

ON THE SET OF LIMIT POINTS OF CONDITIONALLY CONVERGENT SERIES

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ABSTRACT. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space and let τ be a permutation of natural numbers. We study the set $\text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$ of all limit points of a sequence $(\sum_{n=1}^p x_{\tau(n)})_{p=1}^{\infty}$ of partial sums of a rearranged series $\sum_{n=1}^{\infty} x_{\tau(n)}$. We give full characterization of limit sets in finite dimensional spaces. Namely, a limit set in \mathbb{R}^m is either compact and connected or it is closed and all its connected components are unbounded. On the other hand each set of one of these types is a limit set of some rearranged conditionally convergent series. Moreover, this characterization does not hold in infinite dimensional spaces.

We show that if $\sum_{n=1}^{\infty} x_n$ has the Rearrangement Property and A is a closed subset of the closure of the $\sum_{n=1}^{\infty} x_n$ sum range and it is ε -chainable for every $\varepsilon > 0$, then there is a permutation τ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. As a byproduct of this observation we obtain that series having the Rearrangement Property have closed sum ranges.

1. INTRODUCTION

Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series on the real line \mathbb{R} . For any $a < b$ one can find a permutation $\sigma \in S_{\infty}$ of natural numbers such that the sequence $(\sum_{n=1}^k x_{\sigma(n)})_{k=1}^{\infty}$ of partial sums of the rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$ oscillates between a and b . Consequently, a and b are limit points of a sequence of rearranged partial sums $(\sum_{n=1}^k x_{\sigma(n)})_{k=1}^{\infty}$. Since $|x_{\sigma(n)}|$ tends to zero, the whole interval $[a, b]$ consists of limit points of $(\sum_{n=1}^k x_{\sigma(n)})_{k=1}^{\infty}$. This simple observation shows that the set of all limit points of a sequence of rearranged partial sums $(\sum_{n=1}^k x_{\sigma(n)})_{k=1}^{\infty}$ is closed and connected, and for any close connected subset I of real line and any conditionally convergent series $\sum_{n=1}^{\infty} x_n$ one can find a rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$ such that the set of all limit points of its partial sums equals I . If the rearranged series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges to ∞ or to $-\infty$, then the set of all its limit points is empty.

The situation becomes more complicated if limit sets of rearrangements of conditionally convergent series are considered in multidimensional Euclidean spaces. One could expect that such limit sets would be connected or even arcwise connected. It turns out that this is not the case. However, some result concerning connectedness can be proved for multidimensional spaces, see Theorem 3.5.

Now, let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in the Euclidean space \mathbb{R}^n . By Steinitz Theorem the sum range $\text{SR}(\sum_{n=1}^{\infty} x_n) = \{\sum_{n=1}^{\infty} x_{\sigma(n)} : \sigma \in S_{\infty}\}$ of $\sum_{n=1}^{\infty} x_n$, where S_{∞} is a symmetric group of all permutation of natural numbers, is an affine subspace of \mathbb{R}^n . Denote by $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ the set of all limit points of a sequence of rearranged partial sums $(\sum_{n=1}^k x_{\sigma(n)})_{k=1}^{\infty}$. Such limit sets were studied by Victor Klee in [3], where the author claimed that if A is a limit set $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$, then for every $\varepsilon > 0$ an ε -shell $A(\varepsilon) = \{x : \|x - y\| < \varepsilon \text{ for some } y \in A\}$ of A is connected. Our Example 2.2 shows that this claim is not true. Note that connectedness of $A(\varepsilon)$ means that any two points $a, b \in A$ can be joined by a path $x_0, x_1, \dots, x_k \in A$ such that $x_0 = a$, $x_k = b$ and $\|x_i - x_{i-1}\| < \varepsilon$, and if A has this property, then we say that A is ε -chainable. Klee also proved that if $A \subseteq \text{SR}(\sum_{n=1}^{\infty} x_n)$ is closed and ε -chainable for every $\varepsilon > 0$, then there is $\sigma \in S_{\infty}$ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$.

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In this article we complete the Klee's result by giving the full characterization of limit sets $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ in Euclidean spaces. Namely we prove the following dichotomy (Theorem 3.5): the limit set is either compact and connected or any its component is unbounded; moreover, the closure of the limit set in the one-point compactification of \mathbb{R}^m is connected. The proof uses the fact that underlying space has a finite dimension. Moreover, this dichotomy does not hold for all Banach spaces. More precisely, we construct an example of a conditionally convergent series in c_0 such that the limit set of some of its rearrangement consists of two points.

Theorem 3.5 cannot be reversed in the sense that there is an unbounded, closed set in the one-dimensional Euclidean space \mathbb{R} whose every component is unbounded but it cannot be a limit set. Namely, consider the union $X := (-\infty, -1] \cup [1, \infty)$ of two unbounded connected sets. As we have mentioned in the beginning, any limit set on the real line must be connected, and therefore X is not a limit set. However, Theorem 3.5 can be reversed in higher dimensions. This means that any compact connected set (or even any closed ε -chainable set for every $\varepsilon > 0$) in \mathbb{R}^m , $m \geq 1$, and any closed set in \mathbb{R}^m , $m \geq 2$, whose every component is unbounded are limit sets of some rearrangement of a conditionally convergent series.

In the last Section we show that if $\sum_{n=1}^{\infty} x_n$ has the Rearrangement Property and $A \subseteq \overline{\text{SR}(\sum_{n=1}^{\infty} x_n)}$ is closed and ε -chainable for every $\varepsilon > 0$, then there is $\tau \in S_{\infty}$ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. As a byproduct of this observation we obtain that series having the Rearrangement Property have closed sum ranges.

2. COUNTEREXAMPLE FOR KLEE'S CLAIM

As we have mentioned in the Introduction, Victor Klee in [3] claimed that if $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$, then its ε -shell $A(\varepsilon) = \{x : \|x - y\| < \varepsilon \text{ for some } y \in A\}$ is connected for every $\varepsilon > 0$. It is equivalent to saying that A is ε -chainable for every $\varepsilon > 0$. The author used quite a different notation than the one used by us, but the gap in his argument can be translated into our language as follows. Klee argued that $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ cannot intersect two sets X and Y having disjoint ε -shells $X(\varepsilon)$ and $Y(\varepsilon)$; it is supposed to be "evident". However, the following example shows that this is simply not true.

For natural numbers $n < m$ by $[n, m]$ we denote discrete interval $\{n, n+1, n+2, \dots, m\}$ and by $[n, \infty)$ we denote the set $\{n, n+1, \dots\}$. Let $\sum_{n=1}^{\infty} y_n$ be a conditionally convergent series and let $\sum_{n=1}^{\infty} x_n$ be its rearrangement, that is there is $\sigma \in S_{\infty}$ with $x_n = y_{\sigma(n)}$. A partial sums sequence (s_n) , $s_n = \sum_{k=1}^n x_k$, will be called a *walk*. Note that $a \in \text{LIM}(\sum_{n=1}^{\infty} x_n)$ if for every $\varepsilon > 0$ the walk (s_n) hits the ball $B(a, \varepsilon)$. If $(s_n)_{n=1}^{\infty}$ is a sequence in \mathbb{R}^m , then we call it a walk, if some rearrangement of a series $\sum_{n=1}^{\infty} (s_{n+1} - s_n)$ is convergent.

A sequence (s_n) of elements of set X is called an X -walk if

- (i) the set $\{s_n : n \in \mathbb{N}\}$ is dense in X ;
- (ii) there are positive integers n_1, n_2, \dots such that $s_{n_1+i} = s_{n_1-i}$ for $i \in [1, n_1 - 1]$ and

$$s_{\sum_{j=1}^k 2n_j - 1 + n_{k+1} - i} = s_{\sum_{j=1}^k 2n_j - 1 + n_{k+1} + i}$$

for $k > 0$ and $i \in [1, n_{k+1}]$;

- (iii) $\|s_{n+1} - s_n\| \rightarrow 0$.

Proposition 2.1. *Suppose that (s_n) is an X -walk. Then there is a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ and a permutation $\sigma \in S_{\infty}$ such that $s_n = \sum_{k=1}^n x_{\sigma(k)}$. Moreover, $\text{LIM}(\sum_{k=1}^{\infty} x_{\sigma(k)}) = X$.*

Proof. Note that $s_{\sum_{j=1}^k 2n_j - 1} = s_1$ for every k . That means that the X -walk (s_n) gets from s_1 to s_{n_1} and back, using the same points, to $s_{2n_1-1} = s_1$, then it walks to $s_{2n_1-1+n_2}$ and back to $s_{2n_1+2n_2-1} = s_1$, and so on. We define $y_n = s_{n+1} - s_n$. Let $p = \sum_{j=1}^k 2n_j - 1 + n_{k+1} - i$ and $m = \sum_{j=1}^k 2n_j - 1 + n_{k+1} + i$. Then $y_p = -y_{m-1}$. Thus the series $\sum_{n=1}^{\infty} y_n$ can be rearranged into an alternating series $\sum_{n=1}^{\infty} x_n$, which by (iii) is convergent. Since each element of (s_n) is in a closed set X , then $\text{LIM}(\sum_{k=1}^{\infty} x_{\sigma(k)}) \subseteq X$. The opposite inclusion follows from (i). \square

Example 2.2. At first we define elements y_n of a conditionally convergent series $\sum_{n=1}^{\infty} y_n \subseteq \mathbb{R}^2$. The sequence (y_n) will be alternating, that is $y_{2n} = -y_{2n-1}$ for $n \geq 1$. Therefore we will define only the terms y_n with an odd index n .

