DICHOTOMIES FOR $\mathbf{C}_0(X)$ AND $\mathbf{C}_b(X)$ SPACES

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Abstract. Jachymski showed that the set

$$\left\{(x,y)\in \mathbf{c}_0\times\mathbf{c}_0 : \left(\sum_{i=1}^{n}\alpha(i)x(i)y(i)\right)_{n=1}^{\infty} \text{ is bounded}\right\}$$

is either a meager subset of $\mathbf{c}_0\times\mathbf{c}_0$ or is equal to $\mathbf{c}_0\times\mathbf{c}_0$. In the paper we generalize this result by considering more general spaces than $\mathbf{c}_0$, namely $\mathbf{C}_0(X)$, the space of all continuous functions which vanish at infinity, and $\mathbf{C}_b(X)$, the space of all continuous bounded functions. Moreover, we replace the meagerness by σ-porosity.

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1. Introduction

Among linear topological spaces there are spaces X consisting of sequences or functions such that a natural multiplication is defined on pairs $(x_1,x_2)\in X^2$, however, its result need not necessarily belong to X. It is an interesting question about the size of the set of such "bad" pairs, for example from the Baire category point of view. Such a kind of studies was initiated in [1] and [5]. Balcerzak and Wachowicz [1] proved that the set

$$\left\{(x,y)\in \mathbf{c}_0\times\mathbf{c}_0 : \left(\sum_{i=1}^{n}x(i)y(i)\right)_{n=1}^{\infty} \text{ is bounded}\right\}$$

is a meager subset of $\mathbf{c}_0\times\mathbf{c}_0$. This result was generalized by Jachymski in [5]:

Theorem 1.1 ([5]). *Assume that α is any sequence of reals and let*

$$E := \left\{ (x,y) \in \mathbf{c}_0 \times \mathbf{c}_0 : \left(\sum_{i=1}^{n} \alpha(i)x(i)y(i) \right)_{n=1}^{\infty} \text{ is bounded} \right\}.$$

Then the following statements are equivalent

(i) *E is meager in $\mathbf{c}_0 \times \mathbf{c}_0$;*

(ii) *$E \neq \mathbf{c}_0 \times \mathbf{c}_0$;*

(iii) *$\alpha \notin l_1$, that is $\sum_{n=1}^{\infty} |\alpha(n)| = \infty$.*

A natural question arise wheather the above result can be further generalized, by considering more general spaces and replacing Baire category by σ-porosity. In this paper we give an affirmative answer to this question. The major idea is that we can consider $\sum_{i=1}^{n} \alpha(i)x(i)y(i)$ as an integral of the function αxy over the set $\{1, ..., n\}$ with respect to the counting measure on $\mathbb{N}$. According to this we will consider the set E of pairs $(f,g) \in \mathbf{C}_0(X) \times \mathbf{C}_0(X)$ (or $(f,g) \in \mathbf{C}_b(X) \times \mathbf{C}_b(X)$) with a bounded sequence of integrals $\left(\int_{D_n} (fgh) d\mu \right)$ for some fixed sequence (D_n) and fixed function h. We will show that E is equal to $\mathbf{C}_0(X) \times \mathbf{C}_0(X)$ (or $\mathbf{C}_b(X) \times \mathbf{C}_b(X)$), if $\sup \int_{D_n} |h| d\mu < \infty$ or E is small (namely, σ-porous), if $\sup \int_{D_n} |h| = \infty$.

We would like to mention that Balcerzak and Wachowicz in [1] showed also that the set $\{(f,g) \in \mathbf{L}^1[0,1] \times \mathbf{L}^1[0,1] : f \cdot g \in \mathbf{L}^1[0,1]\}$ is a meager subset of $\mathbf{L}^1[0,1] \times \mathbf{L}^1[0,1]$, and that this result was also extended by Jachymski in [5] (he considered general $\mathbf{L}^p(X)$ spaces and obtained an analogous dichotomy as in Theorem 1.1).

In fact, Jachymski's results are applications of his nonlinear version of the Banach–Steinhaus principle. At first we were interested in finding a generalization of this result in the direction of porosity, but it turned out that it is not possible (cf. [2]). That is why we decided to investigate the possibility of generalizing its applications. In particular, in [2] we extended the result from [5] connected with $\mathbf{L}^p(X)$ spaces.

2. Notation and basic facts

Let X be a metric space. $B(x,R)$ stands for the open ball with a radius R centered at a point x. Let $\alpha \in (0,1]$. We say that $M \subset X$ is *α-lower porous* [7], if

$$\forall_{x \in M} \liminf_{R \to 0^+} \frac{\gamma(x,M,R)}{R} \geq \frac{\alpha}{2},$$

where

$$\gamma(x,M,R) = \sup\{r \geq 0 : \exists_{z \in X}\ B(z,r) \subset B(x,R) \backslash M\}.$$

Clearly, M is α-lower porous iff

$$\forall_{x\in M}\ \forall_{\beta\in\left(0,\frac{\alpha}{2}\right)}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{z\in X}\ B(z,\beta R)\subset B(x,R)\backslash M.$$

Now, let $(X,\|\ \|)$ be a normed linear space. We say that M is *strongly ball porous*, if

$$\forall_{R>0}\ \forall_{x\in X}\ \forall_{\alpha\in(0,1)}\ \exists_{y\in X}\ (\|x-y\|=R \text{ and } B(y,\alpha R)\cap M=\emptyset).$$

Finally, we say that M is *σ-α-lower porous* or *σ-strongly ball porous*, respectively, if M is a countable union of α-lower porous sets or strongly ball porous, respectively. The notions of strong ball porosity is closely related to the notion of R-ball porosity (cf. [7]) and were discussed in [6] (cf. condition (2.7) in [6]).
We say that (X,μ) is a *topological measure space*, if X is a topological space, and the measure μ is defined on a σ-algebra subsets of X, containing the family of all Borel subsets of X
We say that a topological measure space (X,μ) is *inner regular*, if $\mu(A)=\sup\{\mu(D):D\subset A,\ D \text{ is closed}\}$ for every $A\in\Sigma$ with $\mu(A)<\infty$.

