{\mu(A) : A \in \Sigma, \mu(A) < \infty\} = \infty$, *there is* $A \in \Sigma$ *with* $1/\varepsilon \leq \mu(A) < \infty$ *and* $\int_A h\, d\mu \leq \varepsilon$.

Proof. (i) follows immediately from the absolute continuity of the function $B \mapsto \int_B h\, d\mu$ ($B \in \Sigma$) with respect to μ.

(ii) Let, for any $n \in \mathbb{N}$, A_n be such that $n < \mu(A_n) < \infty$. Set $F_n = \bigcup_{k=1}^n A_k$. Then (F_n) is increasing, $\mu(F_n) < \infty$ and $\mu(F_n) \to \infty$. Put $F = \bigcup_{n=1}^\infty F_n$. We have

$$\lim_{n\to\infty} \int_{F_n} h\, d\mu = \int_F h\, d\mu < \infty.$$

Then there is $n_0 \in \mathbb{N}$ with

$$\int_{F_{n_0}} h\, d\mu > \int_F h\, d\mu - \varepsilon.$$

Hence

$$\int_{F \setminus F_{n_0}} h\, d\mu < \varepsilon.$$

On the other hand, $\lim_{n\to\infty} \mu(F_n \setminus F_{n_0}) = \infty$, so there is $N \in \mathbb{N}$ such that $\mu(F_N \setminus F_{n_0}) > 1/\varepsilon$. Put $A = F_N \setminus F_{n_0}$. $\square$

Lemma 4. *Let* $p_1, ..., p_n, r \in (0, \infty)$, $(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n}$ *and let* A *be a measurable subset of* X*. Suppose that for some numbers* $a_1, ..., a_n$ *and for each* $i = 1, ..., n$, *the following holds*

$$\int_A |f_i - 1|^{p_i} d\mu \leq a_i.$$

Then for any numbers $c_1, ..., c_n \in (0, 1)$, *we have*

$$\int_A |f_1 \cdot ... \cdot f_n|^r d\mu \geq c_1^r \cdot ... \cdot c_n^r \left(\mu(A) - \frac{a_1}{(1-c_1)^{p_1}} - ... - \frac{a_n}{(1-c_n)^{p_n}} \right).$$

Proof. Observe that the above assumptions imply that $\mu(A) < \infty$. Let $A_i = \{x \in A : f_i(x) < c_i\}$ for $i = 1, ..., n$. Then for any i, we have

$$a_i \geq \int_A |f_i - 1|^{p_i} d\mu \geq \int_{A_i} |f_i - 1|^{p_i} d\mu \geq \int_{A_i} |1 - c_i|^{p_i} d\mu = (1 - c_i)^{p_i} \mu(A_i).$$

Hence

$$\int_A |f_1 \cdot ... \cdot f_n|^r d\mu \geq \int_{A \setminus \bigcup_{i=1}^n A_i} |f_1 \cdot ... \cdot f_n|^r d\mu \geq \int_{A \setminus \bigcup_{i=1}^n A_i} c_1^r \cdot ... \cdot c_n^r d\mu \geq$$

$$c_1^r \cdot \ldots \cdot c_n^r \left(\mu(A) - \mu(\bigcup_{i=1}^n A_i) \right) \geq c_1^r \cdot \ldots \cdot c_n^r \left(\mu(A) - \frac{a_1}{(1-c_1)^{p_1}} - \ldots - \frac{a_n}{(1-c_n)^{p_n}} \right).$$

$\square$

Lemma 5. *Let $A, A_1, ..., A_n$ be measurable with $A_i \subset A$ and $\mu(A_i) > (1 - \frac{1}{n})\mu(A)$ for any $i = 1, ..., n$. Then*

$$\mu \left(\bigcap_{i=1}^n A_i \right) > 0.$$

Proof. Using the induction principle, it is easy to show that

$$\mu \left(\bigcap_{i=1}^k A_i \right) > (1 - k/n)\mu(A) \quad \text{for any } k = 1, ..., n.$$

In particular, for $k = n$, we get that $\mu(\bigcap_{i=1}^n A_i) > 0$. $\square$

The next theorem is a main result of the paper. It is rather technical, but it shows when $E_r^{(p_1,...,p_n)}$ can be σ-porous and how good are porosity estimations in each of the considered cases. For any $n \in \mathbb{N}$ and any $p_1, ..., p_n$, put $c(p_1, ..., p_n) = 2/(1+m)$ if there is at least one finite p_i, where m is the number of finite p_i's, and put $c(p_1, ..., p_n) = 1$ if $p_i = \infty$ for every $i = 1, ..., n$.