Step 1. First two odd y'_n s are $(\frac{1}{2}, 0), (\frac{1}{2}, 0)$.

Step 2. We define next $1 \cdot 4 + 4 + 1 \cdot 4$ odd elements: $(0, \frac{1}{4}), \dots, (0, \frac{1}{4}), (\frac{1}{4}, 0), \dots, (\frac{1}{4}, 0), (0, -\frac{1}{4}), \dots, (0, -\frac{1}{4})$.

Step $k+1$. In this step we define $k \cdot 2^{k+1} + 2^{k+1} + k \cdot 2^{k+1}$ elements

$$\underbrace{(0, \frac{1}{2^{k+1}}), \dots, (0, \frac{1}{2^{k+1}})}_{k \cdot 2^{k+1}}, \underbrace{(\frac{1}{2^{k+1}}, 0), \dots, (\frac{1}{2^{k+1}}, 0)}_{2^{k+1}}, \underbrace{(0, -\frac{1}{2^{k+1}}), \dots, (0, -\frac{1}{2^{k+1}})}_{k \cdot 2^{k+1}}.$$

Since (y_n) is alternating and $\lim_{n \rightarrow \infty} \|y_n\| = 0$, the series $\sum_{n=1}^{\infty} y_n$ is convergent. Now we define our walk, that is a rearrangement of $\sum_{n=1}^{\infty} y_n$, as follows. First two x_1 and x_2 are the elements of (y_n) defined in Step 1 with odd indexes, x_3, x_4 are corresponding elements of (y_n) with even indexes. Next $1 \cdot 4 + 4 + 1 \cdot 4$ of x'_n s are elements of (y_n) defined in Step 2 with odd indexes (in the same order we have defined them above) and the next $1 \cdot 4 + 4 + 1 \cdot 4$ of x'_n s are corresponding elements of (y_n) with even indexes taken with reversed order, and so on. On Figure 1 we present a sequence of partial sums given for x_n 's defined in the first three steps of the construction. Note that $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \{0, 1\} \times [0, \infty)$. Thus the set of limit points of the series $\sum_{n=1}^{\infty} x_n$ has no connected ε -shell for $\varepsilon < \frac{1}{2}$.

Example 2.3. Now we describe a construction in which the limit points of the series are the closure of set of infinitely many pairwise disjoint half-lines $\{a_n : n \in \mathbb{N}\} \times [0, \infty)$ where (a_n) is a sequence of distinct real numbers. This example is similar to Example 2.2, so we prescribe only the walk (s_n) . Since in each step of the construction the walk goes from one point to another and then back through the same path, the steps of the walk can be rearranged to an alternating series. Since the lengths of the walk's steps tend to zero, the obtained series is convergent. We describe the first three steps of the construction:

Step 1. We start the walk at $(a_1, 0)$. Then we move to $(a_2, 0)$ along the line $y = 0$ using steps of length not greater than 1. Then we go back to $(a_1, 0)$ via the same path.

Step 2. We go upward to $(a_1, 1)$, then along the line $y = 1$ to $(a_2, 1)$, next downward to $(a_2, 0)$ and back upward to $(a_2, 1)$, then again along $y = 1$ to $(a_3, 1)$ and downward to $(a_3, 0)$ in each part using steps of length not greater than $\frac{1}{2}$. Finally we go back to $(a_1, 0)$ using the same path.

Step 3. In this step first four points $(a_1, 0), \dots, (a_4, 0)$ are involved, steps are not greater than $\frac{1}{4}$ and to move between vertical lines $x = a_i$ we use a horizontal line $y = 2$, etc.

Clearly $\text{LIM}(\sum_{n=1}^{\infty} x_n) \supseteq \{a_n : n \in \mathbb{N}\} \times [0, \infty)$. Since $\text{LIM}(\sum_{n=1}^{\infty} x_n)$ is closed, we obtain $\text{LIM}(\sum_{n=1}^{\infty} x_n) \supseteq \overline{\{a_n : n \in \mathbb{N}\} \times [0, \infty)} = \{a_n : n \in \mathbb{N}\} \times [0, \infty)$. To show the inverse inclusion let $(u, v) \notin \overline{\{a_n : n \in \mathbb{N}\} \times [0, \infty)}$. If $v < 0$ then $(u, v) \notin \text{LIM}(\sum_{n=1}^{\infty} x_n)$, because our walk is in \mathbb{R}^2 and has a non-negative second coordinate. If $v \geq 0$ and $u \notin \overline{\{a_n : n \in \mathbb{N}\}}$ then $\inf_{n \in \mathbb{N}} |u - a_n| = \delta > 0$. Fix a natural number $m > v + \delta$. Then the ball $B((u, v), \delta)$ does not contain elements of our walk defined in k -th step of construction for any $k \geq m$. Hence $(u, v) \notin \text{LIM}(\sum_{n=1}^{\infty} x_n)$. Finally $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \overline{\{a_n : n \in \mathbb{N}\} \times [0, \infty)}$.

Using Example 2.3 we can show that the limit set of a rearrangement of a conditionally convergent series can have uncountably many unbounded components. Let $E = \{a_1, a_2, \dots\}$ be a countable dense subset of the ternary Cantor set C . By Example 2.3 one can find a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ and a rearrangement σ such that $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)}) = \overline{\{a_n\}_{n=1}^{\infty} \times [0, \infty)} = C \times [0, \infty)$. Since the ternary Cantor set C is totally disconnected, i.e. each its component is a singleton, half-lines $\{x\} \times [0, \infty)$, $x \in C$, are the components of $C \times [0, \infty)$.

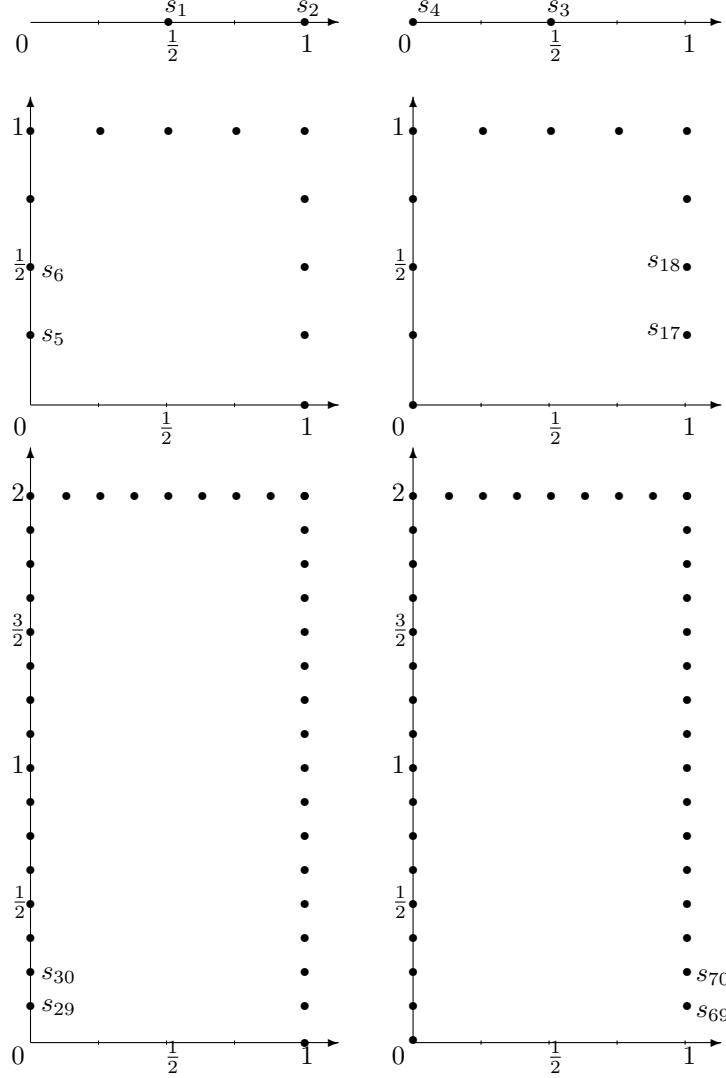


FIGURE 1. The first three steps of the construction of the walk $(\sum_{n=1}^m x_n)_{m=1}^\infty$ from Example 2.2.