Remark 2.1. Most authors define inner regularity by assuming that all measurable sets can be approximated from below by compact sets. However, there are measures (defined on locally compact spaces) which are inner regular in our sense, and some measurable sets can not be approximated from below by compact sets (cf. [4, Sec. 53, Exercise 10]).

The proof of the following lemma is standard and straightforward, so we skip it.

Lemma 2.2. *Let (X,μ) be inner regular and let $h:X\to\mathbb{R}$ be measurable and nonnegative. Then the space (X,η), where $\eta(A):=\int_A hd\mu$ for measurable $A\subset X$, is also inner regular.*

If (X,μ) is a topological Borel measure space, then by $\mathbf{L}^1_{loc}(X,\mu)$ ($\mathbf{L}^1_{loc}$ in short) we denote the set of all locally integrable functions on X, that is, all measurable functions $h:X\to\mathbb{R}$ with $\int_K |h|\ d\mu<\infty$ for every set $K\in\mathbf{K}(X)$ (by $\mathbf{K}(X)$ we denote the set of all compact subsets of X).
By $\mathbf{C}_b(X)$ ($\mathbf{C}_b$ in short) we denote the set of all continuous real functions with bounded images. We consider it as a Banach space with the standard supremum norm:

$$\|f\|:=\sup\{|f(x)|:x\in X\}.$$

By $\mathbf{C}_0(X)$ ($\mathbf{C}_0$ in short) we denote the set of all continuous real functions on X, which vanish at infinity, that is

$$\mathbf{C}_0 := \left\{ f \in \mathbf{C}_b : \forall_{\epsilon>0}\ \exists_{K\in\mathbf{K}(X)}\ \forall_{x\in X\setminus K}\ |f(x)| < \varepsilon \right\}.$$

We consider $\mathbf{C}_0$ also as a Banach space with a supremum norm.
Note that the space $\mathbf{c}_0$ can be viewed as $\mathbf{C}_0(\mathbb{N})$, if we consider the discrete topology on $\mathbb{N}$.
Finally, we consider products $\mathbf{C}_0 \times \mathbf{C}_0$ and $\mathbf{C}_b \times \mathbf{C}_b$ as Banach spaces with the maximum norm:

$$\|(f,g)\| := \max\{\|f\|, \|g\|\}.$$

3. Results for products of $\mathbf{C}_0$ spaces.

If (X,μ) is a topological measure space, $h : X \to \mathbb{R}$ is any measurable function and (D_n) is a sequence of measurable subsets of X, then we define

$$E^0_{h,(D_n)} := \left\{ (f,g) \in \mathbf{C}_0 \times \mathbf{C}_0 : \left(\int_{D_n} fgh\, d\mu \right)_{n=1}^{\infty} \text{ is bounded} \right\}.$$

Remark 3.1. For every measurable function f and every measurable set D, if we say that the integral $\int_D f\, d\mu$ has some properties, then we clearly assume that it is well defined, i.e., the integral of the positive part of f is finite, or the integral of the negative part of f is finite. Hence the statement "$\left(\int_{D_n} fgh\, d\mu\right)_{n=1}^{\infty}$ is bounded" is a shortcut for "for every $n \in \mathbb{N}$, $\int_{D_n} fgh\, d\mu$ is well defined and $\left(\int_{D_n} fgh\, d\mu\right)_{n=1}^{\infty}$ is bounded".

Theorem 3.2. *Assume that (X,μ) is a topological measure space which is inner regular and such that the topological space X is locally compact and σ-compact. Let $h \in \mathbf{L}^1_{loc}$ and let (D_n) be a sequence of measurable subsets of X such that $\sup_{n\in\mathbb{N}} \int_{D_n} |h| d\mu = \infty$. Then the set $E^0_{h,(D_n)}$ is σ-strongly ball porous.*

Proof. Since X is σ-compact and locally compact, it is normal and there exists an increasing sequence of compact sets (K_n) such that for any $n \in \mathbb{N}$, $K_n \subset \operatorname{Int} K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$ ([3, Theorem 3.8.2 and Exercise 3.8.C]). To prove the result, we have to consider two cases.

