ON SOME σ-IDEAL WITHOUT CCC

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Abstract. We prove that σ-ideal $\sigma(a)$ has property (M) and, consequently, fails *ccc* property. It is also shown that $\sigma(a)$ is generated by the family $\{E \setminus \Phi(E)\colon E = cl(E)\}$. Finally, we show that if for any $A \in \sigma(a)$ and U open in density topology, $A \cap U$ is meager in U.

1. Preliminaries

The aim of this paper is to give a complete characterization of σ-ideal $\sigma(a)$. The definition of condition (a) was introduced by Marcin Grande in 2001 in the paper *On the sets of discontinuity points of functions satisfying some approximate quasi-continuity conditions* [G1]. In Theorem 1 Grande proved that the set A is a set of discontinuity points of some approximately contionuous function f with propert (s_1) (see [G1]) if and only if it is an F_σ set of Lebesgue measure zero and satisfies the (a) condition:

(a) *for each nonempty set* $U \in \tau_d$ *contained in the closure* $cl(A)$ *of the set* A *the set* $U \cap A$ *is nowhere dense in* U.

In 2003 Zbigniew Grande and Ewa Strońska proved that the family of all sets satisfying the condition (a) is an ideal of sets, which is $G_{\delta\sigma}$- but not F_σ-generated (see [GS]). They also observed that every set satisfying condition (a) is nowhere dense and of Lebeque measure zero. Answering the question of Grande and Strońska, in 2011 Frankowska and Nowik proved that ideal (a) is not G_δ-generated ([FN2]). They first proved this in the case of the ideal (a) defined on the Cantor set $2^{\mathbb{N}}$ ([FN1]). In fact, they proved that the ideals (a) are not $F_{\sigma\delta}$-generated; it was not stated there explicitly but it can be easily extracted from the proofs.

2. Definitions and notion

Denote by λ the Lebesgue measure in $\mathbb{R}$. For any measurable $A \subseteq \mathbb{R}$ by $\Phi(A)$ we denote the set of density points of A i.e.

$$\Phi(A) = \{x \in \mathbb{R}\colon \lim\inf_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1\}.$$

We say that $U \subseteq \mathbb{R}$ is open in density topology, denoted by τ_d, if and only if U is measurable and $U \subseteq \Phi(U)$ (see [T]).

We use the following characterization of sets $A \in (a)$ rather than original definition.

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Theorem 1 ([N])**.** *A set $A \in (a)$ if and only if for every nonempty $U \in \tau_d$ there exists an open set W in the standard topology such that $U \cap W \neq \emptyset$ and $U \cap W \cap A = \emptyset$.*

Later, we will also need the following:

Theorem 2 ([GS])**.** *For every closed (in the standard topology) set $E \subseteq \mathbb{R}$ we have $E \setminus \Phi(E) \in (a)$.*

3. $\sigma(a)$ has property (M)

Definition 1. Let $\mathcal{I}$ be a proper (e.i. $X \notin \mathcal{I}$) ideal on uncountable Polish space X which contains all singletons and has a Borel basis. We say that $\mathcal{I}$ has *property* (M) if and only if there is a Borel measurable function $f\colon X \to 2^{\mathbb{N}}$ with $f^{-1}[\{x\}] \notin \mathcal{I}$ for each $x \in 2^{\mathbb{N}}$.

Property (M) was introduced and investigated by Balcerzak in [B]. Obviously, an ideal satisfying property (M) fails the *ccc*. Balcerzak, Rosłanowski and Shelah in [BRS] consider whenever it is possible for both the *ccc* and property (M) to fail. For more details we refer the reader to [B] and [BRS].

We will denote by $\mathcal{E}$ the σ-ideal generated by all F_σ subsets of $\mathbb{R}$ of Lebesgue measure zero. In [M] Mauldin proved that there is a Borel measurable function $f\colon [0,1] \to [0,1]^{\mathbb{N}}$ such that for any $x \in [0,1]^{\mathbb{N}}$, $f^{-1}[\{x\}]$ is not a subset of any F_σ set of Lebesgue measure zero. Evidently, $\mathcal{E}$ has (M) property. We will use similar method to that used in Mauldin's paper to prove that $\sigma(a)$ has property (M). We say that a Borel set B is *fat* if for any open subset U of real line such that $U \cap B \neq \emptyset$, the set $U \cap B$ has positive Lebesgue measure. A fat set always contains a fat perfect subset; for details see [Bu]