Theorem 6. *Let $n \in \mathbb{N}$ and let $p_1, ..., p_n, r \in (0, \infty]$. Assume that one of the following conditions holds:*

(i) $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$ *and* $\inf\{\mu(A) : \mu(A) > 0\} = 0$*;*

(ii) $\frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$ *and* $\sup\{\mu(A) : \mu(A) < \infty\} = \infty$*.*

Then for any $u > 0$, the set

$$E_u = \{(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n} : ||f_1 \cdot ... \cdot f_n||_r \leq u\}$$

is c-lower porous, where $c = c(p_1, ..., p_n)$. In particular, the set $E_r^{(p_1,...,p_n)}$ is σ-c-lower porous.

Proof. We will consider two cases.

Case 1. $p_1 = ... = p_n = \infty$.

Then our assumptions imply that $r < \infty$ and $\sup\{\mu(A) : \mu(A) < \infty\} = \infty$. Let $(f_1, ..., f_n) \in L^\infty \times ... \times L^\infty$, $R > 0$, $\alpha \in \left(0, \frac{1}{2}\right)$ (note that in this case $c(p_1, ..., p_n) = 1$). Fix a measurable set A of finite measure such that

$$\mu(A) > \frac{u^r}{\left(\left(\frac{1}{2} - \alpha\right) R\right)^{rn}}.$$

For any $i = 1, ..., n$, we define

$$\tilde{f}_i(x) = \begin{cases} f_i(x) + \frac{1}{2}R, & f_i(x) \geq 0; \\ f_i(x) - \frac{1}{2}R, & f_i(x) < 0. \end{cases}$$

Clearly, for any $i = 1, ..., n$, $||\tilde{f}_i - f_i||_\infty = R/2$ and $B((\tilde{f}_1, ..., \tilde{f}_n), \alpha R) \subset B((f_1, ..., f_n), R)$. Now if $(h_1, ..., h_n) \in B((\tilde{f}_1, ..., \tilde{f}_n), \alpha R)$, then for any $i = 1, ..., n$ and for μ-almost every $x \in A$, we have

$$|h_i(x)| \geq \left(\frac{1}{2} - \alpha\right) R.$$

Hence

$$\int_A |h_1 \cdot ... \cdot h_n|^r \geq \left(\left(\frac{1}{2} - \alpha\right) R\right)^{rn} \cdot \mu(A) > u^r,$$

and

$$||h_1 \cdot ... \cdot h_n||_r > u.$$

This ends the proof in Case 1.

Case 2. For some $i = 1, ..., n$, $p_i < \infty$.

Without loss of generality, we assume that $p_i \in (0, 1)$ for $i = 1, ..., m$, $1 \leq p_i < \infty$ for $i = m + 1, ..., m + k$ and $p_i = \infty$ for $i = m + k + 1, ..., m + k + j$, where j is such that $m + k + j = n$ (clearly, m, k or j can be equal to zero, but $m + k \neq 0$). Additionally define $q_i = p_{m+i}$ for $i = 1, ..., k$. Then the product space $L^{p_1} \times ... \times L^{p_n}$ can be written in the following way:

$$L^{p_1} \times ... \times L^{p_m} \times L^{q_1} \times ... \times L^{q_k} \times L^\infty \times ... \times L^\infty.$$