Case 1. $\int_{D_{n_0}} |h|\, d\mu = \infty$ for some $n_0 \in \mathbb{N}$.
Note that

$$E^0_{h,(D_n)} \subset \left\{ (f,g) \in \mathbf{C}_0 \times \mathbf{C}_0 : \int_{D_{n_0}} |fgh|\, d\mu < \infty \right\} = \bigcup_{u \in \mathbb{N}} F^0_u,$$

where for any $u > 0$,

$$F^0_u := \left\{ (f,g) \in \mathbf{C}_0 \times \mathbf{C}_0 : \int_{D_{n_0}} |fgh|\, d\mu < u \right\}.$$

Hence it is enough to show that for every $u > 0$, the set F^0_u is strongly ball porous. Let $u > 0$, $R > 0$, $(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0$ and $\alpha \in (0,1)$. Put

$$A^1_f := \{x \in X : f(x) \geq 0\} \text{ and } A^{-1}_f := X \setminus A^1_f = \{x \in X : f(x) < 0\}.$$

In the same way we define A^1_g and A^{-1}_g. Then for some $s \in \{-1,1\}^2$, we have

$$\int_{A^{s(1)}_f \cap A^{s(2)}_g \cap D_{n_0}} |h|\, d\mu = \infty.$$

Assume, without loss of generality, that $s = (1,1)$ and set $C := A^1_f \cap A^1_g \cap D_{n_0}$. By the properties of the sequence (K_n), there is $n \in \mathbb{N}$ such that

$$\int_{C \cap K_n} |h|\, d\mu > \frac{u}{((1-\alpha)R)^2}. \tag{3.1}$$

Now since K_n and $X \setminus \operatorname{Int} K_{n+1}$ are closed and disjoint, by the Tietze theorem, there exists a continuous function $w : X \to [0,R]$ such that $w(x) = R$ for $x \in K_n$ and $w(x) = 0$ for $x \notin \operatorname{Int} K_{n+1}$. Put

$$\tilde{f} := f + w \text{ and } \tilde{g} := g + w.$$

Since w is equal to 0 outside the compact set K_{n+1}, we get $\left(\tilde{f}, \tilde{g}\right) \in \mathbf{C}_0 \times \mathbf{C}_0$. Moreover, since $K_n \neq \emptyset$, we have $\|f - \tilde{f}\| = \|g - \tilde{g}\| = R$. It is enough to show that $B\left(\left(\tilde{f}, \tilde{g}\right), \alpha R\right) \cap F^0_u = \emptyset$. Let $(a,b) \in B\left(\left(\tilde{f}, \tilde{g}\right), \alpha R\right)$ and observe that for any $x \in C \cap K_n$,

$$a(x) \geq \tilde{f}(x) - \alpha R = f(x) + R - \alpha R \geq R(1-\alpha).$$

In the same way we get $b(x) \geq (1-\alpha)R$. Hence and by (3.1),

$$\int_{D_{n_0}} |abh|\, d\mu \geq \int_{C\cap K_n} ((1-\alpha)R)^2 |h|\, d\mu \overset{(3.1)}{>} u,$$

so $(a,b) \notin F_u^0$ and the proof in Case 1 is finished.

Case 2. $\int_{D_n} |h| d\mu < \infty$ for every $n \in \mathbb{N}$.

Note that for every $(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0$, there exists $M > 0$ such that for every $x \in X$, $|f(x)|, |g(x)| < M$. Hence for every $n \in \mathbb{N}$, $\int_{D_n} |fgh|\, d\mu \leq M^2 \int_{D_n} |h|\, d\mu < \infty$. Thus for every $(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0$ and every $n \in \mathbb{N}$, the integral $\int_{D_n} fgh\, d\mu$ is well defined.

It is enough to show that for each $u > 0$, the set

$$E_u^0 = \left\{ (f,g) \in \mathbf{C}_0(X) \times \mathbf{C}_0(X) : \left| \int_{D_n} fgh\, d\mu \right| \leq u \text{ for any } n \in \mathbb{N} \right\}$$

is strongly ball porous. Let $u > 0$, $R > 0$, $(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0$ and $\alpha \in (0,1)$. Then there is a compact set K such that for any $x \in X \setminus K$,

$$|f(x)| \leq \frac{1-\alpha}{2} R \quad \text{and} \quad |g(x)| \leq \frac{1-\alpha}{2} R. \tag{3.2}$$

Let $M > 0$ be such that $|f(x)|, |g(x)| < M$ for all $x \in X$. Now since $\int_K |h|\, d\mu < \infty$, $\int_{D_n} |h|\, d\mu < \infty$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \int_{D_n} |h|\, d\mu = \infty$, there exists $n \in \mathbb{N}$ such that

$$\infty > \int_{D_n \setminus K} |h|\, d\mu > \frac{u + (2 + \int_{D_n \cap K} |h|\, d\mu)(M + 2R)^2}{\frac{(1-\alpha)^2}{4} R^2} + 2. \tag{3.3}$$

By the properties of the sequence (K_n), there exists $n_1 \in \mathbb{N}$ such that

$$\int_{(D_n \setminus K) \cap K_{n_1}} |h|\, d\mu > \int_{D_n \setminus K} |h|\, d\mu - 1. \tag{3.4}$$

Put $C := (D_n \setminus K) \cap K_{n_1}$, $A_h^1 := \{x \in X : h(x) \geq 0\}$ and $A_h^{-1} := X \setminus A_h^1$. By Lemma 2.2, there exist closed sets $C^+ \subset C \cap A_h^1$ and $C^- \subset C \cap A_h^{-1}$ with

$$\int_{C \setminus (C^+ \cup C^-)} |h|\, d\mu < 1. \tag{3.5}$$

Then by (3.3), (3.4) and (3.5),

(3.6)
$$\int_{C^+\cup C^-} |h|\, d\mu \overset{(3.5)}{>} \int_C |h| d\mu - 1 \overset{(3.4)}{>} \int_{D_n\setminus K} |h|\, d\mu - 2 \overset{(3.3)}{>} \frac{u + (2 + \int_{D_n\cap K} |h|\, d\mu)(M+2R)^2}{\frac{(1-\alpha)^2}{4} R^2}.$$

Since C^+, C^- and $X \setminus \operatorname{Int} K_{n_1+1}$ are closed and disjoint, by the Tietze theorem, there exist continuous functions $w_1 : X \to [-R, R]$ and $w_2 : X \to [0, R]$ such that

- $w_1(x) = w_2(x) = R$ for $x \in C^+$;
- $w_1(x) = -R$ for $x \in C^-$;
- $w_2(x) = R$ for $x \in C^-$;
- $w_1(x) = w_2(x) = 0$ for $x \notin \operatorname{Int} K_{n_1+1}$.

