ON PARAMETRIC LIMIT SUPERIOR OF A SEQUENCE OF ANALYTIC SETS

Abstract

Let \( A_x \) stand for \( x \)-section of a set \( A \subset 2^\omega \times 2^\omega \). We prove that any sequence \( A_j \subset 2^\omega \times 2^\omega \), \( j \in \omega \), of analytic sets, with uncountable \( \limsup_{j \in H} A_j \) for all \( x \in 2^\omega \) and \( H \in [\omega]^{<\omega} \), admits a perfect set \( P \subset 2^\omega \) and \( H \in [\omega]^{<\omega} \) with uncountable \( \bigcap_{j \in H} A_j \) for all \( x \in P \). This is a parametric version of the Komjáth theorem [2].

1 Main Result.

In [2] Komjáth proved that if the sets \( A^0, A^1, \ldots \) are analytic sets in a Polish space, and \( \limsup_{j \in H} A_j \) is uncountable for each \( H \in [\omega]^{<\omega} \), then there exists a set \( G \in [\omega]^{<\omega} \) for which the intersection \( \bigcap_{j \in G} A_j \) is uncountable. The previous version of this statement was proved by Laczkovich in [3] for a sequence of Borel sets. Komjáth, assuming \( MA(\omega_1) \), proved that this statement holds if the analicity of sets \( A^j \) is skipped, but assuming the axiom of constructibility, he proved that it is false for a sequence of coanalytic sets.

In this paper we prove a parametric version of the Komjáth result. The Parametrized Ellentuck theorem due to Pawlikowski [4] is our basic tool in the proof. We discuss examples which show that some stronger versions of our theorem are impossible.

We use standard set theoretical notation (see [1]). A subset \( P \) of a Polish space is called perfect if it is nonempty, closed, and dense in itself. For \( \alpha \in [\omega]^{<\omega} \) and \( H \in [\omega]^{<\omega} \), let \( [\alpha, H] \) be an Ellentuck neighbourhood; i.e., a set of the form \( \{ G \in [\omega]^{<\omega} : \alpha \subset G \subset \alpha \cup (H \setminus \max(\alpha)) \} \). A set \( A \subset 2^\omega \times [\omega]^{<\omega} \) is called perfectly Ramsey if for any perfect set \( P \subset 2^\omega \) and any Ellentuck neighbourhood \([\alpha, H]\), there exists a perfect set \( Q \subset P \) and an infinite set

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\( G \subset H \) such that either \( Q \times [\alpha, G] \subset A \) or \( (Q \times [\alpha, G]) \cap A = \emptyset \). For \( A \subset 2^\omega \times 2^\omega \) and \( x \in 2^\omega \), put \( A_x = \{y \in 2^\omega : (x, y) \in A\} \); this is called \( x \)-section of \( A \).

**Theorem 1.** Let \((A^j)_{j \in \omega}\) be a sequence of analytic subsets of \( 2^\omega \times 2^\omega \) such that

\[
\forall x \in 2^\omega \ \forall H \in [\omega]^\omega \ \text{card}(\limsup_{j \in H} A^j_x) > \omega.
\]

Then there exist a perfect set \( P \subset 2^\omega \) and \( H \in [\omega]^\omega \) such that

\[
\forall x \in P \ \text{card}(\bigcap_{j \in H} A^j_x) > \omega.
\]

**Proof.** We treat \([\omega]^\omega\) as a Polish subspace of \( 2^\omega \), identifying \( H \in [\omega]^\omega \) with its characteristic function. Define

\[
A = \{(x, H) \in 2^\omega \times [\omega]^\omega : \text{card}(\bigcap_{j \in H} A^j_x) > \omega\}.
\]

Consider

\[
B = \{(x, H, y) \in 2^\omega \times [\omega]^\omega \times 2^\omega : (x, y) \in \bigcap_{j \in H} A^j \} = \{(x, H, y) \in 2^\omega \times [\omega]^\omega \times 2^\omega : \forall j \in \omega \ (j \notin H \text{ or } (x, y) \in A^j)\}
\]

and note that \( B \) is analytic. Thus

\[
A = \{(x, H) \in 2^\omega \times [\omega]^\omega : \text{card}(B_{(x, H)}(x, H)) > \omega\}
\]

is analytic, by the Mazurkiewicz–Sierpiński theorem [1, 29.19]. Now by [4], the set \( A \) is perfectly Ramsey. Hence, there exist a perfect set \( P \subset 2^\omega \) and \( H \in [\omega]^\omega \) such that either \( P \times [\emptyset, H] \subset A \) or \( (P \times [\emptyset, H]) \cap A = \emptyset \). This last case is impossible since for each \( x \in P \), there is \( G \in [H]^\omega \) such that \( \text{card}(\bigcap_{j \in G} A^j_x) > \omega \) (see [2, Theorem 1]). Finally we obtain that

\[
\forall x \in P \ \text{card}(\bigcap_{j \in H} A^j_x) > \omega. \quad \square
\]
2 Examples.

Note that it is impossible to improve Theorem 1 (in ZFC) assuming that sets $A^j$ are coanalytic (see [2, Theorem 4]). The following examples show that we also can not improve it assuming only that all sections of $A^j$ are analytic (even clopen).