Theorem 3. *$\sigma(a)$ has property (M).*

Proof. We define by induction the system of perfect subsets of real line $\langle P_s\colon s \in \mathbb{N}^{<\mathbb{N}}\rangle$ such that for each $s \in \mathbb{N}^{<\mathbb{N}}$:

(1) P_s is perfect.
(2) P_s is fat.
(3) $P_{s^\wedge\langle i\rangle}$ is a nowhere dense subset of P_s with the diameter less than $\frac{1}{1+|s|}$ for each $i \in \mathbb{N}$.
(4) If $U \subseteq \mathbb{R}$ is open and $i \in \mathbb{N}$, then the following implication is true:

$$U \cap P_s \neq \emptyset \Rightarrow \exists_{i\in\mathbb{N}} P_{s^\wedge\langle 2i\rangle} \cup P_{s^\wedge\langle 2i+1\rangle} \subseteq U.$$

(5) $P_{s^\wedge\langle i\rangle} \subseteq \Phi(P_s)$ for each $i \in \mathbb{N}$.
(6) $P_{s^\wedge\langle i\rangle} \cap P_{s^\wedge\langle j\rangle} = \emptyset$ for all natural $i \neq j$.

First, we choose nowhere dense perfect fat set $P_\emptyset$ (the construction of such set you can find in [Bu]). Let fix any $k \in \mathbb{N}$ and suppose that we have defined P_s for all $s \in \mathbb{N}^{<\mathbb{N}}$ such that $|s| \leq k$. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = k$. Let $\langle U_i\colon i \in \mathbb{N}\rangle$ be the family of all basis open sets such that $U_i \cap P_s \neq \emptyset$. Fix $i \in \mathbb{N}$ and suppose that we have already defined sets $\{P_{s^\wedge\langle 2j\rangle}, P_{s^\wedge\langle 2j+1\rangle}\colon j = 0, 1, \ldots, i-1\}$. Since $\bigcup_{j=0}^{i-1} \left(P_{s^\wedge\langle 2j\rangle} \cup P_{s^\wedge\langle 2j+1\rangle}\right)$ is nowhere dense subset of P_s, we can find nonempty open $W \subseteq U$ such that $W \cap P_s \neq \emptyset$ and

$$W \cap \bigcup_{j=0}^{i-1} \left(P_{s^\frown\langle 2j\rangle} \cup P_{s^\frown\langle 2j+1\rangle}\right) = \emptyset$$

.

Moreover, P_s is fat set and $W \cap P_s \neq \emptyset$, so we conclude that $W \cap P_s$ has positive Lebesgue measure. Let Q_s be the countable set dense in P_s. Finally, we can find two disjoint perfect fat $P_{s^\frown\langle 2i\rangle}, P_{s^\frown\langle 2i+1\rangle} \subseteq W \cap \Phi(P_s) \setminus Q_s$ with diameters less than $\frac{1}{1+|s|}$.

Now a function $\phi\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is defined by imposing the following condition:

$$\{\phi(x)\} = \bigcap_{n\in\mathbb{N}} P_{x\restriction n}.$$

Note that ϕ is an embedding and $\phi^{-1}\colon \phi[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ is continuous. Next we define $\sigma\colon \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\sigma(x)(n) = x(n) \mod 2$. Finally, the function $f\colon \phi[\mathbb{N}^{\mathbb{N}}] \to 2^{\mathbb{N}}$ is given by $f = \sigma \circ \phi^{-1}$.

Let us fix $x_0 \in 2^{\mathbb{N}}$. We will show that $f^{-1}[\{x_0\}] \notin \sigma(a)$. Suppose that $\langle X_n\colon n \in \mathbb{N}\rangle$ is any partition of

$$f^{-1}[\{x_0\}] = \phi[\sigma^{-1}[\{x_0\}]].$$

Then

$$\sigma^{-1}[\{x_0\}] = \bigcup_{n\in\mathbb{N}} \phi^{-1}[X_n].$$

The homeomorphism $\tau\colon \sigma^{-1}[\{x_0\}] \to \mathbb{N}^{\mathbb{N}}$ is defined by imposing the following condition

$$\tau^{-1}(y)(n) = 2\cdot y(n) + x_0(n).$$

$\mathbb{N}^{\mathbb{N}}$ is Baire space, hence there is $n_0 \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$ such that

$$cl([s] \cap \tau[\phi^{-1}[X_{n_0}]]) = [s].$$

Let us consider $s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_1| = |s|$ and $s_1(n) = 2\cdot s(n) + x_0(n)$ for each $n < |s|$. Note that $\tau[[s_1] \cap \sigma^{-1}[\{x_0\}]] = [s]$ and hence $[s_1] \cap \phi^{-1}[X_{n_0}]$ is dense in $[s_1] \cap \sigma^{-1}[\{x_0\}]$. Next, we find