Put
$$\tilde{f} := f + w_1 \quad \text{and} \quad \tilde{g} := g + w_2.$$

Since w_1 and w_2 are equal to zero outside the compact set K_{n_1+1}, we get $\tilde{f}, \tilde{g} \in \mathbf{C}_0$. Since $C^+ \cup C^-$ is non-empty, we have that $||\tilde{f} - f|| = R$ and $||\tilde{g} - g|| = R$. To prove the theorem it is enough to show that $B\left(\left(\tilde{f}, \tilde{g}\right), \alpha R\right) \cap E_u^0 = \emptyset$. Let $(a, b) \in B\left(\left(\tilde{f}, \tilde{g}\right), \alpha R\right)$ and note that for any $x \in C^+$, we have by (3.2),

$$a(x) \geq \tilde{f}(x) - \alpha R = f(x) + R - \alpha R \overset{(3.2)}{\geq} -\frac{1-\alpha}{2} R + (1-\alpha)R = \frac{1-\alpha}{2} R,$$

and (by the same computations) $b(x) \geq \frac{1-\alpha}{2} R$. Moreover, for any $x \in C^-$, we have

$$a(x) \leq \tilde{f}(x) + \alpha R = f(x) - R + \alpha R \overset{(3.2)}{\leq} \frac{1-\alpha}{2} R - (1-\alpha)R = -\frac{1-\alpha}{2} R,$$

$$b(x) \geq \tilde{g}(x) - \alpha R = g(x) + R - \alpha R \overset{(3.2)}{\geq} -\frac{1-\alpha}{2} R + (1-\alpha)R = \frac{1-\alpha}{2} R.$$

Hence, for every $x \in C^+ \cup C^-$,

$$a(x)b(x)h(x) \geq \frac{(1-\alpha)^2}{4} R^2 |h(x)|. \tag{3.7}$$

On the other hand, for any $x \in X$, we have (recall that $|f(x)|, |g(x)| < M$ for $x \in X$):

(3.8)
$$\max\{|a(x)|, |b(x)|\} \leq \max\{|\tilde{f}(x)|, |\tilde{g}(x)|\} + \alpha R \leq \max\{|f(x)|, |g(x)\} + 2R < M + 2R.$$

Finally, we obtain, by (3.4)–(3.8)

$$\int_{D_n} abh\ d\mu = \int_{D_n\setminus K} abh\ d\mu + \int_{D_n\cap K} abh\ d\mu =$$

$$= \int_{(D_n\setminus K)\setminus K_{n_1}} abh\ d\mu + \int_{C} abh\ d\mu + \int_{D_n\cap K} abh\ d\mu =$$

$$= \int_{(D_n\setminus K)\setminus K_{n_1}} abh\ d\mu + \int_{C^+\cup C^-} abh\ d\mu + \int_{C\setminus(C^+\cup C^-)} abh\ d\mu + \int_{D_n\cap K} abh\ d\mu \overset{(3.7),(3.8)}{\geq}$$

$$\geq -(M+2R)^2 \int_{(D_n\setminus K)\setminus K_{n_1}} |h|\ d\mu + \frac{(1-\alpha)^2}{4} R^2 \int_{C^+\cup C^-} |h|\ d\mu -$$

$$-(M+2R)^2 \int_{C\setminus(C^+\cup C^-)} |h|\ d\mu - (M+2R)^2 \int_{D_n\cap K} |h|\ d\mu \overset{(3.4),(3.5)}{\geq}$$

$$\geq \frac{(1-\alpha)^2}{4} R^2 \int_{C^+\cup C^-} |h|\ d\mu - 2(M+2R)^2 - (M+2R)^2 \int_{D_n\cap K} |h|\ d\mu \overset{(3.6)}{>} u.$$

Hence $(a,b) \notin E^0_u$. □

As an immediate corollary, we have the following strengthening of Theorem 1.1:

Corollary 3.3. *Assume that (X,μ) and h are as in the formulation of Theorem 3.2. Let (D_n) be a sequence of measurable sets. Then the following statements are equivalent:*

(i) $E^0_{h,(D_n)}$ *is σ-strongly ball porous in $\mathbf{C}_0 \times \mathbf{C}_0$;*

(ii) $E^0_{h,(D_n)} \neq \mathbf{C}_0 \times \mathbf{C}_0$;

(iii) $\sup_{n\in\mathbb{N}} \int_{D_n} |h| d\mu = \infty$.