Let $(f_1, ..., f_m, g_1, ..., g_k, l_1, ..., l_j)$ be a member of that space, and let $R > 0$, $\delta \in (0, \frac{1}{m+k+1})$ (note that in this case $c(p_1, ..., p_n) = 2/(m + k + 1)$). Then, clearly, $1 - \delta > (m + k)\delta$ and hence we can take $\eta \in ((m+k)\delta, 1-\delta)$. Since $\delta/\eta < 1/(m+k)$ and hence $(\delta/\eta)^{q_i} < 1/(m+k)$ for $i = 1, ..., k$, there exist $c \in (0, 1)$ and $\varepsilon > 0$ such that

$$\frac{\delta}{\eta} \leq \frac{(1 - c)^{p_i}}{m + k + \varepsilon} \quad \text{for every } i = 1, ..., m \tag{1}$$

and

$$\left(\frac{\delta}{\eta}\right)^{q_i} \leq \frac{(1 - c)^{q_i}}{m + k + \varepsilon} \quad \text{for every } i = 1, ..., k. \tag{2}$$

Now we will define a positive number β. To define β consider three cases.

If $r < \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_m} + \frac{1}{q_1} + ... + \frac{1}{q_k} > \frac{1}{r}$, then $r\left(\frac{1}{r} - \frac{1}{p_1} - ... - \frac{1}{p_m} - \frac{1}{q_1} - ... - \frac{1}{q_k}\right) < 0$, so we can find $\beta > 0$ be such that for any $\beta' \in (0, \beta]$, we have

$$u^r \left((R(1-2\delta))^{rj} \, (\eta R)^{kr + r\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m}\right)} \cdot c^{(m+k)r} \frac{\varepsilon}{m+k+\varepsilon} \right)^{-1} < \tag{3}$$

$$< (\beta')^{r\left(\frac{1}{r}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}-\frac{1}{q_1}-\ldots-\frac{1}{q_k}\right)} < \infty.$$

If $r < \infty$, $\frac{1}{p_1}+\ldots+\frac{1}{p_m}+\frac{1}{q_1}+\ldots+\frac{1}{q_k} < \frac{1}{r}$, then $r\left(\frac{1}{r}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}-\frac{1}{q_1}-\ldots-\frac{1}{q_k}\right) > 0$, so we can find $\beta > 0$ be such that for any $\beta' \in (0, \beta]$, we have

$$u^r \left((R(1-2\delta))^{rj} \, (\eta R)^{kr + r\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m}\right)} \cdot c^{(m+k)r} \frac{\varepsilon}{m+k+\varepsilon} \right)^{-1} < \tag{4}$$

$$< \left(\frac{1}{\beta'}\right)^{r\left(\frac{1}{r}-\frac{1}{p_1}-\ldots-\frac{1}{p_m}-\frac{1}{q_1}-\ldots-\frac{1}{q_k}\right)} < \infty.$$

If $r = \infty$, then our assumptions imply $\frac{1}{p_1}+\ldots+\frac{1}{p_m}+\frac{1}{q_1}+\ldots+\frac{1}{q_k} > 0 = \frac{1}{r}$, so we can find $\beta > 0$ such that for any $\beta' \in (0, \beta]$, we have

$$u\left((R(1-2\delta))^j c^{m+k} \cdot (\eta R)^{k+\frac{1}{p_1}+\ldots+\frac{1}{p_m}} \right)^{-1} < (\beta')^{-\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m}+\frac{1}{q_1}+\ldots+\frac{1}{q_k}\right)} < \infty. \tag{5}$$

Using Lemma 3 with $h = \max\{|f_1|^{p_1}, .., |f_m|^{p_m}, |g_1|^{q_1}, ..., |g_k|^{q_k}\}$ (note that $h \in L^1$) and

$$\varepsilon = \min\{\beta, (1-\delta-\eta)R, ((1-\delta-\eta)R)^{q_1}, ..., ((1-\delta-\eta)R)^{q_k}\},$$

we infer that there is $A \in \Sigma$ with $0 < \mu(A) \leq \varepsilon$ if $\inf\{\mu(A) : \mu(A) > 0\} = 0$, or with $1/\varepsilon \leq \mu(A) < \infty$ if $\sup\{\mu(A) : \mu(A) < \infty\} = \infty$, such that the following conditions hold:

$$\int_A |f_i|^{p_i} d\mu \leq (1-\delta-\eta)\, R \quad \text{for every } i = 1, ..., m; \tag{6}$$

$$\left(\int_A |g_i|^{q_i} d\mu \right)^{1/q_i} \leq (1-\delta-\eta)\, R \quad \text{for every } i = 1, ..., k. \tag{7}$$

