ON MATRIX SUMMABILITY OF SPLICED SEQUENCES AND A-DENSITY OF POINTS

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ABSTRACT. For $y \in \mathbb{R}$ and a sequence $x = (x_n) \in \ell^\infty$ we define the new notion of A-density $\delta_A(y)$ of indices of those x_n's which are close to y where A is a non-negative regular matrix. We present connections between A-densities $\delta_A(y)$ of indices of (x_n) and the A-limit of (x_n). Our main result states that if the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for any $y \in \mathbb{R}$ where A is a non-negative regular matrix, then $\lim_{n\to\infty}(Ax)_n = \sum_{y\in\mathbb{R}} \delta_A(y) \cdot y$. which presents a different view of Osikiewicz Theorem. On the other hand we also show that the Osikiewicz Theorem can be obtained from the famous Henstock Theorem and finally present an $\mathcal{I}$-analogue of Henstock Theorem for $A^{\mathcal{I}}$-summability method which has been recently introduced.

1. Introduction

For $n, m \in \mathbb{N}$ with $n < m$, let $[n, m]$ denote the set $\{n, n+1, n+2, \dots, m\}$. Let $A \subset \mathbb{N}$. Define

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n} \quad \text{and} \quad \underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

The numbers $\overline{d}(A)$ and $\underline{d}(A)$ are called the upper natural density and the lower natural density of A, respectively. If $\overline{d}(A) = \underline{d}(A)$, then this common value is called the natural density of A and we denote it by $d(A)$. Let $\mathcal{I}_d$ be the family of all subsets of $\mathbb{N}$ which have natural density 0. Then $\mathcal{I}_d$ is a proper nontrivial admissible ideal of subsets of $\mathbb{N}$ (A family $\mathcal{I} \subset 2^{\mathbb{N}}$ of subsets of a nonempty set $\mathbb{N}$ is said to be an ideal in $\mathbb{N}$ if (i) $A,\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$. Further if $\bigcup_{A\in\mathcal{I}} A = \mathbb{N}$ which implies that $\{k\} \in \mathcal{I}$ for each $k \in \mathbb{N}$ then $\mathcal{I}$ is called admissible or free. $\mathcal{I}$ is proper and non-trivial if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$). Let (x_n) be a sequence of reals. We say that (x_n) tends to y statistically provided

$$\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}_d$$

for every $\varepsilon > 0$ [5, 20]. A sequence (x_n) tends to y in the sense of Cesáro if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = y.$$

There is a connection between the Cesáro summability (i.e. the convergence in the sense of Cesáro) and the statistical convergence. Namely if $(x_n) \in \ell^\infty$ is statistically convergent to y, then (x_n) tends to y in the sense of Cesáro [20] (Fridy noted that there is an unbounded sequence (x_n) which is statistically convergent to some y but x_n tends to ∞ in the sense of Cesáro [10]). It was observed by Fast that if (x_n) is a sequence of nonnegative real numbers statistically convergent to zero, then (x_n) tends to zero in the sense of Cesáro [5]. However, in general this implication is not reversible. To see this, consider the following simple example. Let $x_n = a$ if 3 divides n and put $x_n = b$ if 3 does not divide n, $a \neq b$. Clearly (x_n) tends to $(a + 2b)/3$ in the sense of Cesáro, but it is not statistically convergent. However, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = a \cdot d(\{n : x_n = a\}) + b \cdot d(\{n : x_n = b\}).$$

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Osikiewicz developed this idea in [17], where he defined finite and infinite splices. Let $E_1, \dots, E_k$ be a partition of $\mathbb{N}$ into k sequences. Let $y_1, \dots, y_k$ be distinct numbers. Let (x_n) be such that

$$\lim_{n\to\infty, n\in E_i} x_n = y_i.$$

Then (x_n) is called a k-splice. In the same way Osikiewicz defined an infinite splice and he proved the following.

Theorem 1 (Simplified version of Osikiewicz Theorem [17])**.** *Assume that (x_n) is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty, n\in E_i} x_n$. Assume that $d(E_i)$ exists for each i and*

$$\sum_i d(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = \sum_i y_i d(E_i).$$

In fact Osikiewicz considered a more general case, namely matrix summability method and A-density with the use of infinite matrices A.