$$y_k \in \phi^{-1}[X_{n_0}] \cap [s_1^\frown\langle 2k + x_0(|s_1|)\rangle]$$

for each natural k. Then

$$\phi(y_k) \in X_{n_0} \cap P_{s_1^\frown\langle 2k+x_0(|s_1|)\rangle}.$$

If U is open set such that $U \cap \Phi(P_{s_1}) \neq \emptyset$, then there is natural k such that $P_{s_1^\frown\langle 2k+x_0(|s_1|)\rangle} \subseteq U \cap \Phi(P_{s_1})$. Hence $\phi(y_k) \in U \cap \Phi(P_{s_1}) \cap X_{n_0}$. Thus, by Theorem 1, we obtain $X_{n_0} \notin (a)$ and, consequently, $f^{-1}[\{x_0\}] \notin \sigma(a)$.

Finally, we choose $c \in \phi[\mathbb{N}^{\mathbb{N}}]$ and define function $g\colon \mathbb{R} \to 2^{\mathbb{N}}$ by:

$$g(x) = \begin{cases} f(x), & \text{for} \quad x \in \phi[\mathbb{N}^{\mathbb{N}}], \\ f(a), & \text{for} \quad x \in \mathbb{R} \setminus \phi[\mathbb{N}^{\mathbb{N}}]. \end{cases}$$

Obviously, g is Borel function such that $g^{-1}[\{y\}]$ does not belong to $\sigma(a)$ for any $y \in 2^{\mathbb{N}}$ and the proof is complete. □

An immediate consequence of Theorem 3 is the following corollary.

Corollary 4. $\sigma(a)$ *does not satisfy the countable chain condition (*ccc*).*

The following result generalizes a theorem of Marcin Grande ([G2, Theorem 2]) stating that there are sets measurable with respect to the Lebesgue measure and with Baire property which are not in σ-field generated by the union $\mathcal{B}\cup(a)$. In fact, we prove even more i.e. there are analytic sets which are not in σ-field generated by the union $\mathcal{B}\cup(a)$.

Proposition 5. *If σ-ideal $\mathcal{I}\subset\mathcal{P}(\mathbb{R})$ has property (M), then there is an analytic set A such that $A\notin\mathcal{B}\triangle\mathcal{I}$.*

Proof. Suppose that $\mathcal{I}$ is σ-ideal of subsets of real line with property (M). Let $U\subset\mathbb{R}\times\mathbb{R}$ be the universal for the class of analytic sets i.e. for each analytic set A there is $x\in\mathbb{R}$ such that $U_x=A$.

Let $f\colon\mathbb{R}\to\mathbb{R}$ be a Borel measurable fuction such that $f^{-1}[\{y\}]\notin\mathcal{I}$ for any $y\in\mathbb{R}$.

We define $C=\{x\in\mathbb{R}\colon\langle f(x),x\rangle\notin U\}$. Observe that C is coanalytic set which does not belong to $\mathcal{B}\triangle\mathcal{I}$.

On the contrary, suppose that $C=B\triangle M$ and $B\in\mathcal{B}$, $M\in\mathcal{I}$. Then, there is $x_0\in\mathbb{R}$ such that $U_{x_0}=B$. Let x_1 be any element of the $f^{-1}[\{x_0\}]\setminus M$. We consider two cases:

(1) $x_1\in C=B\triangle M$. Since $x_1\in B$, $\langle x_0,x_1\rangle\in U$. It follows immediately that $\langle f(x_1),x_1\rangle\in U$ and hence $x_1\notin C$, which is impossible.
(2) $x_1\notin C=B\triangle M$. Since $x_1\notin B$, $\langle x_0,x_1\rangle\notin U$. It follows easily that $\langle f(x_1),x_1\rangle\notin U$ and hence $x_1\in C$, a contradiction.

□

4. Characterization of (a')

In this section we will be concern with the family of sets (a') given by

$$(a')=\{X\subset\mathbb{R}\colon\exists E\text{ closed and such that }X\subset E\setminus\Phi(E)\}.$$

Grande and Strońska in [GS] showed that for any closed $E\subset\mathbb{R}$ we have $E\setminus\Phi(E)\in(a)$, hence $(a')\subseteq(a)$. The natural question arises whether these two families are equal. We give a negative answer to this question showing that the family (a') is not an ideal.