Next, let $M_1, ..., M_m, N_1, ..., N_k$ be such that:

$$M_i^{p_i} \mu(A) = \eta R \text{ for every } i = 1, ..., m; \tag{8}$$

$$N_i (\mu(A))^{1/q_i} = \eta R \text{ for every } i = 1, ..., k. \tag{9}$$

Now, let us define $\tilde{f}_1, ..., \tilde{f}_m, \tilde{g}_1, ..., \tilde{g}_k, \tilde{l}_1, ..., \tilde{l}_j$ by formulas:

$$\tilde{f}_i(x) = \begin{cases} M_i, & x \in A; \\ f_i(x), & x \notin A, \end{cases} \qquad \tilde{g}_i(x) = \begin{cases} N_i, & x \in A; \\ g_i(x), & x \notin A, \end{cases}$$

$$\tilde{l}_i(x) = \begin{cases} l_i(x) + (1-\delta)R, & \text{if } l_i(x) \geq 0; \\ l_i(x) - (1-\delta)R, & \text{if } l_i(x) < 0. \end{cases}$$

Using (6), (7), (8) and (9) we obtain:

$$d(\tilde{f}_i, f_i) = \int_A |M_i - f_i|^{p_i} d\mu \leq \int_A M_i^{p_i} d\mu + \int_A |f_i|^{p_i} d\mu \leq$$
$$\leq \eta R + (1-\delta-\eta)\, R = R - \delta R,$$
$$||\tilde{g}_i - g_i||_{q_i} = \left(\int_A |N_i - g_i|^{q_i} d\mu\right)^{1/q_i} \leq \left(\int_A N_i^{q_i} d\mu\right)^{1/q_i} + \left(\int_A |g_i|^{q_i} d\mu\right)^{1/q_i} \leq$$
$$\leq \eta R + (1-\delta-\eta)\, R = R - \delta R,$$

and

$$||\tilde{l}_i - l_i||_\infty = (1-\delta)R.$$

Hence $B((\tilde{f}_1, ..., \tilde{f}_m, \tilde{g}_1, ..., \tilde{g}_k, \tilde{l}_1, ..., \tilde{l}_j), \delta R) \subset B((f_1, ..., f_m, g_1, ..., g_k, l_1, ..., l_j), R)$. It is enough to show that $B((\tilde{f}_1, ..., \tilde{f}_m, \tilde{g}_1, ..., \tilde{g}_k, \tilde{l}_1, ..., \tilde{l}_j), \delta R) \cap E_u = \emptyset$. Let

$$(h_1, ..., h_m, s_1, ..., s_k, w_1, ..., w_j) \in B((\tilde{f}_1, ..., \tilde{f}_m, \tilde{g}_1, ..., \tilde{g}_k, \tilde{l}_1, ..., \tilde{l}_j), \delta R).$$

Clearly, since $||\tilde{l}_i||_\infty \geq (1-\delta)R$, for μ-almost every $x \in A$, we have

$$|w_i(x)| \geq R(1-2\delta). \tag{10}$$

Assume now that $r < \infty$. For any $i = 1, ..., m$, we have

$$\delta R \geq \int_A |h_i - \tilde{f}_i|^{p_i} d\mu = \int_A |h_i - M_i|^{p_i} d\mu = M_i^{p_i} \int_A \left|\frac{h_i}{M_i} - 1\right|^{p_i} d\mu.$$

Using (1) and (8) we obtain

$$\int_A \left|\frac{h_i}{M_i} - 1\right|^{p_i} d\mu \leq \frac{\delta R}{M_i^{p_i}} = \frac{\delta}{\eta}\mu(A) \leq \frac{1}{m+k+\varepsilon}\mu(A)(1-c)^{p_i}.$$