3. CHARACTERIZATION OF LIMIT SETS $\text{LIM}(\sum_{n=1}^\infty x_{\sigma(n)})$ IN THE EUCLIDEAN SPACES

Let $B(0, R) = \{v \in \mathbb{R}^m : \|v\| \leq R\}$ and let $S(0, R) = \{v \in \mathbb{R}^m : \|v\| = R\}$. For a topological space X by $\mathcal{K}(X)$ we denote the set of all non-empty compact subsets of X equipped with the Vietoris topology, for details see for example [4, p. 66]. It is well-known that the compactness (metrizability, separability) of X implies the compactness (metrizability, separability) of the hyperspace $\mathcal{K}(X)$ and that the family of all nonempty compact connected subsets of X forms a closed subset of $\mathcal{K}(X)$.

Lemma 3.1. *Let $X \subseteq \mathbb{R}^m$ be a closed set and let $R > 0$. Then*

$$Z := \bigcup \{C : C \text{ is a component of } X \cap B(0, R) \text{ such that } C \cap S(0, R) \neq \emptyset\}$$

is a compact subset of \mathbb{R}^m .

Proof. Let $(v_n) \subseteq Z$. Find components C_n of $X \cap B(0, R)$ such that $C_n \cap S(0, R) \neq \emptyset$ and $v_n \in C_n$. Pick $x_n \in C_n \cap S(0, R)$. Since $\mathcal{K}(X \cap B(0, R))$ is compact, we may assume that C_n tends to some C , $v_n \rightarrow v$ and $x_n \rightarrow x$. Then $v, x \in C$ and C is connected. Therefore v and x are in the same component of $X \cap B(0, R)$ which has a non-empty intersection with the sphere $S(0, R)$. Thus $v \in Z$, and consequently Z is compact. \square

Let $X \subseteq \mathbb{R}^m$ be a closed set. We define an equivalence relation E on X as follows

$$xEy \iff x \text{ and } y \text{ belong to the same component of } X.$$

By X/E we denote the set of all equivalence classes of E and by q we denote the mapping from X to X/E assigning to a point $x \in X$ the equivalence class $[x]_E \in X/E$. On X/E we consider the so-called *quotient topology* consisting of those $U \subseteq X/E$ such that $q^{-1}(U)$ is open in X . The set X/E equipped with this topology is called the *quotient space*, and $q : X \rightarrow X/E$ is called the *natural quotient mapping*. The following result important for us can be found in [1].

Theorem 3.2. *For every compact space X , the quotient space X/E is compact and zero-dimensional.*

For $X \subseteq \mathbb{R}^m$ and $\varepsilon > 0$ put $X(\varepsilon) := \{y \in \mathbb{R}^m : \|x - y\| \leq \varepsilon \text{ for some } x \in X\}$. We will call it an ε -shell or an ε -neighborhood of X .

Lemma 3.3. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in \mathbb{R}^m and let $\sigma \in S_{\infty}$. Assume that Y is a nonempty bounded subset of $X := \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$. If $Y(\varepsilon)$ is disjoint with $X \setminus Y$ for some $\varepsilon > 0$, then $X = Y$.*

Proof. Note that the closure Z of $Y(\varepsilon) \setminus Y(\varepsilon/2)$ is a compact set disjoint with X . Suppose that $X \setminus Y \neq \emptyset$. Consider a set $A := \{\sum_{n=1}^k x_{\sigma(n)} : k \in \mathbb{N}\} \cap Z$ of those partial sums of $\sum_{n=1}^{\infty} x_{\sigma(n)}$ which meet Z . Since all elements of the nonempty sets Y and $X \setminus Y$ are limit points of a rearranged partial sums sequence $\{\sum_{n=1}^k x_{\sigma(n)}\}_{k=1}^{\infty}$, then the elements of that sequence walk from Y to $X \setminus Y$ and back infinitely many times. Since the lengths of steps $\|x_{\sigma(n)}\|$ taken during this walk tend to zero, the set A is infinite. By compactness of Z we obtain that A has a limit point, which in turn is in Z , but this contradicts the fact that $Z \cap X = \emptyset$. Thus $X \setminus Y = \emptyset$ and consequently $X = Y$. \square

By $a(\mathbb{R}^m)$ denote the one-point compactification of \mathbb{R}^m , that is to the underlying set \mathbb{R}^m we add a point ∞ . Neighborhood base at each $x \in \mathbb{R}^m$ consists of open ball centered in x and neighborhood base at ∞ consists of all sets of the form $(\mathbb{R}^m \setminus C) \cup \{\infty\}$ where C is compact in \mathbb{R}^m . For $A \subset a(\mathbb{R}^m)$ by \overline{A}^{∞} denote the closure of A in $a(\mathbb{R}^m)$.

Lemma 3.4. *Let $\{C_i : i \in I\}$ be a family of connected and unbounded subsets of \mathbb{R}^m and let $C := \bigcup_{i \in I} C_i$. Then*

- (1) $\overline{C}^{\infty} = \overline{C} \cup \{\infty\}$;
- (2) \overline{C}^{∞} is connected.

Proof. (1) The set $\overline{C} \cup \{\infty\}$ is closed in $a(\mathbb{R}^m)$, since $(\mathbb{R}^m \cup \{\infty\}) \setminus (\overline{C} \cup \{\infty\}) = \mathbb{R}^m \setminus \overline{C}$ is open in \mathbb{R}^m . Thus $\overline{C}^{\infty} \subseteq \overline{C} \cup \{\infty\}$. Since C is unbounded, then $\infty \in \overline{C}^{\infty}$, and consequently $\overline{C} \cup \{\infty\} \subseteq \overline{C}^{\infty}$.

(2) Note that $\overline{C \cup \{\infty\}}^{\infty} = \overline{C} \cup \{\infty\}$ – it follows from (1) and inclusions $C \subseteq C \cup \{\infty\} \subseteq \overline{C} \cup \{\infty\}$. It is enough to show that $A := C \cup \{\infty\}$ is connected. Suppose to the contrary that there are two nonempty disjoint open sets U and V with $A = (A \cap U) \cup (A \cap V)$ and $\infty \in U$. Put $U' := U \setminus \{\infty\}$. Then U' is open in \mathbb{R}^m . There is a compact set $D \subseteq \mathbb{R}^m$ such that $(X \setminus D) \cup \{\infty\} = U$. Then $X \setminus D = U'$ and $V \subseteq D$. Since V is nonempty, there is $i \in I$ with $V \cap C_i \neq \emptyset$. But then $C_i = (V \cap C_i) \cup (U \cap C_i)$ and by the connectedness of C_i we obtain $C_i \subseteq V \subseteq D$ which contradicts the unboundedness of C_i . \square

Theorem 3.5. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in \mathbb{R}^m and let $\sigma \in S_{\infty}$ be a permutation of indexes. Then the set $X = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ is either compact connected or it is a union (finite, infinite countable or uncountable) of unbounded closed connected sets; in particular \overline{X}^{∞} is compact and connected.*

Proof. Since X is closed in \mathbb{R}^m , $\overline{X}^\infty = X$ if X is bounded and $\overline{X}^\infty = X \cup \{\infty\}$ if X is unbounded. On \overline{X}^∞ define an equivalence relation E given by the decomposition of \overline{X}^∞ into components.

Assume that C is a bounded component of X . There is $R > 0$ such that $C \subseteq B(0, R)$ and $C \cap S(0, R) = \emptyset$. Suppose that the set $Z := \bigcup\{C' : C' \text{ is component of } X \cap B(0, R) \text{ such that } C' \cap S(0, R) \neq \emptyset\}$ is nonempty. Then, by Lemma 3.1, Z is compact in \mathbb{R}^m . Put $U := (\overline{X}^\infty \cap B(0, R)) \setminus Z$. Then U is open in \overline{X}^∞ and $U = q^{-1}(q(U))$; therefore $q(U)$ is open in \overline{X}^∞/E . Since $C \in q(U)$ and \overline{X}^∞/E is zero-dimensional, there is a clopen set $V \subseteq \overline{X}^\infty/E$ with $C \in V \subseteq q(U)$. Since Z and $Y := q^{-1}(V)$ are compact, there is $\varepsilon > 0$ with $Y(\varepsilon) \cap Z = \emptyset$ and $(Y(\varepsilon) \setminus Y) \cap \overline{X}^\infty = \emptyset$; consequently $Y(\varepsilon) \cap X \setminus Y = \emptyset$. By Lemma 3.3 we obtain that $Z \subseteq X \setminus Y = \emptyset$ which gives a contradiction. Thus $Z = \emptyset$. Therefore there are no components of X having nonempty intersection with $S(0, R)$. Thus $X \cap B(0, R) = q^{-1}(q(X \cap B(0, R)))$ and consequently $q(X \cap B(0, R))$ is open in \overline{X}^∞/E . Since \overline{X}^∞/E is zero dimensional, there is a clopen V with $C \in V \subseteq q(X \cap B(0, R))$. Thus $Y := q^{-1}(V)$ is clopen and it contains C . There is $\varepsilon > 0$ such that $Y(\varepsilon) \subseteq B(0, R)$ which means that $Y(\varepsilon)$ is disjoint with $X \setminus Y$. By Lemma 3.3 we obtain that X is bounded.