Proposition 6. *The family (a') is not an ideal.*

Proof. Let $C\subset[0,1]$ be a set homeomorphic to the ternary Cantor set such that C has positive measure and $0,1\in C$. Let $\bigcup_{n\in\mathbb{N}}(a_n,b_n)=[0,1]\setminus C$ be such that intervals (a_n,b_n) are pairwise disjoint. Note that the set $\{a_n,b_n:n\in\mathbb{N}\}$ is dense in C. For any n find $(x^n_k)_{k\in\mathbb{N}},(y^n_k)_{k\in\mathbb{N}}\subset(a_n,b_n)$ such that $x^n_k\to a_n$ and $y^n_k\to b_n$.

Let $X=\{x^n_k,y^n_k:n,k\in\mathbb{N}\}$. Then $cl(X)=X\cup C$ and $X\cap C=\emptyset$. Note that

$$X\subset cl(X)\setminus C\subset cl(X)\setminus\Phi(cl(X)).$$

Therefore $X\in(a')$.

Now fix $p\in\Phi(C)$. Clearly, $p\in C$. Consider the set $Y=X\cup\{p\}$. Observe that any nonempty open in density topology $U\subset cl(X)$ is contained in C. Thus $U\cap X=\{p\}$ is nowhere dense in U. (One can prove it in simpler way, since $X,\{p\}\in(a')\subset(a)$ and (a) is an ideal). We only need to show that $Y\notin(a')$ and hence (a') is not an ideal, so in the consequence $(a')\neq(a)$.

Suppose that there is a closed set E with $Y\subset E\setminus\Phi(E)$. Then $cl(X)\subset E$ and $p\in E$. Then $\Phi(E)$ contains $\Phi(C)$ and in the consequence $p\in\Phi(E)$, a contradiction. □

According to the above proposition, families (a) and (a') are different. However, we will prove that these families generate the same σ-ideal.

Let us first prove the following lemma:

Lemma 7 (folklore)**.** *Let $F \subset \mathbb{R}$ be closed. Then $F \cap \Phi(F)$ is open in density topology.*

Proof. We need to show that $\Phi(F \cap \Phi(F)) = F \cap \Phi(F)$. Since every point of $F \cap \Phi(F)$ is its density point, $F \cap \Phi(F) \subseteq \Phi(F \cap \Phi(F))$. On the other hand let x be a point of density of $F \cap \Phi(F)$. Then x is a limit point of $F \cap \Phi(F)$, and therefore since F is closed, $x \in F$. That means $\Phi(F \cap \Phi(F)) \subseteq F \cap \Phi(F)$. □

Let $X \in (a)$. We define derivative $X' = \{x \in X : x \in \Phi(cl(X))\}$ and inductively $X^\alpha = \{x \in \bigcap_{\xi<\alpha} X^\xi : x \in \Phi(cl(\bigcap_{\xi<\alpha} X^\xi))\}$.

Let λ stand for the Lebesgue measure.

Proposition 8. *Let $X \in (a)$. Then for any $\alpha < \omega_1$ we have*

(i) *the set $X^\alpha \cap \Phi(X^\alpha)$ has measure zero;*
(ii) *$cl(X^\alpha)$ is a nowhere dense subset of $cl(X^\beta) \cap \Phi(cl(X^\beta))$ for every $\beta < \alpha$;*
(iii) *if $\lambda(cl(X^\alpha)) > 0$, then $\lambda(cl(X^\alpha)) < \lambda(cl(X^\beta))$ for every $\beta < \alpha$.*

Proof. (i) follows from the fact that $X^\alpha \subset X$ and $X \in \mathcal{N}$.

(ii) Suppose that $cl(X^\alpha)$ is not nowhere dense subset of $cl(X^\beta) \cap \Phi(cl(X^\beta))$. Then $cl(X^\alpha)$ contains a relatively open subset V of $cl(X^\beta) \cap \Phi(cl(X^\beta))$. Clearly V is open in density topology. Moreover $X \cap V = X^\alpha \cap V$ is dense in V, which contradicts the fact that $X \in (a)$.

(iii) It follows from the fact that open dense subset V of $P \cap \Phi(P)$ where P is perfect with $\lambda(P) > 0$ has also positive measure.

□

Let $X \in (a)$. On account of Proposition 8(iii) there is $\alpha < \omega_1$ such that $\lambda(cl(X^\alpha)) = 0$. Set

$$\mathrm{rank}(X) = \min\{\alpha < \omega_1 : \lambda(cl(X^\alpha)) = 0\}.$$

Then $X^{\mathrm{rank}(X)} = \emptyset$ and $X^\alpha \neq \emptyset$ for each $\alpha < \mathrm{rank}(X)$.