Proof. Implication (i)⇒(ii) is trivial, implication (iii)⇒(i) is stated in Theorem 3.2. Now let $N > 0$ be such that $\sup_{n\in\mathbb{N}} \int_{D_n} |h| d\mu < N$ and let $(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0$. Then $\|f\|, \|g\| < M$ for some $M > 0$, so for every $n \in \mathbb{N}$,

$$\left| \int_{D_n} fgh\ d\mu \right| \leq M^2 \int_{D_n} |h|\ d\mu < M^2 N.$$

Hence $(f,g) \in E^0_{h,(D_n)}$. This gives (ii)⇒(iii). □

Corollary 3.4. *Assume that (X,μ) is as in the formulation of Theorem 3.2. Additionally, let $\mu(K) < \infty$ for every compact set $K \subset X$, and let $G^0 := \{(f,g) \in \mathbf{C}_0 \times \mathbf{C}_0 : fg \in \mathbf{L}^1\}$. Then the following statements are equivalent:*

(i) G^0 *is* σ*-strongly ball porous;*

(ii) $G^0 \neq \mathbf{C}_0 \times \mathbf{C}_0$*;*

(iii) $\mu(X) = \infty$.

Proof. The result follows from Corollary 3.3 by taking $h = 1$ and the sequence (D_n) such that $D_n = X$ for every $n \in \mathbb{N}$. $\square$

Remark 3.5. If X is a Banach space, then we say that $M \subset X$ is c-porous, if its convex hull $\operatorname{conv} M$ is nowhere dense. In an obvious way we define σ-c-porous sets. As we proved in [6], every c-porous set is strongly ball porous, and the converse is not true. However, we did not know if there exists a set which is σ-strongly ball porous and is not σ-c-porous.
It turns out that the set

$$E := \left\{ (x,y) \in \mathbf{c}_0 \times \mathbf{c}_0 : \left(\sum_{i=1}^{n} \alpha(i) x(i) y(i) \right)_{n=1}^{\infty} \text{ is bounded} \right\}$$

satisfies this condition. This result will be included in the paper which is under preparation.

4. Results for products of $\mathbf{C}_b$ spaces.

Now we will investigate the case when the space $\mathbf{C}_0$ is replaced by the space $\mathbf{C}_b$. It turns out that very similar results hold, but obtained in a slightly different way. On one hand, the assumptions will be weaker, but on the other, the porosity will be also weaker than the strong ball porosity.
Let (X, μ) be a topological measure space. If $h : X \to \mathbb{R}$ is a measurable function and (D_n) is a sequence of measurable sets, then we define:

$$E^b_{h,(D_n)} := \left\{ (f,g) \in \mathbf{C}_b \times \mathbf{C}_b : \left(\int_{D_n} fgh \, d\mu \right)_{n=1}^{\infty} \text{ is bounded} \right\}.$$

Theorem 4.1. *Assume that* (X, μ) *is a topological measure space which is inner regular, and such that the topological space* X *is normal. Let* h *be any measurable function on* X *and* (D_n) *be a sequence of measurable sets such that* $\sup_{n \in \mathbb{N}} \int_{D_n} |h| d\mu = \infty$*. Then the set* $E^b_{h,(D_n)}$ *is* σ*-*$\frac{1}{2}$*-lower porous in* $\mathbf{C}_b \times \mathbf{C}_b$*.*

Proof. Consider two cases:

Case 1. $\int_{D_{n_0}} |h|\, d\mu = \infty$ for some $n_0 \in \mathbb{N}$.

We deal with this case in a similar (but even simpler) way as with the proof of Case 1 of Theorem 3.2, so we skip the proof.

Case 2. $\int_{D_{n_0}} |h|\, d\mu < \infty$ for any $n \in \mathbb{N}$. Clearly, for every $(f,g) \in \mathbf{C}^b \times \mathbf{C}^b$ and every $n \in \mathbb{N}$, the integral $\int_{D_n} fgh\, d\mu$ is well defined.

It is enough to show that for any $u > 0$, the set

$$E_u := \left\{ (f,g) \in \mathbf{C}_b \times \mathbf{C}_b : \left| \int_{D_n} fgh\, d\mu \right| \leq u \text{ for any } n \in \mathbb{N} \right\}$$

is $\frac{1}{2}$-lower porous. Hence let $u > 0$. It is enough to show that

$$\forall_{(f,g)\in E_u}\, \forall_{R>0}\, \exists_{\tilde{f},\tilde{g}\in \mathbf{C}^b} \left(\|f - \tilde{f}\| = ||g - \tilde{g}|| = \frac{3}{4}R \text{ and } B\left(\left(\tilde{f},\tilde{g}\right), \frac{1}{4}R \right) \cap E_u^b = \emptyset \right).$$

Let $(f,g) \in E_u$ and $R > 0$. Let $n \in \mathbb{N}$ be such that

$$\int_{D_n} |h|\, d\mu > \frac{2u + 2 + 2R + R^2 + \frac{1}{4}R^2}{\frac{1}{8}R^2} > 2. \tag{4.1}$$

Now define A_f^1, A_f^{-1}, A_g^1, A_g^{-1}, A_h^1 and A_h^{-1} as in the proof of Theorem 3.2. Moreover, for any $s \in \{-1,1\}^3$, define

$$A^s := D_n \cap A_f^{s(1)} \cap A_g^{s(2)} \cap A_h^{s(3)}.$$

Clearly, the family $\left\{A^s : s \in \{-1,1\}^3\right\}$ is a decomposition of D_n into eight pairwise disjoint measurable sets. For any $s \in \{-1,1\}^3$, let $sgn(s) = s(1) \cdot s(2) \cdot s(3)$. In this way we obtain a natural decomposition of D_n into two sets

$$C := \bigcup_{sgn(s)=1} A^s \quad \text{and} \quad F := \bigcup_{sgn(s)=-1} A^s.$$