We have already proved that if C is a bounded component of X , then X is bounded itself. That means that if X has an unbounded component, then each its component is unbounded and, by Lemma 3.4, \overline{X}^∞ is connected – equivalently $q(\overline{X}^\infty) = [\infty]_E$. Thus \overline{X}^∞ is connected in $a(\mathbb{R}^m)$ if X is unbounded. To finish the proof we need to show that if X is bounded, then it is connected. If not, there would be two disjoint nonempty clopen subsets Y and $X \setminus Y$ of X . But then there would be $\varepsilon > 0$ with $Y(\varepsilon) \cap (X \setminus Y) = \emptyset$ which by Lemma 3.3 leads to a contradiction. \square

4. THEOREM 3.5 DOES NOT HOLD IN INFINITELY DIMENSIONAL SPACES

Now we will define: a conditionally convergent series $\sum_{n=1}^\infty y_n$ in c_0 such that $\sum_{n=1}^\infty y_n = \theta := (0, 0, \dots)$ and its rearrangement σ such that $\text{LIM}(\sum_{n=1}^\infty y_{\sigma(n)})$ consists of two points.

Example 4.1. As in Example 2.2 and Example 2.3 the constructed series $\sum_{n=1}^\infty y_n$ will be alternating, so we will define only elements with odd indexes. Let $\{e_i : i \in \mathbb{N}\}$ be a standard basis in c_0 . We define (y_n) inductively:

Step 1. At first we define three odd elements; the first equals e_2 , the second equals e_1 , and the third equals $-e_2$;

Step $k+1$. In this step we define $3 \cdot 2^k$ elements of the series with odd indexes: the first 2^k of them are equal to $\frac{1}{2^k}e_{k+2}$, next 2^k of y_n 's equal $\frac{1}{2^k}e_1$ and the last 2^k of them are equal to $-\frac{1}{2^k}e_{k+2}$.

We define a rearrangement $\sum_{n=1}^\infty x_n$ of $\sum_{n=1}^\infty y_n$ in the similar way as in Example 2.2. Namely, x_1, x_2, x_3 are consecutive elements of (y_n) defined in Step 1 with odd indexes, that is $x_1 = y_1, x_2 = y_3, x_3 = y_5$. Next three of x_n 's are the elements of (y_n) with even indexes corresponding to the odd indexes defined in Step 1 with reversed order, that is $x_4 = y_6, x_5 = y_4, x_6 = y_2$. In Step $k+1$ the first $3 \cdot 2^k$ x_n 's are consecutive elements of (y_n) defined in Step $k+1$ with odd indexes, and next $3 \cdot 2^k$ x_n 's are consecutive elements of (y_n) with even indexes with reversed order. The sequence of partial sums $s_m = \sum_{n=1}^m x_n$ is the following

$$\begin{aligned} & e_2, e_2 + e_1, e_1, e_2 + e_1, e_2, \theta, \\ & \frac{1}{2}e_3, e_3, e_3 + \frac{1}{2}e_1, e_3 + e_1, \frac{1}{2}e_3 + e_1, e_1, \frac{1}{2}e_3 + e_1, e_3 + e_1, e_3 + \frac{1}{2}e_1, e_3, \frac{1}{2}e_3, \theta, \dots \\ & \frac{1}{2^k}e_{k+2}, \frac{2}{2^k}e_{k+2}, \dots, e_{k+2}, e_{k+2} + \frac{1}{2^k}e_1, \dots, e_{k+2} + e_1, \frac{2^k - 1}{2^k}e_{k+2} + e_1, \dots, e_1, \\ & e_1 + \frac{1}{2^k}e_{k+2}, \dots, e_1 + e_{k+2}, e_{k+2} + \frac{2^k - 1}{2^k}e_1, \dots, e_{k+2}, \frac{2^k - 1}{2^k}e_{k+2}, \dots, \theta, \dots \end{aligned}$$

The walk (s_m) has the following properties:

- (i) θ and e_1 appear infinitely many times in a partial sums sequence (s_m) ;
- (ii) for every natural number $j \geq 2$ there exists $p \in \mathbb{N}$ such that $s_m(j) = 0$ for every natural $m \geq p$;
- (iii) the distance between the point $z = (z(1), z(2), \dots)$ with $z(1) \notin [0, 1]$ and the set $\{s_m : m \in \mathbb{N}\}$ is positive;
- (iv) if $s_m(1) \notin \{0, 1\}$ then there exists a natural $k \geq 2$ such that $s_m(k) = 1$.

We claim that $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \{\theta, e_1\}$. By (i) we get $\theta, e_1 \in \text{LIM}(\sum_{n=1}^{\infty} x_n)$. Conditions (ii) and (iii) give us the inclusion $\text{LIM}(\sum_{n=1}^{\infty} x_n) \subseteq \{(a, 0, 0, \dots) : a \in [0, 1]\}$. Indeed, since $z = (z(1), z(2), \dots) \in \text{LIM}(\sum_{n=1}^{\infty} x_n)$ then by (ii) we get $z(i) = 0$ for every $i \geq 2$. Moreover, if $z(1) > 1$ or $z(1) < 0$ then by (iii) we have $z \notin \text{LIM}(\sum_{n=1}^{\infty} x_n)$. Now, let $a \in (0, 1)$. We will show that $(a, 0, 0, \dots) \notin \text{LIM}(\sum_{n=1}^{\infty} x_n)$. One can find $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \cap \{0, 1\} = \emptyset$. We consider the ball $B((a, 0, 0, \dots), \varepsilon)$ in c_0 . If $z \in B((a, 0, 0, \dots), \varepsilon) \cap \{s_m : m \in \mathbb{N}\}$ then $z(1) \in (a - \varepsilon, a + \varepsilon)$, hence the first coordinate of z is neither 0 nor 1. Then by (iv) there exists a natural number $k \geq 2$ such that $z(k) = 1$ which contradicts the fact that $z \in B((a, 0, 0, \dots), \varepsilon)$. Hence $B((a, 0, 0, \dots), \varepsilon) \cap \{s_m : m \in \mathbb{N}\} = \emptyset$, so $(a, 0, 0, \dots) \notin \overline{\{s_m : m \in \mathbb{N}\}}$. Since $\text{LIM}(\sum_{n=1}^{\infty} x_n)$ is contained in $\overline{\{s_m : m \in \mathbb{N}\}}$, we have $(a, 0, 0, \dots) \notin \text{LIM}(\sum_{n=1}^{\infty} x_n)$. Finally $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \{\theta, e_1\}$.

Remark. Roman Wituła reminded us that he had found a very similar example of series with two-point limit set, see [5].

5. ON THE REVERSE OF THEOREM 3.5

In this Section we will prove that Theorem 3.5 can be reversed. It means that for any compact and connected subset X of a Euclidean space \mathbb{R}^m there is a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ and a permutation $\sigma \in S_{\infty}$ with $X = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$, and for any closed subset Y of \mathbb{R}^m whose each component is unbounded there is a conditionally convergent series $\sum_{n=1}^{\infty} y_n$ and permutation $\tau \in S_{\infty}$ with $Y = \text{LIM}(\sum_{n=1}^{\infty} y_{\tau(n)})$. This shows that Theorem 3.5 gives a full characterization of limit sets in finitely dimensional Banach spaces.

Theorem 5.1. *Let $m \in \mathbb{N}$. Assume that $X \subseteq \mathbb{R}^m$ is closed and ε -chainable for every $\varepsilon > 0$. Then there is a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ in \mathbb{R}^m such that $X = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ for some $\sigma \in S_{\infty}$. In particular, the assertion holds if X is compact and connected.*

Proof. Let (d_n) be dense in X . We will construct an X -walk. In the first step we find a 1-chain inside X between points d_1 and d_2 and denote it $a_1 = d_1, a_2, \dots, a_p = d_2$. We define $s_i = a_i$ for every $i \in \{1, \dots, p\}$. Then we go back to d_1 using the same way, which means that $s_i = a_{2p-i}$ for $i \in \{p+1, \dots, 2p-1\}$. In the second step let $a_{2p-1} = d_1, a_{2p}, \dots, a_{2p-1+r} = d_3$ be a 2^{-1} -chain between d_1 and d_3 . We define y_n 's in the same way, that means they are the following elements of the next chain, $s_i = a_i$ for $i \in \{2p-1, \dots, 2p-1+r\}$ and then we go back to d_1 via the same elements. In the third step we consider a 2^{-2} -chain between d_1 and d_4 and define the next s_n 's as before, and so on. By Proposition 2.1, we obtain the assertion. Finally, note that connected sets are ε -chainable for every $\varepsilon > 0$. \square

Theorem 5.2. *Let $m \geq 2$. Assume that $X \subseteq \mathbb{R}^m$ is closed and any component of X is unbounded. Then there is a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ in \mathbb{R}^m such that $X = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ for some $\sigma \in S_{\infty}$.*

Proof. Let $X = \bigcup_{t \in T} A_t$, where for every $t \in T$ the set A_t is an unbounded component of X . Clearly each A_t is closed and ε -chainable for every $\varepsilon > 0$. Let (d_n) be dense in X . If $d_i \in A_s, d_j \in A_t, A_t \cap A_s = \emptyset$, then, by the connectedness and the unboundedness of A_s and A_t , there is a sphere $S(0, R)$ which has non-empty intersections with A_s and A_t . Let $a_s \in A_s \cap S(0, R)$ and $a_t \in A_t \cap S(0, R)$. By an ε -chain via $S(0, R)$ from d_i to d_j we mean a concatenation of three ε -chains: from d_i to a_s using elements of A_s , from a_s to a_t using elements of S_R and from d_j to a_t using elements of A_t . If $A_t = A_s$, then by an ε -chain via $S(0, R)$ from d_i to d_j

we mean just an ε -chain from d_i to d_j using elements of A_s . Let R_n be a sequence of radii tending to ∞ such that $S(0, R_n)$ has a nonempty intersection with each component containing d_1, \dots, d_{n+1} . Now let us describe a walk (s_n) , which, in general, does not need to be an X -walk: The first elements of the walk (s_n) is a $1/2$ -chain via $S(0, R_1)$ from d_1 to d_2 , and then back via the same elements. In the k -th step of the construction the next elements of (s_n) are elements of a concatenation of 2^{-k} -chains via $S(0, R_k)$ from d_i to d_{i+1} , $i = 1, \dots, k$, and then back from d_{k+1} to d_1 via the same elements.