If $x = (x_n)$ is a sequence and $A = (a_{n,k})$ is a summability matrix, then by Ax we denote the sequence $((Ax)_1, (Ax)_2, (Ax)_3, \dots)$ where $(Ax)_n = \sum\limits_{k=1}^{\infty} a_{n,k}x_k$. The matrix A is called regular if $\lim\limits_{n\to\infty} x_n = L$ implies $\lim\limits_{n\to\infty} (Ax)_n = L$. The well-known Silverman-Töeplitz theorem characterizes regular matrices in the following way. A matrix A is regular if and only if

(i) $\lim\limits_{n\to\infty} a_{n,k} = 0$,

(ii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{n,k} = 1$,

(iii) $\sup\limits_{n\in\mathbb{N}} \sum\limits_{k=1}^{\infty} |a_{n,k}| < \infty$.

For a non-negative regular matrix A and $E \subset \mathbb{N}$, following Freedman and Sember [9], we define the A-density of E, denoted by $\delta_A(E)$, as follows

$$\overline{\delta_A}(E) = \limsup_{n\to\infty} \sum_{k\in E} a_{n,k} = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\mathbb{1}_E(k) = \limsup_{n\to\infty} (A\mathbb{1}_E)_n,$$

$$\underline{\delta_A}(E) = \liminf_{n\to\infty} \sum_{k\in E} a_{n,k} = \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\mathbb{1}_E(k) = \liminf_{n\to\infty} (A\mathbb{1}_E)_n$$

where $\mathbb{1}_E$ is a 0-1 sequence such that $\mathbb{1}_E(k) = 1 \iff k \in E$. If $\overline{\delta_A}(E) = \underline{\delta_A}(E)$ then we say that the A-density of E exists and it is denoted by $\delta_A(E)$. Clearly, if A is the Cesaro matrix i.e.

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

then δ_A coincides with the natural density.

Throughout we assume that A is a non-negative regular summability matrix.

We here recall The original Osikiewicz Theorem.

Theorem 2 (Osikiewicz[17])**.** *Assume that A is non-negative regular summability matrix. Assume that $(x_n) \in \ell^\infty$ is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty, n\in E_i} x_n$. Assume that $\delta_A(E_i)$ exists for each i and*

$$\sum_i \delta_A(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}x_k = \sum_i y_i \delta_A(E_i).$$

In this paper we are interested when the assertion of Osikiewicz Theorem holds. However, we do not want to assume that the set of indices of a sequence (x_n) is divided into appropriate splices. In our approach we define for a sequence (x_n) a density $\delta_A(y)$ of indices of those x_n which are close to y which seems to be a new idea not dealt with so far in the literature. This is a more general approach than that of Osikiewicz and our treatment is not at all analogous to that of Osikiewicz and involves essentially new methods of proofs.

By ℓ^∞ we denote the set of all bounded sequences of reals. Fix $(x_n) \in \ell^\infty$. For $y \in \mathbb{R}$ let

$$\overline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \overline{\delta_A}(\{n : |x_n - y| \leq \varepsilon\})$$

and

$$\underline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \underline{\delta_A}(\{n : |x_n - y| \leq \varepsilon\}).$$

If $\overline{\delta_A}(y) = \underline{\delta_A}(y)$, then the common value is denoted by $\delta_A(y)$. Formally we should write $\delta_A^{(x_n)}(y)$ but it would be always clear which sequence (x_n) is considered.

The main result of this paper is the following.

Theorem 3. *Let $x = (x_n) \in \ell^\infty$. Suppose that the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for any $y \in \mathbb{R}$ where A is a non-negative regular matrix. Then*

$$\lim_{n \to \infty} (Ax)_n = \sum_{y \in \mathbb{R}} \delta_A(y) \cdot y.$$

The paper is organized as follows. In Section 2 we show a connection between the A-limit $\lim_{n\to\infty} (Ax)_n$ and the A-densities $\delta_A(y)$ of (x_n). We consider situation when some A-densities, $\delta_A(y)$ or $\overline{\delta_A}(y)$, are positive. In Section 3 we prove that if all densities $\delta_A(y)$ are zero, then the set of limit points of (x_n) is uncountable. In fact we prove a more general statement, namely that if (x_n) does not have any $\mathcal{I}$-limit point for some P-ideal $\mathcal{I}$, then its set of limit points is uncountable. In the process we also give a characterization of A-statistical cluster points which are not A-statistical limit points of (x_n), in terms of $\overline{\delta_A}(y)$. In Section 4 we show that a sequence $(x_n) \in \ell^\infty$ with $\sum_{y \in \mathbb{R}} \delta_A(y) = 1$ is an infinite splice for which the assumptions of Osikiewicz Theorem are fulfilled. Finally, combining a number of the previous results from this paper, we present the proof of Theorem 3. In Section 5, in another direction we show that the Osikiewicz Thereom [17] is actually a particular case of Henstock Theorem and it can be easily concluded from it. In the last section of the paper we use the notion of ideal convergence to generalize the Henstock Theorem for $A^{\mathcal{I}}$-summability method which has been recently introduced in [19].