Similarly for any $i = 1, ..., k$,

$$(\delta R)^{q_i} \geq \int_A |s_i - \tilde{g}_i|^{q_i} d\mu = N_i^{q_i} \int_A \left|\frac{s_i}{N_i} - 1\right|^{q_i} d\mu,$$

and using (2) and (9) we have

$$\int_A \left|\frac{s_i}{N_i} - 1\right|^{q_i} d\mu \leq \left(\delta \frac{R}{N_i}\right)^{q_i} = \left(\frac{\delta}{\eta}\right)^{q_i} \mu(A) \leq \frac{1}{m+k+\varepsilon}\mu(A)(1-c)^{q_i}.$$

By (3),(4), (8), (9), (10) and Lemma 4 used for $c_i = c$, we obtain the following

$$\int_X |h_1 \cdot ... \cdot h_m \cdot s_1 \cdot ... \cdot s_k \cdot w_1 \cdot ... \cdot w_j|^r d\mu \geq (R(1-2\delta))^{rj} \int_A |h_1 \cdot ... \cdot h_m \cdot s_1 \cdot ... \cdot s_k|^r d\mu =$$
$$= (R(1-2\delta))^{rj} M_1^r \cdot ... \cdot M_m^r \cdot N_1^r \cdot ... \cdot N_k^r \int_A \left|\frac{h_1}{M_1} \cdot ... \cdot \frac{h_m}{M_m} \cdot \frac{s_1}{N_1} \cdot ... \cdot \frac{s_k}{N_k}\right|^r d\mu \geq$$

$$\geq (R(1-2\delta))^{rj} M_1^r \cdot \ldots \cdot M_m^r \cdot N_1^r \cdot \ldots \cdot N_k^r \cdot c^{(m+k)r} \left(\mu(A) - (m+k)\frac{1}{m+k+\varepsilon}\mu(A)\right) =$$

$$= (R(1-2\delta))^{rj} M_1^r \cdot \ldots \cdot M_m^r \cdot N_1^r \cdot \ldots \cdot N_k^r \cdot c^{(m+k)r} \frac{\varepsilon}{m+k+\varepsilon}\mu(A) =$$

$$= (R(1-2\delta))^{rj} \left[M_1^{p_1}\mu(A)\right]^{\frac{r}{p_1}} \cdot \ldots \cdot \left[M_m^{p_m}\mu(A)\right]^{\frac{r}{p_m}} \cdot \left[N_1 \mu(A)^{\frac{1}{q_1}}\right]^r \cdot \ldots \cdot \left[N_k \mu(A)^{\frac{1}{q_k}}\right]^r \cdot c^{(m+k)r} \cdot$$

$$\cdot (\mu(A))^{r\left(\frac{1}{r} - \frac{1}{p_1} - \ldots - \frac{1}{p_m} - \frac{1}{q_1} - \ldots - \frac{1}{q_k}\right)} \cdot \frac{\varepsilon}{m+k+\varepsilon} =$$

$$= (R(1-2\delta))^{rj} (\eta R)^{\frac{r}{p_1}} \cdot \ldots \cdot (\eta R)^{\frac{r}{p_m}} \cdot (\eta R)^r \cdot \ldots \cdot (\eta R)^r \cdot c^{(m+k)r} \cdot$$

$$\cdot (\mu(A))^{r\left(\frac{1}{r} - \frac{1}{p_1} - \ldots - \frac{1}{p_m} - \frac{1}{q_1} - \ldots - \frac{1}{q_k}\right)} \cdot \frac{\varepsilon}{m+k+\varepsilon} =$$

$$= (R(1-2\delta))^{rj} (\eta R)^{kr + r\left(\frac{1}{p_1} + \ldots + \frac{1}{p_m}\right)} \cdot c^{(m+k)r} \cdot (\mu(A))^{r\left(\frac{1}{r} - \frac{1}{p_1} - \ldots - \frac{1}{p_m} - \frac{1}{q_1} - \ldots - \frac{1}{q_k}\right)} \cdot \frac{\varepsilon}{m+k+\varepsilon} > u^r.$$