The following shows that the defined hierarchy of sets in the ideal (a) is non-trivial.

Theorem 9. *For every $\alpha < \omega_1$ there is a countable set $X \in (a)$ with $\mathrm{rank}(X) = \alpha$.*

Proof. Our proof starts with the observation that for each $\alpha < \omega_1$ there is a sequence $\{C_\xi : \xi < \alpha\}$ of perfect sets such that

- $\lambda(C_\xi) > 0$;
- if $\beta < \alpha$, then C_β is nowhere dense subset of $\bigcap_{\xi<\beta} C_\xi$;
- $\bigcap_{\xi<\alpha} C_\xi$ is a singleton.

Let $X_0 \subset \mathbb{R}$ be such that $cl(X_0) \supseteq C_0$, $X_0 \cap C_0 = \emptyset$ and $C_0 = (X_0)'$, where $(X_0)'$ denotes the Cantor-Bendixson derivative of X_0. It means that C_0 is a set of limit points of X_0 and X_0 consists of all isolated points of $cl(X_0)$. The construction of such X_0 is the same as that of X in the proof of Proposition 6. We define by induction on $\beta < \alpha$ X_β such that:

(1) $X_\beta \subset \bigcap_{\xi<\beta} C_\xi$;
(2) $cl(X_\beta) \supseteq C_\beta$;

(3) $(X_\beta)' = C_\beta$;
(4) $X_\beta \cap C_\beta = \emptyset$.

Let $X = \bigcup_{\beta<\alpha} X_\alpha$.

We need following lemmas:

Lemma 10. *Let $Z \subset \mathbb{R}$, $V \in \tau_d$ and $V \neq \emptyset$. If $Z \cap V$ is dense in V, with respect to the natural topology on $\mathbb{R}$, and $x \in Z$ is an isolated point of Z, then $x \notin V$.*

Proof. Suppose that $x \in V$. Since V is open in τ_d, x is a density point of V. Thus for any open interval $I \ni x$ with $I \cap Z = \{x\}$, we have $(I \setminus \{x\}) \cap V \neq \emptyset$. Therefore $I \cap Z$ is not dense in $I \cap V$. Since $Z \cap V$ is dense in V, $x \notin V$. □

Lemma 11. *Let $V \in \tau_d$, $V \neq \emptyset$. Then $X \cap V$ is nowhere dense in V.*

Proof. Suppose that $X \cap V$ is not nowhere dense in V. Then we may assume that $X \cap V$ is dense in V (if not we will find $V' \subset V$ such that $X \cap V'$ is dense in V' and we will work with Y' instead of Y). By Lemma 10 we obtain that V is disjoint with X_0. Thus $(X \setminus X_0) \cap V$ is dense in V. But X_1 is set of isolated point of $X \setminus X_0$. Using Lemma 10 we obtain that $V \cap X_1 = \emptyset$. Proceeding inductively we obtain that $V \cap X_\beta = \emptyset$. Hence $V \cap X = V \cap \bigcup X_\beta = \bigcup (V \cap X_\beta) = \emptyset$, and we reach a contradiction. □

From the Lemma 11 we obtain that $X \in (a)$. By the construction we obtain that $X_\beta = X^\beta$. Hence $\mathrm{rank}(X) = \alpha$.

□

Theorem 12. *$\sigma(a') = \sigma(a)$.*

Proof. Since $(a') \subset (a)$, obviously we only need to show that $\sigma(a) \subset \sigma(a')$.

Let $X \in (a)$. Let $\alpha < \omega_1$ be a rank of X. Then

$$X \setminus X' \subset cl(X) \setminus \Phi(cl(X)).$$

In general

$$X^\xi \setminus X^{\xi+1} \subset cl(X^\xi) \setminus \Phi(cl(X^\xi)).$$

Since $X = \bigcup_{\xi<\alpha} X^\xi \setminus X^{\xi+1}$ and $X^0 = X$, $X \in \sigma(a')$.

□

A possible way to prove that the ideal generated by (a') and (a) are not equal is to show that an union of two sets with finite ranks has a finite rank. Then any set from the ideal generated by (a') would have finite rank, while, by Theorem 9, there is a set in (a) with infinite rank. However, the authors are not able to prove it. Therefore the following problem remains unsolved.