2. A-limit for sequences with positive densities $\delta_A(y)$ or $\overline{\delta_A}(y)$

Lemma 4. *Suppose that $\delta_A(y)$ exists for any $y \in \mathbb{R}$. Then the set $D = \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is countable and $\sum_{y \in D} \delta_A(y) \leq 1$.*

Proof. Let (r_n) be a strictly monotonically decreasing sequence converging to 1. For $m \in \mathbb{N}$ let $D_m = \{y \in \mathbb{R} : \delta_A(y) \geq 1/m\}$. Let $y_1, \ldots, y_l \in D_m$ be distinct. Then for $\varepsilon = \min_{i \neq j} \frac{|y_i - y_j|}{3} > 0$ the sets $E_i = \{n : |x_n - y_i| \leq \varepsilon\}$ are pairwise disjoint and $\underline{\delta_A}(E_i) \geq 1/m$. Since A is also regular so we can choose a n_0 such that

$$\sum_{k \in E_i} a_{n,k} \geq \frac{1}{m} \text{ and } \sum_{k=1}^{\infty} a_{n,k} \leq r_p$$

for $n \geq n_0$ and for all $i = 1, \ldots, l$ where p is fixed. Since $E_1, \ldots, E_l$ are pairwise disjoint, so

$$\sum_{k \in E_1 \cup E_2 \cup \cdots \cup E_l} a_{n,k} = \sum_{j=1}^{l} \sum_{k \in E_j} a_{n,k} \geq \frac{l}{m}$$

for $n \geq n_0$. Therefore we must have $l \leq m[r_p]$ where as usual $[r_p]$ denotes the largest positive integer less or equal to r_p. Hence D_m must be finite and also

$$\sum_{y\in D_m} \delta_A(y) \leq r_p.$$

Since $D_1 \supset D_2 \supset D_3 \dots$ and $D = \bigcup_m D_m$, we obtain

$$\sum_{y\in D} \delta_A(y) = \lim_{m\to\infty} \sum_{y\in D_m} \delta_A(y) \leq r_p.$$

Since this is true for every r_p and $r_p \to 1$ so we must have

$$\sum_{y\in D} \delta_A(y) \leq 1.$$

Clearly D must be countable. $\square$

Lemma 4 would not remain true if one would change $\delta_A(y)$ to $\overline{\delta_A}(y)$, that is $\overline{D} := \{y \in \mathbb{R} : \overline{\delta_A}(y) > 0\}$ need not be countable. Note that a point y with $\overline{\delta_A}(y) > 0$ is an A-statistical limit point (which will be proved later).

Proposition 5. *There is a bounded sequence (x_n) such that $\overline{\delta_A}(\{y \in \mathbb{R} : |x_n - y| \leq \varepsilon\}) = \overline{d}(\{y \in \mathbb{R} : |x_n - y| \leq \varepsilon\}) = 1$ for any $\varepsilon > 0$ and any $y \in [0, 1]$ where A is the Cesaro matrix.*

Proof. Let (z_n) be a sequence such that its set of limit points equals $[0, 1]$. One can define (z_n) in such a way that any rational number from $[0, 1]$ appears infinitely many times in the sequence (z_n).

Let $n_k = 10^{k^2}$. Then

$$\frac{|[n_k + 1, n_{k+1}]|}{n_{k+1}} = \frac{10^{(k+1)^2} - 10^{k^2} - 1}{10^{(k+1)^2}} = 1 - \frac{1}{10^{2k+1}} - \frac{1}{10^{(k+1)^2}} \to 1.$$

Let $B_0 = [0, n_1]$ and $B_k = [n_k + 1, n_{k+1}]$ for $k \geq 1$. Clearly if A consists of infinitely many B_k's, then $\overline{d}(A) = 1$. Let $x_n = z_k$ if $n \in B_k$. Let $y \in [0, 1]$. Then for every $\varepsilon > 0$ the set $C := \{k : |z_k - y| < \varepsilon\}$ is infinite. Note that

$$A := \{n : |x_n - y| < \varepsilon\} = \bigcup_{k\in C} B_k.$$

Therefore $\overline{d}(A) = 1$. $\square$

The next result is a slight improvement of Osikiewicz Theorem. We will show in Section 4 that the condition $\sum_{y\in D} \delta_A(y) = 1$ implies that the set of indices of (x_n) can be divided into appropriate splices. The method which we use in our proof is similar to that of Osikiewicz, but not analogous as we use essentially new arguments.