**Example 1.** We will construct a sequence $(A^j)_{j \in \omega}$ of subsets of $2^\omega \times 2^\omega$ such that

- $\forall H \in [\omega]^{\omega} \forall x \in 2^\omega \ card(\limsup_{j \in H} A^j_x) > \omega$,
- $A^j_x$ is clopen for all $x \in 2^\omega$,

and there is no perfect set $P \subset 2^\omega$, and no $H \in [\omega]^{\omega}$ such that

$$\forall x \in P \ card(\bigcap_{j \in H} A^j_x) > \omega.$$  

Let $\{N_i : i \in \omega\}$ be a family of almost disjoint infinite subsets of $\omega$ such that for every $s \in [\omega]^{<\omega}$, there exists $i, j \in \omega$ with $N_i \cap N_j = s$. Let $\{B_i : i \in \omega\}$ be a partition of $2^\omega$ into pairwise disjoint Bernstein subsets. Fix two disjoint clopen sets $C^0, C^1 \subset 2^\omega$. Put

$$A^j = \bigcup_{i \in \omega} B_i \times C^{\chi_{N_i}(j)},$$

where $\chi_{N_i}$ is the characteristic function of $N_i$. Immediately from the definition of $A^j$ we obtain that for every $x \in 2^\omega$, $A^j_x$ is a clopen set, and for any $x \in 2^\omega$ and $H \in [\omega]^{\omega}$, there exists $G \in [H]^{\omega}$ such that $card(\bigcap_{j \in G} A^j_x) > \omega$.

Suppose that there exists a perfect set $P$ and $H \in [\omega]^{\omega}$ such that

$$\forall x \in P \ card(\bigcap_{j \in H} A^j_x) > \omega.$$  

Then $P$ intersects every set $B_i, \ i \in \omega$. For each $i$, since $P$ intersects $B_i$, it follows that either $H \subset N_i$ or $H \supset \omega \setminus N_i$. Since $N_i \cap N_j$ is finite for $i \neq j$, there exists $i_0 \in \omega$ such that $H \subset \omega \setminus N_i$ for all $i \neq i_0$. Let $s \in H$ be finite and nonempty. There exists $i, j \in \omega$ such that $N_i \cap N_j = s$. Then $H \cap N_i \neq \emptyset$, and $H \cap N_j \neq \emptyset$. This implies that $i = j = i_0$, which is a contradiction.\[\square\]

**Remark.** If the axiom of constructibility holds that there is a partition $\{B_i : i \in \omega\}$ of $2^\omega$ into pairwise disjoint Bernstein sets, such that $B_i \in \Delta^1_2(2^\omega)$ for all $i \in \omega$. Hence, we can construct a sequence $(A^j)_{j \in \omega}$ of $\Delta^1_2$ sets in $2^\omega$ with the same properties as in Example 1.

Now we will show that under CH there is a more pathological example.
Lemma 2. Assume CH and list all sets in $[\omega]^\omega$ as $H_\alpha$, $\alpha < \omega_1$. Then there are sets $G_\alpha \in \bigcup_{\beta < \alpha} H_\beta$, $\alpha < \omega_1$, such that for each $\alpha < \omega_1$ we have
\[ \forall \beta < \alpha \ (G_\alpha \cap H_\beta \neq \emptyset \text{ and } H_\beta \setminus G_\alpha \neq \emptyset). \]

Proof. Let $G_0 \in [\omega]^\omega$ be such that $G_0 \subset H_0$ and $H_0 \setminus G_0 \neq \emptyset$. For $\alpha < \omega_1$, let $(F_n)_{n \in \omega}$ be an enumeration of $\{H_\beta : \beta < \alpha\}$. For $n \in \omega$, let $F_n = \{a_n^0, a_n^1, a_n^2, \ldots\}$ and fix $m_n \in F_n \setminus \{a_i^j : i, j < 2n\}$. Put $G_\alpha = \{m_n : n \in \omega\}$ and notice that
\[ \forall n \in \omega \ (G_\alpha \cap F_n \neq \emptyset \text{ and } F_n \setminus G_\alpha \neq \emptyset). \]

Example 2. Assume CH. Let $G_\alpha$, $\alpha < \omega_1$, be sets from Lemma 2. Let $\{r_\alpha : \alpha < \omega_1\}$ be an enumeration of $2^\omega$. Fix two disjoint clopen sets $C_0, C_1 \subset 2^\omega$. Put
\[ A_j^\alpha = \bigcup_{r_\alpha} \{r_\alpha\} \times C^{\chi_{G_\alpha}(j)}. \]
Suppose that there is an uncountable set $E \subset \omega_1$ and $H \in [\omega]^\omega$ such that for all $\alpha \in E$, we have
\[ \text{card}(\bigcap_{j \in H} A_j^\alpha) = \omega_1. \]
Since $A_j^0 = C^0$ or $A_j^1 = C_1$, we obtain
\[ [\forall \alpha \in E \ \forall j \in H \ (A_j^{\alpha_0} = C^0)] \text{ or } [\forall \alpha \in E \ \forall j \in H \ (A_j^{\alpha_0} = C_1)]. \]
There exists $\alpha_0 < \omega_1$ for which $H = H_{\alpha_0}$. Let $\alpha \in E$ be such that $\alpha_0 < \alpha$. Since $G_\alpha \cap H \neq \emptyset$ and $H \setminus G_\alpha \neq \emptyset$, pick $j_1 \in G_\alpha \cap H$ and $j_0 \in H \setminus G_\alpha$. See that $A_j^{\alpha_1} = C^1$ and $A_j^{\alpha_0} = C^0$, which yields a contradiction. Hence for every uncountable $P \subset 2^\omega$ and $H \in [\omega]^\omega$, there is $r \in P$ with $\bigcap_{j \in H} A_j^r = \emptyset$. □

The Referee claims that “it might be interesting to have a direct forcing proof of the Theorem 1. For example, is it true that if $S$ denotes Sacks forcing and $M$ denotes Mathias forcing, and if $(x, H)$ is generic for $S \times M$, then $\bigcap_{j \in H} A_j^x$ is uncountable?”

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References