For the last inequality, observe that if $\frac{1}{p_1} + \ldots + \frac{1}{p_m} + \frac{1}{q_1} + \ldots + \frac{1}{q_k} > \frac{1}{r}$, then by hypothesis, we infer that $\mu(A) \leq \varepsilon \leq \beta$, so we may use (3) with $\beta' = \mu(A)$. If $\frac{1}{p_1} + \ldots + \frac{1}{p_m} + \frac{1}{q_1} + \ldots + \frac{1}{q_k} < \frac{1}{r}$, then $\frac{1}{\mu(A)} \leq \varepsilon \leq \beta$, and we may use (4) with $\beta' = \frac{1}{\mu(A)}$. Hence

$$||h_1 \cdot \ldots \cdot h_m \cdot s_1 \cdot \ldots \cdot s_k \cdot w_1 \cdot \ldots \cdot w_j||_r > u.$$

Assume now that $r = \infty$. As was mentioned, this case is possible only if $\inf\{\mu(A) : \mu(A) > 0\} = 0$. For any $i = 1, \ldots, m$, we define:

$$A_i^1 = \{x \in A : h_i(x) \geq cM_i\}, \quad A_i^2 = A \setminus A_i^1,$$

and for any $i = 1, \ldots, k$, we define

$$B_i^1 = \{x \in A : s_i(x) \geq cN_i\} \text{ and } B_i^2 = A \setminus B_i^1.$$

Then

$$\delta R > \int_A |h_i - M_i|^{p_i} d\mu \geq \int_{A_i^2} |h_i - M_i|^{p_i} d\mu \geq M_i^{p_i}(1-c)^{p_i}\mu(A_i^2).$$

Hence by (1) and (8), we have

$$\mu(A_i^2) < \frac{\delta R}{M_i^{p_i}(1-c)^{p_i}} = \frac{\delta}{\eta}\frac{1}{(1-c)^{p_i}}\mu(A) \leq \frac{1}{m+k}\mu(A).$$

Then $\mu(A_i^1) > (1 - \frac{1}{m+k})\mu(A)$ for each $i = 1, \ldots, m$. The same estimations (by (2) and (9)) hold for s_i:

$$(\delta R)^{q_i} > \int_A |s_i - N_i|^{q_i} d\mu \geq \int_{B_i^2} |s_i - N_i|^{q_i} d\mu \geq N_i^{q_i}(1-c)^{q_i}\mu(B_i^2).$$

Then

$$\mu(B_i^2) < \left(\frac{\delta R}{N_i(1-c)}\right)^{q_i} \leq \left(\frac{\delta}{\eta(1-c)}\right)^{q_i} \mu(A) \leq \frac{1}{m+k}\mu(A).$$

Hence $\mu(B_i^1) > (1 - \frac{1}{m+k})\mu(A)$ for each $i = 1, ..., k$. Now by Lemma 5 we obtain that $\mu(A_1^1 \cap ... \cap A_m^1 \cap B_1^1 \cap ... \cap B_k^1) > 0$. Also, for μ-almost every $x \in A_1^1 \cap ... \cap A_m^1 \cap B_1^1 \cap ... \cap B_k^1$, using (8),(9),(10) and (5) we have

$$|h_1(x) \cdot ... \cdot h_m(x) \cdot s_1(x) \cdot ... \cdot s_k(x) \cdot w_1(x) \cdot ... \cdot w_j(x)| \geq (R(1-2\delta))^j c^{m+k} M_1 \cdot ... \cdot M_m \cdot N_1 \cdot ... \cdot N_k =$$

$$= (R(1-2\delta))^j c^{m+k} (\eta R)^{\frac{1}{p_1}+...+\frac{1}{p_m}} (\eta R)^k (\mu(A))^{-\left(\frac{1}{p_1}+...+\frac{1}{p_m}+\frac{1}{q_1}+...+\frac{1}{q_k}\right)} > u,$$

and hence

$$||h_1 \cdot ... \cdot h_m \cdot s_1 \cdot ... \cdot s_k \cdot w_1 \cdot ... \cdot w_j||_r > u.$$

This ends the proof. □

Lemma 7. *Assume that*

$$\inf\{\mu(A) : \mu(A) > 0\} > 0.$$

Then:

(i) *for every $r \in (1, \infty)$, $L^1 \subset L^r$;*
(ii) *for every $p > 0$, $L^p \subset L^\infty$.*

The proof of Lemma 7 is known (see, e.g. [F, 224X(e)]).