Open question. *Are the ideals generated by (a) and (a') equal?*

Note that the continuous image of a null and perfect set may be of positive measure. Therefore the sets from (a) and (a') are not preserved by continuous functions. We say that a mapping $f : \mathbb{R} \to \mathbb{R}$ is a *density preserving homeomorphism* if f is autohomeomorphism of the real line with the natural topology and for any measurable $S \subset \mathbb{R}$, $f(x)$ is a density point of the set $f(S)$ whenever x is a density point of S. This notion was introduced by Bruckner in [Br] and afterwards studied by Niewiarowski in [Ni]. The class of density preserving homeomorphisms contains continuously differentiable homeomorphisms whose derivatives never vanish, and it is contained in the class of absolutely continuous functions [Br]. Clearly, density preserving homeomorphisms preserve (a), (a') and ranks of sets from (a), i.e. if $X \in (a)$, then $\text{rank}(X) = \text{rank}(f(X))$ for any density preserving homeomorphism f. Since the hierarchy of sets in (a) is nontrivial, there not exists a universal set in (a) which contains a copy of every set from (a).

Proposition 13. *There is no universal set $A \in (a)$ such that for any set $B \in (a)$ there is a density preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(B) \subset A$.*

Proof. Let $A \in (a)$. By Theorem 9 there is a set $B \in (a)$ such that $\text{rank}(B) > \text{rank}(A)$. Suppose that there is a density preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(B) \subset A$. Then $\text{rank}(f(B)) > \text{rank}(A)$ which contradicts the monotonicity of the rank operator. □

5. $\sigma(a)$ and porous sets

Andrzej Nowik in [N] showed that every porous set satisfies condition (a). We will prove that every porous set satisfies condition (a'). Let us remind the definition of porosity and porous set.

Definition 2. Let $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be any open interval. We denote by $l(A, I)$ the maximal length of open subinterval $J \subseteq I$ disjoint with A.

If the limit

$$\limsup_{\epsilon \to 0^+} \frac{l(A,(x-\epsilon, x+\epsilon))}{\epsilon}$$

exists, we donote it by $\rho(A, x)$ and call it *the porosity of set A in point x.*

A set A is *porous* if and only if it has a positive porosity in every point $x \in A$. The σ- ideal generated by the family of porous set we denote by $\sigma\mathcal{P}$.s

The following lemma will be useful.

Lemma 14. *Let $X \in \mathbb{R}$ The following conditions are equivalent:*

(1) $X \in (a')$;
(2) $\Phi(cl(X)) \cap X = \emptyset$.

Proof. Assume that $X \in (a')$. Let $E \subseteq \mathbb{R}$ be a closed set such that $X \subseteq E \setminus \Phi(E)$. Then $cl(X) \subseteq E$ and thus $\Phi(cl(X)) \subseteq \Phi(E)$. Clearly, $\Phi(cl(X)) \cap X = \emptyset$.

On the other hand, suppose that $X \subseteq \mathbb{R}$ is such that $\Phi(cl(X)) \cap X = \emptyset$. Then take $E = cl(X)$. Obviously, $X \subseteq E \setminus \Phi(E)$. □

Theorem 15. *For every porous set $A \subseteq \mathbb{R}$ there is a closed $E \subseteq \mathbb{R}$ such that $A \subseteq E \setminus \Phi(E)$.*

Proof. Suppose that $A \subseteq \mathbb{R}$ is porous set. We only need to show that $A \cap \Phi(cl(A)) = \emptyset$. Let fix $a \in A$. Since A is porous set,

$$\limsup_{\epsilon \to 0^+} \frac{l(A,(x-\epsilon, x+\epsilon))}{\epsilon} > 0.$$

Choose a decreasing seguence of numbers $\{\epsilon_k\}_{k\in\mathbb{N}}$ tending to 0 and $\eta > 0$ such that

$$l(A,(a-\epsilon_k, a+\epsilon_k)) \geq \eta \cdot \epsilon_k.$$

Let I_k be an open interval of length $\eta \cdot \epsilon_k$ such that $I_k \subseteq (a-\epsilon_k, a+\epsilon_k) \setminus A$. Then $cl(A) \cap I_k = \emptyset$ and thus

$$\frac{\lambda((a-\epsilon_k,a+\epsilon_k)\cap cl(A))}{2\epsilon_k}\leq\frac{2\epsilon_k-\eta\cdot\epsilon_k}{2\epsilon_k}=1-\frac{\eta}{2}.$$

Clearly, a is not a density point of $cl(A)$ and by Lemma 14 A is in (a'). □

Note that if a set A is porous, then its closure clA is porous as well. Moreover, if A is porous at a point $a\in A$, then a is not a density point of A. Therefore if A is porous, then $\Phi(A)=\emptyset$, and by Lebesgue Theorem A has measure zero. Consequently any porous set can be covered by closed set of measure zero. But any such set is in (a') that is in the first level of hierarchy on (a). We will show that there is a set S with property (a') which is not in $\mathcal{E}$.