Then for every $x \in C$,

$$f(x)g(x)h(x) = |f(x)g(x)h(x)|, \tag{4.2}$$

and for every $x \in F$,

$$f(x)g(x)h(x) = -|f(x)g(x)h(x)|. \tag{4.3}$$

Clearly, we have that either $\int_C |h| d\mu \geq \frac{1}{2}\int_{D_n} |h| d\mu$ or $\int_F |h| d\mu \geq \frac{1}{2}\int_{D_n} |h| d\mu$. Assume, without loss of generality, that

$$\int_C |h| d\mu \geq \frac{1}{2}\int_{D_n} |h| d\mu. \tag{4.4}$$

Now we will define some auxiliary sets

$$A^1_{f,1} := \left\{x \in A^1_f : |f(x)| \geq \frac{1}{2}R\right\} \quad \text{and} \quad A^1_{f,2} := \left\{x \in A^1_f : |f(x)| < \frac{1}{2}R\right\}.$$

In the same way we define $A^{-1}_{f,1}$, $A^{-1}_{f,2}$, $A^1_{g,1}$, $A^1_{g,2}$, $A^{-1}_{g,1}$ and $A^{-1}_{g,2}$.

Now put

$$A^s_p := D_n \cap A^{s(1)}_{f,p(1)} \cap A^{s(2)}_{g,p(2)} \cap A^{s(3)}_h$$

for $s \in \{1,-1\}^3$ and $p \in \{1,2\}^2$. Clearly, for any $s \in \{1,-1\}^3$, the family $\left\{A^s_p : p \in \{1,2\}^2\right\}$ is a decomposition of A^s into 4 pairwise disjoint measurable sets. By the regularity of (X,μ) and Lemma 2.2, we can find closed sets $C^s \subset A^s$ for each s with $sgn(s) = 1$, and closed sets $F^s_p \subset A^s_p$ for each s with $sgn(s) = -1$ and $p \in \{1,2\}^2$, such that

$$\int_{D_n \setminus (C' \cup F')} |h| \, d\mu < 1, \tag{4.5}$$

$$\int_{D_n \setminus (C' \cup F')} |fh| \, d\mu < 1, \tag{4.6}$$

$$\int_{D_n \setminus (C' \cup F')} |gh| \, d\mu < 1, \tag{4.7}$$

$$\int_{D_n \setminus (C' \cup F')} |fgh| \, d\mu < 1, \tag{4.8}$$

where

$$C' := \bigcup \{ C^s : s \in \{-1,1\}^3,\ sgn(s) = 1 \} \subset C$$

and

$$F' := \bigcup \{ F^s_p : s \in \{-1,1\}^3,\ sgn(s) = -1,\ p \in \{1,2\}^2 \} \subset F.$$

Clearly, $C' \cup F'$ is a closed subset of X. Hence and by the fact that sets from the family $\{C^s : sgn(s) = 1\} \cup \{F^s_p : sgn(s) = -1,\ p \in \{1,2\}^2\}$ are closed and pairwise disjoint, by the Tietze theorem, we can define continuous functions $w_1 : X \to \left[-\frac{3}{4}R, \frac{3}{4}R\right]$ and $w_2 : X \to \left[-\frac{3}{4}R, \frac{3}{4}R\right]$ such that

- if $sgn(s) = 1$, then for $x \in C^s$,

$$w_1(x) = \begin{cases} \frac{3}{4}R, & \text{if } f(x) \geq 0, \\ -\frac{3}{4}R, & \text{if } f(x) < 0, \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} \frac{3}{4}R, & \text{if } g(x) \geq 0, \\ -\frac{3}{4}R, & \text{if } g(x) < 0, \end{cases}$$

- if $sgn(s) = -1$ and $p = (1,1)$, then for $x \in F^s_p$,

$$w_1(x) = \begin{cases} -\frac{1}{4}R, & \text{if } f(x) \geq 0, \\ \frac{1}{4}R, & \text{if } f(x) < 0, \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} -\frac{1}{4}R, & \text{if } g(x) \geq 0, \\ \frac{1}{4}R, & \text{if } g(x) < 0, \end{cases}$$

- if $sgn(s) = -1$ and $p = (1,2)$, then for $x \in F^s_p$,

$$w_1(x) = \begin{cases} \frac{1}{4}R, & \text{if } f(x) \geq 0, \\ -\frac{1}{4}R, & \text{if } f(x) < 0, \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} -\frac{3}{4}R, & \text{if } g(x) \geq 0, \\ \frac{3}{4}R, & \text{if } g(x) < 0, \end{cases}$$

- if $sgn(s) = -1$ and $p = (2,1)$ or $p = (2,2)$, then for $x \in F^s_p$,

$$w_1(x) = \begin{cases} -\frac{3}{4}R, & \text{if } f(x) \geq 0, \\ \frac{3}{4}R, & \text{if } f(x) < 0, \end{cases} \quad \text{and} \quad w_2(x) = \begin{cases} \frac{1}{4}R, & \text{if } g(x) \geq 0, \\ -\frac{1}{4}R, & \text{if } g(x) < 0. \end{cases}$$

We are ready to define functions $\tilde{f}$ and $\tilde{g}$. Put

$$\tilde{f} := f + w_1 \quad \text{and} \quad \tilde{g} := g + w_2.$$

By (4.1), (4.4) and (4.5),

$$\int_{C'} |h| d\mu \overset{(4.5)}{>} \int_C |h| d\mu - 1 \overset{(4.4)}{\geq} \frac{1}{2} \int_{D_n} |h| d\mu - 1 \overset{(4.1)}{>} 0, \tag{4.9}$$

so C' is nonempty and therefore $\|\tilde{f} - f\| = \frac{3}{4}R$ and $\|\tilde{g} - g\| = \frac{3}{4}R$. To complete the proof, it is enough to show that $B\left(\left(\tilde{f}, \tilde{g}\right), \frac{1}{4}R\right) \cap E_u = \emptyset$. To do this, take any $(a, b) \in B\left(\left(\tilde{f}, \tilde{g}\right), \frac{1}{4}R\right)$.