Theorem 6. *Suppose that $x = (x_n)$ is a bounded sequence, $\delta_A(y)$ exists for every $y \in \mathbb{R}$ and $\sum_{y\in D} \delta_A(y) = 1$. Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in D} \delta_A(y) \cdot y.$$

Proof. Since (x_n) is bounded, there is $M > 0$ such that $|x_n| \leq M$ for every $n \in \mathbb{N}$. Let $D = \{y_i\}_i$ i.e. y_i's are distinct. Let $\varepsilon > 0$ be given and let $r \in \mathbb{N}$ be such that $\sum_{i=1}^{r} \delta_A(y_i) > 1 - \varepsilon$ and $\sum_{i=r+1}^{\infty} \delta_A(y_i) \cdot y_i < \varepsilon$. Let $N \in \mathbb{N}$ be such that $\frac{1}{3} \min_{1\leq i\neq j\leq r} |y_i - y_j| > 1/N$ and such that the set $E_i := \{j : |x_j - y_i| < 1/N\}$ have the following property

$$\delta_A(y_i) \leq \underline{\delta_A}(E_i) \leq \overline{\delta_A}(E_i) \leq \delta_A(y_i) + \frac{\varepsilon}{rM_0}$$

for $i=1,\dots,r$ where $M_0=\max\{|y_1|,|y_2|,\dots|y_r|\}$. Note that $E_1,\dots,E_r$ are pairwise disjoint. Now choose a $m_0\in\mathbb{N}$ such that

$$\underline{\delta_A}(E_i)-\frac{1}{N}<\sum_{k\in E_i}a_{n,k}<\overline{\delta_A}(E_i)+\frac{1}{N}$$

for every $n\geq m_0$ and $i=1,\dots,r$. Therefore

$$\delta_A(y_i)-\frac{1}{N}-\frac{\varepsilon}{rM_0}<\sum_{k\in E_i}a_{n,k}<\delta_A(y_i)+\frac{1}{N}+\frac{\varepsilon}{rM_0}$$

for every $n\geq m_0$ and $i=1,\dots,r$. Then for $n\geq m_0$ we have

$$(Ax)_n=\sum_{k=1}^{\infty}a_{n,k}x_k\leq\sum_{k\in E_1}a_{n,k}\cdot\Big(y_1+\frac{1}{N}\Big)+\dots+\sum_{k\in E_r}a_{n,k}\cdot\Big(y_r+\frac{1}{N}\Big)+\sum_{k\in(E_1\cup\dots\cup E_r)^c}a_{n,k}\cdot M.$$

Since A is regular, we can choose a $m_1\geq m_0$ such that for all $n\geq m_1$

$$\sum_{k=1}^{\infty}a_{n,k}<1+\varepsilon.$$

Now observe that

$$1+\varepsilon>\sum_{k=1}^{\infty}a_{n,k}=\sum_{k\in E_1\cup\dots\cup E_r}a_{n,k}+\sum_{k\in(E_1\cup\dots\cup E_r)^c}a_{n,k}$$

where from above we have

$$\sum_{k\in E_1\cup\dots\cup E_r}a_{n,k}=\sum_{j=1}^{r}\sum_{k\in(E_j}a_{n,k}>\sum_{j=1}^{r}\delta_A(y_j)-\frac{r}{N}-\frac{\varepsilon}{M_0}>1-\frac{r}{N}-(1+\frac{1}{M_0})\cdot\varepsilon.$$

Therefore for $n\geq m_0$ we have

$$(Ax)_n\leq\sum_{k\in E_1}a_{n,k}\cdot\Big(y_1+\frac{1}{N}\Big)+\dots+\sum_{k\in E_r}a_{n,k}\cdot\Big(y_r+\frac{1}{N}\Big)+\frac{Mr}{N}+(2+\frac{1}{M_0})M\varepsilon$$

and analogously

$$(Ax)_n\geq\sum_{k\in E_1}a_{n,k}\cdot\Big(y_1-\frac{1}{N}\Big)+\dots+\sum_{k\in E_r}a_{n,k}\cdot\Big(y_r-\frac{1}{N}\Big)-\frac{Mr}{N}-(2+\frac{1}{M_0})M\varepsilon.$$