Using the same argument as in Proposition 2.1 we can find an alternating series $\sum_{n=1}^{\infty} x_n$ and $\sigma \in S_{\infty}$ such that $s_n = \sum_{i=1}^n x_{\sigma(i)}$. Clearly $X \subseteq \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)}) \subseteq X \cup \bigcup_{k=1}^{\infty} S(0, R_k)$. Since $R_k \rightarrow \infty$ and the sequence (s_n) contains at most finitely many elements of $S(0, R_k) \setminus X$, we obtain the reverse inclusion $X \supseteq \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$. \square

As we have mentioned in Introduction, the assertion of Theorem 5.2 is not true if $m = 1$.

6. WHEN A LIMIT SET IS A SINGLETON

By definition, if $\sum_{n=1}^{\infty} x_n = x_0$, then $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \{x_0\}$, since every subsequence of the sequence of partial sums is convergent to x_0 . In general the inverse implication does not need to be true which is illustrated, in c_0 , by Proposition 6.2. However, in finitely dimensional spaces the above implication can be reversed.

Theorem 6.1. *Let $\sum_{n=1}^{\infty} x_n$ be a series in \mathbb{R}^m with $x_n \rightarrow 0$. If $\text{LIM}(\sum_{n=1}^{\infty} x_n)$ is a singleton, then $\sum_{n=1}^{\infty} x_n$ is convergent and $\{\sum_{n=1}^{\infty} x_n\} = \text{LIM}(\sum_{n=1}^{\infty} x_n)$.*

Proof. Let $\text{LIM}(\sum_{n=1}^{\infty} x_n) = \{x_0\}$. Suppose that $\sum_{n=1}^{\infty} x_n$ does not converge to x_0 , so there exists $\varepsilon > 0$ such that for every $k_0 \in \mathbb{N}$ one can find $l \geq k_0$ such that $\|\sum_{n=1}^l x_n - x_0\| > \varepsilon$. That means that there are infinitely many indexes p such that $\sum_{n=1}^p x_n \notin \overline{B}(x_0, \varepsilon)$. On the other hand, since x_0 is a limit point of the series $\sum_{n=1}^{\infty} x_n$, there exist infinitely many $r \in \mathbb{N}$ such that $\sum_{n=1}^r x_n \in B(x_0, \varepsilon/2)$. Hence there are infinitely many elements of a walk (s_n) of partial sums inside the ball $B(x_0, \varepsilon/2)$ and infinitely many outside the closed ball $\overline{B}(x_0, \varepsilon)$. Since $x_n \rightarrow 0$, there are infinitely many s_n 's in $B = \overline{B}(x_0, \varepsilon) \setminus B(x_0, \varepsilon/2)$. By the compactness of B , it contains a limit point of (s_n) which contradicts that $\text{LIM}(\sum_{n=1}^{\infty} x_n)$ is a singleton. \square

Note that the assumption $x_n \rightarrow 0$ cannot be omitted. To see this consider the series $2^{-1} + 2^1 - 2^1 + 2^{-2} + 2^2 - 2^2 + 2^{-3} + 2^3 - 2^3 + \dots$. Clearly 1 is its only limit point, but the series is not convergent.

Proposition 6.2. *There is a conditionally convergent series $\sum_{n=1}^{\infty} x_n$ in c_0 and $\sigma \in S_{\infty}$ such that $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)}) = \{\theta\}$ but $\sum_{n=1}^{\infty} x_{\sigma(n)}$ diverges.*

Proof. To define an alternating series $\sum_{n=1}^{\infty} x_n$ it suffices to prescribe only the elements with odd indexes:

Step 1. Firstly we define $x_1 = e_1$.

Step k. The next 2^{k-1} elements with odd indexes are equal to $\frac{1}{2^{k-1}} e_k$.

The series $\sum_{n=1}^{\infty} x_n$ is the following

$$e_1 - e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_2 + \frac{1}{2}e_2 - \frac{1}{2}e_2 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 + \dots$$

Now we define a rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$. Firstly we use elements with odd indexes defined in Step 1, and then the corresponding elements with even indexes; secondly we use elements with odd indexes defined in Step 2, and then the corresponding elements with even indexes, and so on. The rearranged series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ has the following form

$$e_1 - e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_2 - \frac{1}{2}e_2 - \frac{1}{2}e_2 + \frac{1}{4}e_3 + \frac{1}{4}e_3 + \frac{1}{4}e_3 + \frac{1}{4}e_3 - \frac{1}{4}e_3 - \frac{1}{4}e_3 - \frac{1}{4}e_3 - \frac{1}{4}e_3 - \frac{1}{4}e_3 + \dots$$

Thus the walk $s_n = \sum_{k=1}^n x_{\sigma(k)}$ has the following properties:

- (1) $s_{2^{k+1}-2} = \theta$ for every $k \in \mathbb{N}$;
- (2) for every natural number j exists $p \in \mathbb{N}$ such that $s_m(j) = 0$ for every $m \geq p$;
- (3) $s_{2^{k+2^{k-1}}-2} = e_k$ for every $k \in \mathbb{N}$.

From (1) we have the inclusion $\{\theta\} \subseteq \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ and from (2) we get the inverse inclusion. Hence $\text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)}) = \{\theta\}$. From (1) and (3) we get $\|s_{2^{k+2^{k-1}}-2}(k) - s_{2^{k+1}-2}(k)\| = \|e_k\| = 1$ for every $k \in \mathbb{N}$. That means that the sequence of partial sums of the rearranged series is not a Cauchy sequence, and consequently it diverges. \square

7. REARRANGEMENT PROPERTY

Klee proved that if $A \subseteq \text{SR}(\sum_{n=1}^{\infty} x_n)$ is closed and ε -chainable for every $\varepsilon > 0$, then there is $\tau \in S_{\infty}$ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. This is a strengthening of Theorem 5.1 – to see it take any conditionally convergent series $\sum_{n=1}^{\infty} x_n$ with $\text{SR}(\sum_{n=1}^{\infty} x_n) = \mathbb{R}^m$. We show that this fact holds true in every Banach space provided $\sum_{n=1}^{\infty} x_n$ has the so-called Rearrangement Property. In fact we prove that if $A \subseteq \overline{\text{SR}(\sum_{n=1}^{\infty} x_n)}$ is closed and ε -chainable for every $\varepsilon > 0$, then there is $\tau \in S_{\infty}$ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$. As a byproduct of this observation we obtain that if $\sum_{n=1}^{\infty} x_n$ has the Rearrangement Property, then its sum range $\text{SR}(\sum_{n=1}^{\infty} x_n)$ is closed.

Lemma 7.1. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space X . Then $\text{SR}(\sum_{n=1}^{\infty} x_n) = \text{SR}(\sum_{n=k+1}^{\infty} x_n) + \sum_{n=1}^k x_n$ for every $k \in \mathbb{N}$.*

Proof. "⊇" Let $k \in \mathbb{N}$ and $x \in \text{SR}(\sum_{n=k+1}^{\infty} x_n) + \sum_{n=1}^k x_n$. Then there exists a permutation $\sigma : [k+1, \infty) \rightarrow [k+1, \infty)$ such that $x = \sum_{n=k+1}^{\infty} x_{\sigma(n)} + \sum_{n=1}^k x_n$. Define $\pi(n) = n$ for $n \leq k$ and $\pi(n) = \sigma(n)$ for $n \geq k+1$. Hence $x = \sum_{n=1}^{\infty} x_{\pi(n)}$, so $x \in \text{SR}(\sum_{n=1}^{\infty} x_n)$.