Let $s \in \{-1, 1\}^3$ be such that $sgn(s) = 1$. Then for any $x \in C^s$, we have:

$$\text{if } \ f(x) \geq 0, \ \text{ then } \ a(x) \geq \tilde{f}(x) - \frac{1}{4}R = f(x) + \frac{1}{2}R,$$

$$\text{if } \ f(x) < 0, \ \text{ then } \ a(x) \leq \tilde{f}(x) + \frac{1}{4}R = f(x) - \frac{1}{2}R,$$

$$\text{if } \ g(x) \geq 0, \ \text{ then } \ b(x) \geq \tilde{g}(x) - \frac{1}{4}R = g(x) + \frac{1}{2}R,$$

$$\text{if } \ g(x) < 0, \ \text{ then } \ b(x) \leq \tilde{g}(x) + \frac{1}{4}R = g(x) - \frac{1}{2}R.$$

Hence by (4.2), for every $x \in C^s$, $a(x)b(x)h(x) = |a(x)b(x)h(x)|$, so

$$\int_{C^s} abh \, d\mu = \int_{C^s} |abh| \, d\mu \geq \int_{C^s} \left(|f(x)| + \frac{1}{2}R\right)\left(|g(x)| + \frac{1}{2}R\right)|h| \, d\mu \geq$$

$$\geq \int_{C^s} |fgh| \, d\mu + \int_{C^s} \frac{1}{4}R^2 |h| \, d\mu \overset{(4.2)}{=} \int_{C^s} fgh \, d\mu + \frac{1}{4}R^2 \int_{C^s} |h| \, d\mu.$$

Therefore we get

$$\int_{C'} abh \, d\mu \geq \int_{C'} fgh \, d\mu + \frac{1}{4}R^2 \int_{C'} |h| \, d\mu. \tag{4.10}$$

Let $s \in \{-1, 1\}^3$ be such that $sgn(s) = -1$ and let $p = (1, 1)$. By the definition of A_p^s, for any $x \in F_p^s$, we have $|f(x)| \geq \frac{1}{2}R$ and $|g(x)| \geq \frac{1}{2}R$. Then by the definition of w_1 and w_2, for every $x \in F_p^s$, we have:

$$\text{if } \ f(x) \geq 0, \text{ then } \ 0 \leq a(x) \leq f(x) \quad \text{ and if } \ f(x) < 0, \text{ then } \ 0 \geq a(x) \geq f(x),$$

$$\text{if } \ g(x) \geq 0, \text{ then } \ 0 \leq b(x) \leq g(x) \quad \text{ and if } \ g(x) < 0, \text{ then } \ 0 \geq b(x) \geq g(x).$$

Hence by (4.3), for every $x \in F_p^s$, $a(x)b(x)h(x) = -|a(x)b(x)c(x)|$, so

$$\int_{F_p^s} abh \, d\mu = -\int_{F_p^s} |abh| \, d\mu \geq -\int_{F_p^s} |fgh| \, d\mu \overset{(4.3)}{=} \int_{F_p^s} fgh \, d\mu.$$

Let $s \in \{-1,1\}^3$ be such that $sgn(s) = -1$ and let $p = (1,2)$. Then for $x \in F_p^s$, we obtain

$$\text{if } \ f(x) \geq 0, \ \text{ then } \ a(x) \geq \tilde{f}(x) - \frac{1}{4}R = f(x) + \frac{1}{4}R - \frac{1}{4}R \geq 0,$$

$$\text{if } \ f(x) < 0, \ \text{ then } \ a(x) \leq \tilde{f}(x) + \frac{1}{4}R = f(x) - \frac{1}{4}R + \frac{1}{4}R \leq 0,$$

and

$$\text{if } \ g(x) \geq 0, \ \text{ then } \ b(x) \leq \tilde{g}(x) + \frac{1}{4}R = g(x) - \frac{3}{4}R + \frac{1}{4}R \leq 0,$$
$$\text{if } \ g(x) < 0, \ \text{ then } \ b(x) \geq \tilde{g}(x) - \frac{1}{4}R = g(x) + \frac{3}{4}R - \frac{1}{4}R \geq 0.$$

Hence by (4.3), for every $x \in F_p^s$, $a(x)b(x)h(x) = -|a(x)b(x)h(x)|$, so

$$\int_{F_p^s} abh \; d\mu \geq 0 \geq -\int_{F_p^s} |fgh| \; d\mu \overset{(4.3)}{=} \int_{F_p^s} fgh \; d\mu.$$