Proposition 8. *Let $p_1, ..., p_n, r \in (0, \infty]$. If one of the following conditions holds:*

(i) $\sup\{\mu(A) : \mu(A) < \infty\} < \infty$ *and* $0 < \frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$;
(ii) $\inf\{\mu(A) : \mu(A) > 0\} > 0$ *and* $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$,

then $E_r^{(p_1,...,p_n)} = L^{p_1} \times ... \times L^{p_n}$.

Proof. Assume (i). Then r is finite and at least one $p_i < \infty$.

Let $M = \sup\{\mu(A) : \mu(A) < \infty\}$. For any $k \in \mathbb{N}$, let D_k be a measurable set with $M - 1/k \leq \mu(D_k) \leq M$. Set $D = \bigcup_{k=1}^\infty D_k$. Since $\mu(\bigcup_{s=1}^k D_s) \leq M$ for any k, then $\mu(D) = M$ and for a measurable $F \subset X \setminus D$ we have $\mu(F) = 0$ or $\mu(F) = \infty$. Hence if $p < \infty$ and $f \in L^p$, then $\mu(\{x \in X \setminus D : f(x) \neq 0\}) = 0$.

Assume that for some $1 \leq m \leq n$, we have $p_1, ..., p_m < \infty$ and $p_{m+1}, ..., p_n$ are equal to ∞. Let $M > 0$ be such that $|f_i| \leq M$ μ-a.e. on X for $i = m+1, ..., n$, and set $h = \max\{|f_1|^{p_1}, ..., |f_m|^{p_m}\}$. Then $h \in L^1$. Since $f_1 \in L^{p_1}$ and $p_1 < \infty$, we have that

$$\mu(\{x \in X \setminus D : f_1(x) \cdot ... \cdot f_n(x) \neq 0\}) = 0.$$

Hence

$$\int_X |f_1 \cdot ... \cdot f_n|^r d\mu = \int_D |f_1 \cdot ... \cdot f_n|^r d\mu \leq M^{n-m} \int_D |f_1 \cdot ... \cdot f_m|^r d\mu \leq$$

$$\leq M^{n-m} \int_D h^{r\left(\frac{1}{p_1}+...+\frac{1}{p_m}\right)} d\mu.$$

We only have to observe that $\int_D h^{r\left(\frac{1}{p_1}+...+\frac{1}{p_m}\right)} d\mu < \infty$, but this follows from the fact that $\mu(D) < \infty$ and $r\left(\frac{1}{p_1} + ... + \frac{1}{p_m}\right) < 1$.

Now assume (ii). We have to consider two cases:

Case 1. $r < \infty$. Then at least one of $p_1, ..., p_n$ is finite. Assume again, that for some $1 \leq m \leq n$, we have $p_1, ..., p_m < \infty$ and $p_{m+1} = ... = p_n = \infty$. Let $(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n}$. Set $h = \max\{|f_1|^{p_1}, ..., |f_m|^{p_m}\}$. Then $h \in L^1$. Let $M > 0$ be such that $|f_i| \leq M$ μ-a.e. on X for all $i = m+1, ..., n$. Then by Lemma 7, we obtain

$$\int_X |f_1 \cdot ... \cdot f_n|^r \, d\mu \leq M^{n-m} \int_X h^{r\left(\frac{1}{p_1}+...+\frac{1}{p_n}\right)} \, d\mu < \infty,$$

since $r\left(\frac{1}{p_1} + ... + \frac{1}{p_n}\right) > 1$.

Case 2. $r = \infty$. By Case 1, we obtain that for $r' < \infty$ with

$$\frac{1}{r'} < \frac{1}{p_1} + ... + \frac{1}{p_n},$$

if $(f_1, ..., f_n) \in L^{p_1} \times ... \times L^{p_n}$, then $f_1 \cdot ... \cdot f_n \in L^{r'}$. Hence by Lemma 7, we have $||f_1 \cdot ... \cdot f_n||_\infty < \infty$. □

Note that Proposition 8 is not valid if each p_i is infinite. Indeed, if we consider the following measure

$$\mu(A) = 0 \text{ if } A = \emptyset \text{ and } \mu(A) = \infty \text{ if } A \neq \emptyset,$$

and we set $f = g = 1$, then $(f, g) \in L^\infty \times L^\infty$, but $(f, g) \notin E_r^{(\infty,\infty)}$.