"⊆" Let $x \in \text{SR}(\sum_{n=1}^{\infty} x_n)$ and $k \in \mathbb{N}$. Then there exists a permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $x = \sum_{n=1}^{\infty} x_{\pi(n)}$. Let $M = \pi^{-1}(\{1, \dots, k\})$. Then for every $\varepsilon > 0$ there exists $m_0 \geq \max M$ such that for every $m > m_0$ the following inequality is true: $\|x - \sum_{n=1}^m x_{\pi(n)}\| < \varepsilon$. It means that $\|x - \sum_{n=1}^k x_n - \sum_{n \in \{1, \dots, m\} \setminus M} x_{\pi(n)}\| < \varepsilon$ for every $m > m_0$. Define a permutation $\sigma : [k+1, \infty) \rightarrow [k+1, \infty)$ as follows $\sigma(k+l) = \pi(n)$ where n is the l -th number in the set $\mathbb{N} \setminus M$. Then $x = \sum_{n=k+1}^{\infty} x_{\sigma(n)} + \sum_{n=1}^k x_n$. Hence $x \in \text{SR}(\sum_{n=k+1}^{\infty} x_n) + \sum_{n=1}^k x_n$. \square

We say that a conditionally convergent series $\sum_{k=1}^{\infty} x_k$ has the Rearrangement Property, or (RP), if for every $\varepsilon > 0$ there are: a natural number $N(\varepsilon)$ and a positive real number $\delta(\varepsilon)$ such that the implication

$$\left\| \sum_{i=1}^n y_i \right\| < \delta(\varepsilon) \Rightarrow \left(\max_{j \leq n} \left\| \sum_{i=1}^j y_{\sigma(i)} \right\| < \varepsilon \text{ for some permutation } \sigma \in S_n \right)$$

holds for every finite sequence $(y_i)_{i=1}^n \subseteq (x_i)_{i=N(\varepsilon)}^{\infty}$. Note that if $\varepsilon > \varepsilon' > 0$, then we can find numbers $\delta(\varepsilon), N(\varepsilon)$ and $\delta(\varepsilon'), N(\varepsilon')$ from the definition of (RP) used for ε and ε' , respectively, such that $\delta(\varepsilon) \geq \delta(\varepsilon')$ and $N(\varepsilon) \leq N(\varepsilon')$. Similarly, having a decreasing sequence (ε_n) of positive real numbers, we find $\delta(\varepsilon_n), N(\varepsilon_n)$ from the definition of (RP) such that $\delta(\varepsilon_n) \geq \delta(\varepsilon_{n+1})$ and $N(\varepsilon_n) \leq N(\varepsilon_{n+1})$ for every $n \in \mathbb{N}$.

Lemma 7.2. *Assume that $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series with (RP) in a Banach space X . Let $\varepsilon \geq \varepsilon' > 0$ and let $\delta(\frac{\varepsilon}{2}), N(\frac{\varepsilon}{2})$ and $\delta(\frac{\varepsilon'}{2}), N(\frac{\varepsilon'}{2})$ be numbers from the definition of (RP) used for $\frac{\varepsilon}{2}$ and $\frac{\varepsilon'}{2}$, respectively. Let $k \in \mathbb{N}$, $a, b \in \text{SR}(\sum_{n=1}^{\infty} x_n)$ with $\|a - b\| < \min\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon}{2})\}$, and $\tau : [1, k] \rightarrow \mathbb{N}$ be a partial permutation such that $\|\sum_{n=1}^k x_{\tau(n)} - a\| \leq \min\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon}{2})\}$ and $\text{rng } \tau \supseteq [1, N(\frac{\varepsilon}{2})]$. Then there exist $k' > k$ and a partial permutation $\tau' : [1, k'] \rightarrow \mathbb{N}$ such that the following conditions hold:*

- (1) $\tau'|_{[1, k]} = \tau$ and $[1, \max \text{rng } \tau] \subseteq \text{rng } \tau'$;
- (2) $\|\sum_{n=1}^p x_{\tau'(n)} - a\| \leq \varepsilon$ for $p \in [k+1, k']$;

- (3) $\|\sum_{n=1}^{k'} x_{\tau'(n)} - b\| \leq \min\{\frac{\varepsilon'}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon'}{2})\};$
(4) $\text{rng}(\tau') \supseteq [1, N(\frac{\varepsilon'}{2})].$

Proof. Let $k_0 = \max\{N(\frac{\varepsilon'}{2}), N(\frac{\varepsilon}{2}), \max \text{rng } \tau\}$. Define $y = \sum_{n=1}^k x_{\tau(n)}$ and $z = \sum_{n \in \{1, \dots, k_0\} \setminus \{\tau(1), \dots, \tau(k)\}} x_n$. Hence $y + z = \sum_{n=1}^{k_0} x_n$. From the assumption that $b \in \text{SR}(\sum_{n=1}^{\infty} x_n)$ by Lemma 7.1 we obtain $b - (y + z) \in \text{SR}(\sum_{n=k_0+1}^{\infty} x_n)$. Thus we can find $k_0 < n_1 < n_2 < \dots < n_l$ such that $y + z + w \in B(b, \min\{\frac{\varepsilon'}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon'}{2}), \frac{1}{3} \cdot \delta(\frac{\varepsilon}{2})\})$, where $w = x_{n_1} + \dots + x_{n_l}$.

Enumerate the set $([1, k_0] \setminus \{\tau(1), \dots, \tau(k)\}) \cup \{n_1, \dots, n_l\}$ as $\{m_1 < m_2 < \dots < m_{k'-k}\}$, where $k' = k_0 + l$. Hence,

$$\left\| \sum_{i=1}^{k'-k} x_{m_i} \right\| = \|z + w\| \leq \|y - a\| + \|a - b\| + \|b - (y + z + w)\|.$$

Consequently,

$$\left\| \sum_{i=1}^{k'-k} x_{m_i} \right\| \leq \min\left\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta\left(\frac{\varepsilon}{2}\right)\right\} + \min\left\{\frac{\varepsilon'}{12}, \frac{1}{3} \cdot \delta\left(\frac{\varepsilon'}{2}\right)\right\} + \min\left\{\frac{\varepsilon'}{12}, \frac{1}{3} \cdot \delta\left(\frac{\varepsilon'}{2}\right), \frac{1}{3} \cdot \delta\left(\frac{\varepsilon}{2}\right)\right\} \leq \delta\left(\frac{\varepsilon}{2}\right).$$

Since $m_i \geq N(\frac{\varepsilon}{2})$ for $i \in [1, k' - k]$ and $\|\sum_{i=1}^{k'-k} x_{m_i}\| \leq \delta(\frac{\varepsilon}{2})$, by (RP) there is a permutation $\sigma \in S_{k'-k}$ such that $\|\sum_{i=1}^j x_{m_{\sigma(i)}}\| \leq \frac{\varepsilon}{2}$ for every $j \in [1, k' - k]$. Let us define $\tau'(n) = \tau(n)$ for $n \leq k$ and $\tau'(n) = m_{\sigma(n-k)}$ for $n \in [k+1, k']$. Then for every $p \in [k+1, k']$ we have the following:

$$\left\| \sum_{n=1}^p x_{\tau'(n)} - a \right\| = \left\| \sum_{n=1}^k x_{\tau(n)} + \sum_{n=k+1}^p x_{\tau'(n)} - a \right\| \leq \|y - a\| + \left\| \sum_{n=k+1}^p x_{\tau'(n)} \right\| \leq \min\left\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta\left(\frac{\varepsilon}{2}\right)\right\} + \frac{\varepsilon}{2} < \varepsilon$$

which gives us (2).

Now we check (1), (3) and (4). Note that the numbers $1, \dots, k_0$ are among $\tau'(1), \dots, \tau'(k')$ and $k_0 \geq \max \text{rng } \tau$. Therefore we have (1). Since $\sum_{n=1}^{k'} x_{\tau'(n)} = y + z + w$ and $\|y + z + w - b\| \leq \min\{\frac{\varepsilon'}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon'}{2})\}$, we obtain (3). Condition (4) follows from the fact that if $n \notin \text{rng}(\tau')$, then $n > k_0 \geq N(\frac{\varepsilon'}{2})$. \square

Lemma 7.3. *Let A be a subset of a Banach space such that A is separable and ε -chainable for every $\varepsilon > 0$. Let (η_i) be a sequence of positive numbers. Then there is a sequence (d_n) dense in A with the property that there is an increasing sequence (l_i) such that $\{d_{l_i}, d_{l_i+1}, \dots, d_{l_{i+1}}\}$ is an η_i -chain for every i .*

Proof. Since A is separable, there are v_1, v_2, \dots such that $A = \overline{\{v_n : n \in \mathbb{N}\}}$. Then one can find an η_i -chain: $d_{l_i}, d_{l_i+1}, \dots, d_{l_{i+1}}$ of elements of A with $d_{l_i} = v_i$ and $d_{l_{i+1}} = v_{i+1}$ for any $i \in \mathbb{N}$. Clearly the sequence $\{d_n\}_{n=1}^{\infty}$ fulfills the desired condition. \square

Lemma 7.4. *Let A be a separable and ε -chainable for every $\varepsilon > 0$ subset of a Banach space. Assume that $\{d_i : i \in \mathbb{N}\}$ is a dense subset of A and (ε_i) is a sequence of positive numbers tending to zero. If (x_i) is such that $\|x_i - d_i\| < \varepsilon_i$ for every $i \in \mathbb{N}$, then $\text{LIM}(x_i) = \overline{A}$ where $\text{LIM}(x_i)$ denotes the set of all limit points of the sequence (x_i) .*

Proof. If A is a singleton, then $d_i = a$, $A = \{a\}$ and $x_i \rightarrow a$. Then $\text{LIM}(x_i) = \{a\} = \overline{A}$. Assume that A has at least two elements. Clearly A is dense-in-itself. Fix $i \in \mathbb{N}$. There is a sequence $(d_{j_k})_{k=1}^{\infty}$ such that $j_1 < j_2 < \dots$ and $\|d_{j_k} - d_i\| < \varepsilon_{j_k}$. Then $x_{j_k} \rightarrow d_i$ and consequently $d_i \in \text{LIM}(x_i)$. Since the set $\text{LIM}(x_i)$ is closed, we have $\overline{A} \subseteq \text{LIM}(x_i)$.