In the same way we can show that for any s with $sgn(s) = -1$ and $p = (2,1)$ or $p = (2,2)$, we have that

$$\int_{F_p^s} abh \; d\mu \geq 0 \geq \int_{F_p^s} fgh \; d\mu.$$

As a consequence, we obtain

$$\int_{F'} abh \; d\mu \geq \int_{F'} fgh \; d\mu. \tag{4.11}$$

Finally, by (4.1) and (4.5) – (4.11), we get:

$$\int_{D_n} abh \; d\mu = \int_{C'} abh \; d\mu + \int_{F'} abh \; d\mu + \int_{D_n \setminus (C' \cup F')} abh \; d\mu \overset{(4.10),(4.11)}{\geq}$$

$$\geq \frac{1}{4}R^2 \int_{C'} |h| d\mu + \int_{C'} fgh \; d\mu + \int_{F'} fgh \; d\mu + \int_{D_n \setminus (C' \cup F')} abh \; d\mu \overset{(4.9)}{\geq}$$

$$\geq \frac{1}{8}R^2 \left(\int_{D_n} |h| \; d\mu - 2 \right) + \int_{D_n} fgh \; d\mu - \int_{D_n \setminus (C' \cup F')} |fgh| \; d\mu - \int_{D_n \setminus (C' \cup F')} |abh| \; d\mu \overset{(4.8)}{\geq}$$

$$\geq \frac{1}{8}R^2 \left(\int_{D_n} |h| \; d\mu - 2 \right) - u - 1 - \int_{D_n \setminus (C' \cup F')} (|f| + R)(|g| + R)|h| \; d\mu \geq$$

$$\geq \frac{1}{8}R^2 \left(\int_{D_n} |h| d\mu - 2 \right) - u - 1 - \int_{D_n \setminus (C' \cup F')} |fgh| \; d\mu -$$

$$R\left(\int_{D_n\setminus(C'\cup F')}|fh|\,d\mu+\int_{D_n\setminus(C'\cup F')}|gh|\,d\mu\right)-R^2\int_{D_n\setminus(C'\cup F')}|h|\,d\mu\overset{(4.5)-(4.8)}{\geq}$$

$$\geq\frac{1}{8}R^2\int_{D_n}|h|\,d\mu-\frac{1}{4}R^2-u-1-1-2R-R^2\overset{(4.1)}{>}u.$$

Hence $(a,b)\notin E_u$ and the proof of part (ii) is finished. □

As an immediate corollary, we have the following dichotomies (we skip the proofs since they are very similar to the proofs of analogous corollaries in the previous Section).

Corollary 4.2. *Assume that (X,μ) and h are as in the formulation of Theorem 4.1, and let (D_n) be a sequence of measurable sets. The following statements are equivalent:*

(i) $E^b_{h,(D_n)}$ *is σ-$\frac{1}{2}$-lower porous in $\mathbf{C}_b\times\mathbf{C}_b$;*
(ii) $E^b_{h,(D_n)}\neq\mathbf{C}_b\times\mathbf{C}_b$;
(iii) $\sup_{n\in\mathbb{N}}\int_{D_n}|h|\,d\mu=\infty$.

Corollary 4.3. *Assume that (X,μ) is as in the formulation of Theorem 4.1. Let*

$$G^b:=\left\{(f,g)\in\mathbf{C}_b\times\mathbf{C}_b:fg\in\mathbf{L}^1\right\}.$$

Then the following statements are equivalent:

(i) G^b *is σ-$\frac{1}{2}$-lower porous in $\mathbf{C}_b\times\mathbf{C}_b$;*
(ii) $G^b\neq\mathbf{C}_b\times\mathbf{C}_b$;
(iii) $\mu(X)=\infty$.

5. Final remarks

Let (X,Σ,μ) be a signed measure on X, i.e., μ is a countably additive functional such that either $\sup\{\mu(A):A\in\Sigma\}<\infty$ or $\inf\{\mu(A):A\in\Sigma\}>-\infty$. Then there exist measurable disjoint sets X^+ and X^- such that $X=X^+\cup X^-$ and for all $A\subset X^+$, $\mu(A)\geq 0$ and for all $A\subset X^-$, $\mu(A)\leq 0$ (this decomposition is called a *Hahn decomposition*). Now let $|\mu|$ be a variation of μ, that is

$$|\mu|(A)=\mu(A\cap X^+)-\mu(A\cap X^-)\text{ for a measurable set }A.$$

(cf. [4, Sec. 28 and 29] for a more information on signed measures). Set $h(x) := 1$ for $x \in X^+$ and $h(x) := -1$ for $x \in X^-$. Then for every measurable function f, we have that

$$\int_X fh\, d|\mu| = \int_X f\, d\mu.$$

It means that if one of the above integrals is defined, then the second is also defined and they are equal.

This shows that every signed measure can be generated by some measurable function h, and hence presented results can easily be adapted to signed measures. However, note that the function $h(x) = 1$ for $x > 0$ and $h(x) = -1$ for $x \leq 0$ does not generate any signed measure on $\mathbb{R}$. Hence our approach is more general.

Also, it can easily be seen that presented results remain valid with a very similar but more technically complicated proofs, if we write them in a more general way, namely, if we consider the sets (here $k \geq 2$):

$$\left\{(f_1, ..., f_k) \in \mathbf{C}_0 \times ... \times \mathbf{C}_0 : \left(\int_{D_n} f_1 \cdots f_k h\, d\mu\right)_{n=1}^{\infty} \text{ is bounded}\right\}$$

and

$$\left\{(f_1, ..., f_k) \in \mathbf{C}_b \times ... \times \mathbf{C}_b : \left(\int_{D_n} f_1 \cdots f_k h\, d\mu\right)_{n=1}^{\infty} \text{ is bounded}\right\}.$$

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