Now we can summarize our results in the two following theorems. We write c instead of $c(p_1, ..., p_n)$, where $c(p_1, ..., p_n)$ was defined before the statement of Theorem 6.

Theorem 9. *Let (X, Σ, μ) be a measure space. The following conditions are equivalent:*

(i) *for any $n \in \mathbb{N}$ and $p_1, ..., p_n, r > 0$ such that $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$, the set $E_r^{(p_1,...,p_n)}$ is σ-c-lower porous;*

(ii) *for any $n \in \mathbb{N}$ and $p_1, ..., p_n, r > 0$ such that $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$, the set $E_r^{(p_1,...,p_n)}$ is not equal to $L^{p_1} \times ... \times L^{p_n}$;*

(iii) *there are* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$ *and the set* $E_r^{(p_1,...,p_n)}$ *is* σ*-c-lower porous;*

(iv) *there are* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $\frac{1}{p_1} + ... + \frac{1}{p_n} > \frac{1}{r}$ *and the set* $E_r^{(p_1,...,p_n)}$ *is not equal to* $L^{p_1} \times ... \times L^{p_n}$*;*

(v) $\inf\{\mu(A) : \mu(A) > 0\} = 0$.

Proof. The following implications are trivial: (i)⇒(ii), (i)⇒(iii), (ii)⇒(iv) and (iii)⇒(iv). Implication (iv)⇒(v) follows from Proposition 8. Finally, (v)⇒(i) follows from Theorem 6. □

Theorem 10. *Let* (X, Σ, μ) *be a measure space. The following conditions are equivalent:*

(i) *for any* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $0 < \frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$*, the set* $E_r^{(p_1,...,p_n)}$ *is* σ*-c-lower porous;*

(ii) *for any* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $0 < \frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$*, the set* $E_r^{(p_1,...,p_n)}$ *is not equal to* $L^{p_1} \times ... \times L^{p_n}$*;*

(iii) *there are* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $0 < \frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$ *and the set* $E_r^{(p_1,...,p_n)}$ *is* σ*-c-lower porous;*

(iv) *there are* $n \in \mathbb{N}$ *and* $p_1, ..., p_n, r > 0$ *such that* $0 < \frac{1}{p_1} + ... + \frac{1}{p_n} < \frac{1}{r}$ *and the set* $E_r^{(p_1,...,p_n)}$ *is not equal to* $L^{p_1} \times ... \times L^{p_n}$*;*

(v) $\sup\{\mu(A) : \mu(A) < \infty\} = \infty$.

Proof. The following implications are trivial: (i)⇒(ii), (i)⇒(iii), (ii)⇒(iv) and (iii)⇒(iv). Implication (iv)⇒(v) follows from Proposition 8. Finally, (v)⇒(i) follows from Theorem 6. □

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References

[BW] Balcerzak, M.; Wachowicz, A.: Some examples of meager sets in Banach spaces. Real Anal. Exchange 26 (2000/01), no. 2, 877–884.

[F] Fremlin, D. H.: Measure theory. Vol. 2. Broad foundations. Torres Fremlin, Colchester, 2001.

[G] Grafakos, L.: Classical Fourier analysis, Graduate Texts in Mathematics, 249 (2nd ed.), Berlin, New York: Springer-Verlag, 2008.

[J] Jachymski, J.: A nonlinear Banach-Steinhaus theorem and some meager sets in Banach spaces. Studia Math. 170 (2005), no. 3, 303–320.

[Z1] Zajíček, L.: Porosity and σ-porosity, Real Anal. Exchange 13 (1987/1988), 314–350.

[Z2] Zajíček, L.: On σ-porous sets in abstract spaces, Abstr. Appl. Anal. 5 (2005), 509–534.

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