Note that for every k almost every element of (x_i) is in ε_k -shell of A . Thus $\overline{A} \supseteq \text{LIM}(x_i)$. \square

Theorem 7.5. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series with (RP) in a Banach space X . Then for every $A \subseteq \overline{\text{SR}(\sum_{n=1}^{\infty} x_n)}$ which is closed and ε -chainable for every $\varepsilon > 0$, there exists a permutation $\tau \in S_{\infty}$ such that $A = \text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)})$.*

Proof. Let $\varepsilon_i = \frac{1}{2^i}$. We fix numbers $\delta(\frac{\varepsilon_i}{2}), N(\frac{\varepsilon_i}{2})$ from the definition of (RP) such that $\delta(\frac{\varepsilon_i}{2}) \geq \delta(\frac{\varepsilon_{i+1}}{2})$ and $N(\frac{\varepsilon_i}{2}) \leq N(\frac{\varepsilon_{i+1}}{2})$ for every $i \in \mathbb{N}$. Since A is separable and ε -chainable for every $\varepsilon > 0$, using Lemma 7.3, let $A = \overline{\{d_n : n \in \mathbb{N}\}}$, where for every $i \in \mathbb{N}$ the elements $d_{l_i}, d_{l_i+1}, \dots, d_{l_{i+1}}$ form a η_i -chain for some $1 = l_1 < l_2 < \dots$, where $\eta_i = \min\{\frac{\varepsilon_i}{48}, \frac{1}{12} \cdot \delta(\frac{\varepsilon_i}{2})\}$. Note that (η_i) is a non-increasing sequence of positive real numbers.

Inductively we define natural numbers $1 = k_1 < k_2 < \dots$, one-to-one functions $\tau_i : [1, k_{i+1}] \rightarrow \mathbb{N}$ and d'_1, d'_2, \dots fulfilling the following conditions

- (i) $\tau_i \subseteq \tau_{i+1}$;
- (ii) $[1, \max \text{rng } \tau_i] \subseteq \text{rng}(\tau_{i+1})$;
- (iii) $\|\sum_{n=1}^p x_{\tau_i(n)} - d'_{i-1}\| < \varepsilon_j$ for $p \in [k_i + 1, k_{i+1}]$ and $i \in [l_j + 1, l_{j+1}]$;
- (iv) $d'_i \in \text{SR}(\sum_{n=1}^\infty x_n)$, $\|d'_i - d_i\| < \eta_j$ where $i \in [l_j, l_{j+1} - 1]$;
- (v) $\|\sum_{n=1}^{k_{i+1}} x_{\tau_i(n)} - d'_i\| < 4\eta_j$ where $i \in [l_j, l_{j+1} - 1]$;
- (vi) $\text{rng } \tau_i \supseteq [1, N(\frac{\varepsilon_i}{2})]$ where $i \in [l_j, l_{j+1} - 1]$.

Define $x = \sum_{n=1}^{N(\frac{\varepsilon_1}{2})} x_n$. Let $d'_1 \in \text{SR}(\sum_{n=1}^\infty x_n)$ be such that $\|d_1 - d'_1\| < \eta_1$. Hence from Lemma 7.1 we get $d'_1 - x \in \text{SR}(\sum_{n=N(\frac{\varepsilon_1}{2})+1}^\infty x_n)$. Let $\pi : [N(\frac{\varepsilon_1}{2}) + 1, \infty) \rightarrow [N(\frac{\varepsilon_1}{2}) + 1, \infty)$ be a bijection such that $d'_1 - x = \sum_{n=N(\frac{\varepsilon_1}{2})+1}^\infty x_{\pi(n)}$. One can find a natural number $k_2 > N(\frac{\varepsilon_1}{2})$, which satisfies the inequality $\|d'_1 - x - \sum_{n=N(\frac{\varepsilon_1}{2})+1}^{k_2} x_{\pi(n)}\| \leq \eta_1$. Define $\tau_1(k) = k$ for $k \leq N(\frac{\varepsilon_1}{2})$ and $\tau_1(k) = \pi(k)$ for $k \in [N(\frac{\varepsilon_1}{2}) + 1, k_2]$. Conditions (i)–(vi) are fulfilled for τ_1, d'_1, k_1, k_2 . We do not need to check (i) and (ii), condition (iii) needs to be checked for $i \geq l_1 + 1 = 2$ and conditions (iv)–(vi) are fulfilled since $l_1 = 1$.

Assume now, that we have already defined $\tau_1, \dots, \tau_i, k_1 < \dots < k_{i+1}$ and d'_1, \dots, d'_i fulfilling (i)–(vi). Find d'_{i+1} such that (iv) holds. We use Lemma 7.2 for $a = d'_i, b = d'_{i+1}, \tau = \tau_i, \varepsilon = \varepsilon_j$ where $l_j \leq i \leq l_{j+1} - 1, \varepsilon' = \varepsilon_q$ where $l_q - 1 \leq i \leq l_{q+1} - 2$, and $k = k_{i+1}$; note that $j = q$ if $l_j \leq i < l_{j+1} - 1$, that is if $i \notin \{l_s - 1 : s \geq 1\}$, otherwise $i = l_{j+1} - 1$ implies that $q = j + 1$. Let us check the assumptions of Lemma 7.2. By (iv) and (vi) we obtain $a, b \in \text{SR}(\sum_{n=1}^\infty x_n)$ and $\text{rng } \tau \supseteq [1, N(\frac{\varepsilon}{2})]$. Since $d_{l_j}, d_{l_j+1}, \dots, d_{l_{j+1}}$ form a η_j -chain, by (iv) we obtain

$$\|a - b\| \leq \|d_i - d'_i\| + \|d_i - d_{i+1}\| + \|d_{i+1} - d'_{i+1}\| \leq \eta_j + \eta_j + \eta_j < 4\eta_j = \min\left\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta\left(\frac{\varepsilon}{2}\right)\right\}.$$

By (v) we obtain $\|\sum_{n=1}^k x_{\tau(n)} - a\| \leq 4\eta_j = \min\{\frac{\varepsilon}{12}, \frac{1}{3} \cdot \delta(\frac{\varepsilon}{2})\}$. Now, using Lemma 7.2 we find $k_{i+2} > k_{i+1}$ and function $\tau_{i+1} : [1, k_{i+2}] \rightarrow \mathbb{N}$ such that

- (1) $\tau_{i+1}|_{[1, k_{i+1}]} = \tau_i$ and $[1, \max \text{rng } \tau_i] \subseteq \text{rng } \tau_{i+1}$;
- (2) $\|\sum_{n=1}^p x_{\tau_{i+1}(n)} - d'_i\| \leq \varepsilon_j$ where $p \in [k_{i+1} + 1, k_{i+2}]$ and $i \in [l_j, l_{j+1} - 1]$;
- (3) $\|\sum_{n=1}^{k_{i+2}} x_{\tau_{i+1}(n)} - d'_{i+1}\| \leq 4\eta_q$ where $i \in [l_q - 1, l_{q+1} - 2]$;
- (4) $\text{rng}(\tau_{i+1}) \supseteq [1, N(\frac{\varepsilon_q}{2})]$ where $i \in [l_q - 1, l_{q+1} - 2]$.

Note that $\tau_1, \dots, \tau_{i+1}, k_1 < \dots < k_{i+2}$ and d'_1, \dots, d'_{i+1} fulfill (i)–(vi): By (1) we obtain conditions (i) and (ii). Since the condition $i + 1 \in [l_j + 1, l_{j+1}]$ is equivalent to $i \in [l_j, l_{j+1} - 1]$, we obtain (iii). The element d'_{i+1} has already been chosen to fulfill (iv). Conditions (3) and (4) are exactly (v) and (vi) for $i + 1$.