To prove it, take a fat perfect set P. By Theorem 16 the set S of all points in P which are not density points of P is a comeager in P. Since $S=P\setminus\Phi(P)$, S has property (a'). Let $G\subset S$ be a G_δ subset of S which is dense in P. Suppose to the contrary that G is in $\mathcal{E}$. Then $G\subset\bigcup_{n\in\mathbb{N}}F_n$ where F_n are closed null sets and $F_1\subset F_2\subset\dots$. Then there is n with $int_G(F_n)\neq\emptyset$. Thus $F_n\cap P$ is open in P and by fatness of P we obtain that $F_n\cap P$ has positive measure. A contradiction.

What we prove does not mean that every set in $\mathcal{E}$ has property (a'). By Theorem 9 in each level of hierarchy on (a) there is a countable set, which in turn is in $\mathcal{E}$. Moreover, there is a countable set which is not in (a). Therefore $\mathcal{E}$ is not contained in (a) but it is a proper subset of $\sigma(a)$.

6. $\sigma(a)$ and meager sets

Let us recall that Nowik proved in [N] that $A\in(a)$ *if and only if for any nonempty U open in density topology $A\cap U$ is nowhere dense in U*. The natural question arises whether the analogous fact is true for $\sigma(a)$? Obviously, if $A\in\sigma(a)$, then $A\cap U$ is meager in U. We shall prove that the converse implication is not true.

Let $\mathcal{M}$ denotes the family of meager subsets of $\mathbb{R}$ and for any $X\subset\mathbb{R}$, $\mathcal{M}(X)$ stands for the family of meager subsets of X.

We define the σ-ideal $AFC(\tau_d)$ by a condition:

$$X\in AFC(\tau_d)\quad\text{iff}\quad\forall_{U\in\tau_d\setminus\emptyset}X\cap U\in\mathcal{M}(U).$$

Theorem 16 (Bu)**.** *If $\gamma>0.5$ and $P\subseteq\mathbb{R}$ is nowhere dense perfect set, then $D_\gamma(P)$ is a meager set in P.*

In the above theorem of Buczolich $\gamma\in(0,1]$ and $D_\gamma(P)$ is the set of points of lower density greater or equal to γ i.e.

$$D_\gamma(P)=\{x\in\mathbb{R}\colon\liminf_{h\to0}\frac{\lambda((x-h,x+h)\cap P)}{2h}\geq\gamma\}.$$

Particularly, if $\gamma=1$, then $\Phi(P)$ is meager in P for any nowhere dense perfect set $P\subseteq\mathbb{R}$.

Lemma 17. *Let $N\subseteq\mathbb{R}$ be a closed, nowhere dense set. Then for any nonempty $D\in\tau_d$ the set*

$$N\cap D\cap cl(int(cl(D)))$$

is nowhere dense subset of D (in euclidian topology).

Proof. Suppose that $N \subseteq \mathbb{R}$ is closed and nowhere dense (in euclidian topology) and $D \in \tau_d \setminus \{\emptyset\}$. Suppose, contrary to our claim, that $N \cap D \cap cl(int(cl(D)))$ is not a nowhere dense subset of D. Then there exists an open $W \subseteq \mathbb{R}$ such that

$$\emptyset \neq W \cap D \subseteq N \cap D \cap cl(int(cl(D))).$$

Since W is open and $W \cap D \subseteq cl(int(cl(D)))$, $W \cap int(cl(D)) \neq \emptyset$.

Let $W^* = W \cap int(cl(D))$. Clearly, W^* is open and nonempty and hence $W^* \subseteq W \cap cl(D)$. Moreover,

$$W \cap cl(D) \subseteq cl(W \cap D) \subseteq N$$

and hence $W^* \subseteq N$, since N is closed and W is open. This contradicts our assumption that N is nowhere dense.

□

Theorem 18. *For any nonempty $D \in \tau_d$ and any meager set $M \subseteq \mathbb{R}$ we have $M \cap D \in \mathcal{M}(D)$.*

Proof. Consider $D \in \tau_d \setminus \{\emptyset\}$ and $M \in \mathcal{M}$. Define $D_1 = D \setminus cl(int(cl(D)))$. Obviously, $D_1 \in \tau_d$. We shall prove that $int(cl(D_1)) = \emptyset$. Conversely, suppose that there exists an open $W \subseteq \mathbb{R}$ such that $W \subseteq cl(D_1)$. Since $D_1 \cap int(cl(D)) = \emptyset$, $cl(D_1) \cap int(cl(D)) = \emptyset$ and hence $W \cap int(cl(D))$ is empty. On the other hand, $W \subseteq cl(D_1) \subseteq cl(D)$ and thus $W \subseteq int(cl(D))$. We obtain a contradiction.

We will show that $D_1 \in \mathcal{M}(D)$. Observe that $cl(D_1)$ is a perfect set, since $D_1 \in \tau_d$. What is more, $int(cl(D_1)) = \emptyset$ and hence $cl(D_1)$ is nowhere dense. According to Theorem 16, $\Phi(cl(D_1))$ is meager in $cl(D_1)$. Hence there exists a family $\{E_n\}_{n\in\mathbb{N}}$ such that:

1. $E_n \subseteq cl(D_1)$ for each $n \in \mathbb{N}$;
2. $int_{cl(D_1)}(E_n) = \emptyset$, where $int_{cl(D_1)}(E_n)$ denotes the interior of E_n in $cl(D_1)$ in euclidian topology;
3. $\Phi(cl(D_1)) \subseteq \bigcup_{n\in\mathbb{N}} E_n$.

We will prove that $int_D(D \cap E_n)$ is empty. To obtain a contradiction, suppose that there is an open $W \subseteq \mathbb{R}$ such that $\emptyset \neq D \cap W \subseteq D \cap E_n$. Recall now that $cl(D_1) \cap int(cl(D)) = \emptyset$. Since $D \cap W \subseteq E_n \subseteq cl(D_1)$, $D \cap W \cap int(cl(D)) = \emptyset$. Clearly, $W \cap D_1 \subseteq W \cap D$. We will show that $W \cap D_1$ is nonempty. Indeed, if $W \cap D_1 = W \cap D \setminus cl(int(cl(D)))$ is empty, then would be

$$W \cap D \subseteq cl(int(cl(D))) \setminus int(cl(D)) = Fr(int(cl(D))),$$

since $W \cap D \cap int(cl(D)) = \emptyset$. Let $N = Fr(int(cl(D)))$. Of course, N is closed and nowhere dense. By the Lemma 17 $N \cap D \cap cl(int(cl(D)))$ is nowhere dense subset of D. Observe that

$$Fr(int(cl(D))) \cap D \cap cl(int(cl(D))) = Fr(int(cl(D))) \cap D$$

and

$$\emptyset \neq W \cap D \subseteq Fr(int(cl(D))) \cap D \in NWD(D).$$

These contradicts the fact that W is open. Hence $\emptyset \neq W \cap D_1 \subseteq E_n$. Since E_n is closed and W is open, $W \cap cl(D_1) \subseteq cl(W \cap D_1) \subseteq E_n$, contrary to $int_{cl(D_1)}(E_n) = \emptyset$. We obtain a contradiction assuming that $int_D(D \cap E_n)$ is nonempty.

Hence $int_D(D \cap E_n) = \emptyset$. Moreover, since $D_1 \in \tau_d$, $D_1 \subseteq \Phi(D_1) \subseteq \Phi(cl(D_1)) \subseteq \bigcup_{n\in\mathbb{N}} E_n$. Finally, we conclude that $D_1 \subseteq \bigcup_{n\in\mathbb{N}}(D \cap E_n)$, and consequently, $D_1 \in \mathcal{M}(D)$.

Now, let $M \in \mathcal{M}$. Then there exists the family of closed nowhere dense sests $\{N_n\}_{n\in\mathbb{N}}$ such that $M \subseteq \bigcup_{n\in\mathbb{N}} N_n$. Since $D = D_1 \cup cl(int(cl(D)))$,

$$M \cap D \subseteq D_1 \cup \bigcup_{n\in\mathbb{N}} N_n \cap D \cap cl(int(cl(D))) \in \mathcal{M}(D).$$

□

From Theorem 18 it immediately follows that the σ-ideal $AFC(\tau_d)$ is exactly the σ-ideal of meager sets and hence $\sigma(a) \subsetneq AFC(\tau_d)$.

Summarizing, we have the following diagram:

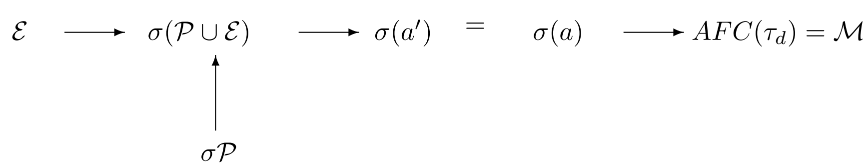

FIGURE 1. Relations between $\sigma(a)$ and others σ-ideals

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