Let $\tau = \bigcup_{i \geq 1} \tau_i : \mathbb{N} \rightarrow \mathbb{N}$. Then (i) implies that τ is one-to-one. Condition (ii) implies that τ is onto \mathbb{N} , and consequently $\tau \in S_\infty$. By (iii) and (iv) we obtain that the distance between A and $\sum_{n=1}^p x_{\tau(n)}$ is less than $1/2^j$ for almost every $p \in \mathbb{N}$. Thus $\text{LIM}(\sum_{n=1}^\infty x_{\tau(n)}) \subseteq \bar{A}$. By (iv) and (v) we obtain that $\|\sum_{n=1}^{k_{i+1}} x_{\tau(n)} - d_i\| < 5\eta_j < \varepsilon_j$ where $i \in [l_j, l_{j+1} - 1]$. Thus by Lemma 7.4 we get $\bar{A} = \text{LIM}((\sum_{n=1}^{k_{i+1}} x_{\tau(n)})_{i=1}^\infty) \subseteq \text{LIM}(\sum_{n=1}^\infty x_{\tau(n)})$. \square

It is well-known that every conditionally convergent series of elements in a finite dimensional Banach space has the (RP), for detail see [2]. Thus, the Klee's result which we have mentioned at the beginning of this Section is a particular case of Theorem 7.5. Combining methods used in proofs of Theorem 7.5 and Theorem 5.2 one can prove the following strengthening of Theorem 5.2.

Corollary 7.6. *Let $m \geq 2$. Assume that $\text{SR}(\sum_{n=1}^{\infty} x_n) = \mathbb{R}^m$, $X \subseteq \mathbb{R}^m$ is closed and any component of X is unbounded. Then $X = \text{LIM}(\sum_{n=1}^{\infty} x_{\sigma(n)})$ for some $\sigma \in S_{\infty}$.*

Note that singletons are trivially ε -chainable for every $\varepsilon > 0$. Fix $a \in \overline{\text{SR}(\sum_{n=1}^{\infty} x_n)}$. Using Theorem 7.5 for $A \subseteq \overline{\text{SR}(\sum_{n=1}^{\infty} x_n)}$ such that $A = \{a\}$, we obtain that there is $\tau \in S_{\infty}$ such that $\text{LIM}(\sum_{n=1}^{\infty} x_{\tau(n)}) = \{a\}$. As we have seen in Proposition 6.2 it does not necessarily mean that $\sum_{n=1}^{\infty} x_{\tau(n)} = a$. However, if we put $d_i = a$, then $d'_i \rightarrow a$ and by condition (iii) we get that almost all elements of the sequence $(\sum_{n=1}^p x_{\tau(n)})_{p=1}^{\infty}$ of partial sums are in every neighborhood of a . Therefore $\sum_{n=1}^{\infty} x_{\tau(n)}$ is convergent to a . Thus $a \in \text{SR}(\sum_{n=1}^{\infty} x_n)$. Hence as a byproduct of the proof of Theorem 7.5 we obtain the following.

Corollary 7.7. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space X , which has the (RP). Then its sum range $\text{SR}(\sum_{n=1}^{\infty} x_n)$ is a closed set.*

Now, we will discuss a problem whether or not Corollary 7.7 can be reversed, namely whether or not the closedness of the sum range $\text{SR}(\sum_{n=1}^{\infty} x_n)$ implies the (RP) for a series $\sum_{n=1}^{\infty} x_n$. By S_n we denote the set of all permutations of the set $[1, n]$.

Lemma 7.8. *Let $k \in \mathbb{N}$ and $n = \binom{2k}{k}$, then there exists a finite sequence $x_1, \dots, x_{2k} \in \mathbb{R}^n$ such that:*

- (1) $\|x_i\|_{\text{sup}} = 1$ for every $i \leq 2k$.
- (2) $\|\sum_{i=1}^k x_{\sigma(i)}\|_{\text{sup}} \geq k$ for every $\sigma \in S_{2k}$.
- (3) $\sum_{i=1}^{2k} x_i = 0$.

Proof. Let $k \in \mathbb{N}$. There are $n = \binom{2k}{k}$ sequences of length $2k$ consisting of k many 1's and k many -1 's. Enumerate all such sequences as t_1, \dots, t_n . Define $x_i(j) = t_j(i)$ for $j = 1, \dots, n$ and $i = 1, \dots, 2k$. Now, if $\sigma \in S_{2k}$, then there is a sequence $t_{j_{\sigma}}$ such that $t_{j_{\sigma}}(\sigma(i)) = 1$ for $i = 1, \dots, k$ and $t_{j_{\sigma}}(\sigma(i)) = -1$ for $i = k+1, \dots, 2k$. Thus

$$\sum_{i=1}^k x_{\sigma(i)}(j_{\sigma}) = k,$$

and consequently

$$\left\| \sum_{i=1}^k x_{\sigma(i)} \right\|_{\text{sup}} \geq k.$$

□

Before we state the last result, first note that if the series $\sum_{i=1}^{\infty} x_i$ does not have (RP), then one can find $\varepsilon > 0$ such that for every $\delta > 0$ and $N \in \mathbb{N}$ there exists the finite subsequence $\{y_i\}_{i=1}^n \subseteq \{x_i\}_{i=1}^{\infty}$ for which two conditions hold:

- $\|\sum_{i=1}^n y_i\| < \delta$.
- for every $\sigma \in S_n$ there is $j \leq n$ such that $\|\sum_{i=1}^j y_{\sigma(i)}\|_{\text{sup}} \geq \varepsilon$.

The following theorem shows that Corollary 7.7 cannot be reversed.

Theorem 7.9. *There is a conditionally convergent series $\sum_{n=1}^{\infty} z_n$ in c_0 such that it does not have (RP) and its sum range $\text{SR}(\sum_{n=1}^{\infty} z_n)$ is a singleton, in particular it is a closed set.*

Proof. Define $e_n = (\delta_{in})_{i=1}^{\infty}$ for every $n \in \mathbb{N}$, where $\delta_{in} = 1$, if $i = n$ and $\delta_{in} = 0$ otherwise. Let $n_0 = 0$ and $n_k = \binom{2^{k+1}}{2^k} + n_{k-1}$. For every $k \in \mathbb{N}$ let $x_1^{(k)}, \dots, x_{2^{k+1}}^{(k)} \in \mathbb{R}^{n_k - n_{k-1}}$ will be the sequence constructed in Lemma 7.8. Define $y_i^{(k)} = \frac{1}{2^k} \cdot \sum_{j=1}^{n_k - n_{k-1}} x_i^{(k)}(j) \cdot e_{n_{k-1}+j}$ for $i, k \in \mathbb{N}$. It easy to see that $y_i^{(k)} \in c_0$. Define the series $\sum_{n=1}^{\infty} z_n$ as follows:

$$z_1 = y_1^{(1)}, z_2 = -y_1^{(1)}, z_3 = y_2^{(1)}, z_4 = -y_2^{(1)}, z_5 = y_3^{(1)}, z_6 = -y_3^{(1)}, z_7 = y_4^{(1)}, z_8 = -y_4^{(1)}, z_9 = y_1^{(2)}, z_{10} = -y_1^{(2)}, \dots$$

It is easy to see that $\sum_{n=1}^{\infty} z_n$ converges to 0.

Let $\varepsilon = 1$ and $N \in \mathbb{N}$, $\delta > 0$. One can find $k \in \mathbb{N}$, such that $n_{k-1} > N$. Then by Lemma 7.8 for $(y_1^{(k)}, \dots, y_{2^{k+1}}^{(k)}) \subseteq (z_i)_{i \geq N}$ and every permutation $\sigma \in S_{2^{k+1}}$ we have

$$\left\| \frac{1}{2^k} \cdot \sum_{i=1}^{2^k} y_{\sigma(i)}^{(k)} \right\|_{\text{sup}} = \left\| \frac{1}{2^k} \cdot \sum_{i=1}^{2^k} x_{\sigma(i)}^{(k)} \right\|_{\text{sup}} \geq \frac{1}{2^k} \cdot 2^k = 1 = \varepsilon.$$

Moreover $\left\| \sum_{i=1}^{2^{k+1}} y_i^{(k)} \right\| = 0 < \delta$. This proves that the series $\sum_{n=1}^{\infty} z_n$ does not have (RP).

Since the projection of the series on each coordinate contains only finitely many nonzero terms and a finite sum does not change under rearrangements, then $\text{SR}(\sum_{n=1}^{\infty} z_n) = \{\theta\}$. \square

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REFERENCES

- [1] Engelking, R. General topology. Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989. viii+529 pp.
- [2] Kadets, M.I.; Kadets, V.M. Series in Banach spaces. Conditional and unconditional convergence. Translated from the Russian by Andrei Iacob. Operator Theory: Advances and Applications, 94. Birkhuser Verlag, Basel, 1997. viii+156 pp.
- [3] Klee, V. Rearrangements of series of vectors. Math. Z. 81, 1963, 46–51.
- [4] Srivastava, S. M. A course on Borel sets. Graduate Texts in Mathematics, 180. Springer-Verlag, New York, 1998. xvi+261 pp.
- [5] Wituła, R.; Hetmaniok, E.; Kaczmarek, K. On series whose rearrangements possess discrete sets of limit points. J. Appl. Anal. 20 (2014), no. 1, 93–96.

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