Hence for $n\geq m_0$

$$\begin{aligned}\Big|(Ax)_n-\sum_i\delta_A(y_i)\cdot y_i\Big|&\leq\Big|(Ax)_n-\sum_{i=1}^{r}\delta_A(y_i)\cdot y_i\Big|+\varepsilon\\
&\leq\sum_{i=1}^{r}\Big|\sum_{k\in E_i}a_{n,k}\cdot\Big(y_i\pm\frac{1}{N}\Big)-\delta_A(y_i)\cdot y_i\Big|+\frac{Mr}{N}+(2M+\frac{M}{M_0}+1)\varepsilon\\
&\leq\sum_{i=1}^{r}\Big|\Big(\sum_{k\in E_i}a_{n,k}-\delta_A(y_i)\Big)\cdot\Big(y_i\pm\frac{1}{N}\Big)\Big|+\frac{r}{N}+\frac{Mr}{N}+(2M+\frac{M}{M_0}+1)\varepsilon\\
&\leq\Big(\frac{1}{N}+\frac{\varepsilon}{rM_0}\Big)\cdot r\cdot(M_0+\frac{1}{N})+\frac{r}{N}+\frac{Mr}{N}+(2M+\frac{M}{M_0}+1)\varepsilon\end{aligned}$$

Since N can be chosen arbitrarily large, we obtain

$$\Big|(Ax)_n-\sum_i\delta_A(y_i)\cdot y_i\Big|\leq(2M+\frac{M}{M_0}+2)\varepsilon$$

for every $\varepsilon>0$. Therefore $\lim_{m\to\infty}(Ax)_n=\sum_i\delta_A(y_i)\cdot y_i$. □

Proposition 7. *Assume that $x=(x_n)$ is bounded. If $\overline{\delta_A}(y)=1$, then y is a limit point of the sequence $\big((Ax)_n\big)$.*

Proof. Since (x_n) is bounded, there is $M>0$ such that $|x_n|\le M$ for every $n\in\mathbb{N}$. Let $y\in\mathbb{R}$ be such that $\overline{\delta_A}(y)=1$. Let $N\in\mathbb{N}$. Let $E_N=\{j\in\mathbb{N}:|x_j-y|<1/P\}$ where $P=P(N)>N$ is such that $1\le\overline{\delta_A}(E_N)\le 1+\frac{1}{2N}$. Then there is $k_N\ge N$ such that

$$\sum_{k\in E_N}a_{k_N,k}>\overline{\delta_A}(E_N)-\frac{1}{2N}\ge 1-\frac{1}{2N}$$

and also from regularity of A

$$\sum_{k=1}^{\infty}a_{k_N,k}<1+\frac{1}{N}.$$

Then we have

$$\sum_{k\in E_N}a_{k_N,k}\cdot\Big(y-\frac{1}{N}\Big)-\sum_{k\notin E_N}a_{k_N,k}\cdot M\le\sum_{k=1}^{\infty}a_{k_N,k}\cdot x_k\le\sum_{k\in E_N}a_{k_N,k}\cdot\Big(y+\frac{1}{N}\Big)+\sum_{k\notin E_N}a_{k_N,k}\cdot M.$$

Hence

$$\begin{aligned}|(Ax)_{k_N}-y|=|\sum_{k=1}^{\infty}a_{k_N,k}-y|&\le(\frac{1}{N}+\frac{1}{N^2})+\sum_{k\notin E_N}a_{k_N,k}\cdot(M+|y|)+\frac{|y|}{N}\\&\le\Big(\frac{1}{2N}+\frac{1}{N}\Big)\cdot(M+|y|)+\frac{|y|+1}{N}+\frac{1}{N^2}.\end{aligned}$$

Therefore

$$\lim_{N\to\infty}(Ax)_{k_N}=y.$$

□

Immediately we obtain the following.

Corollary 8. *Let (x_n) be a bounded sequence. Suppose that there are y and z $(y\neq z)$ with $\overline{\delta_A}(y)=\overline{\delta_A}(z)=1$. Then the A limit*

$$\lim_{n\to\infty}(Ax)_n$$

does not exist.

It turns out that we cannot weaken the Corollary 8 assuming that $\overline{\delta_A}(y),\overline{\delta_A}(z)>r$ for some $r\in(0,1)$.

Proposition 9. *Let $t\in\mathbb{N}$, $r,s\in[1,2^t-1]$ and $L\in\mathbb{R}$. Let $y,z\in\mathbb{R}$ with $y\neq z$. Then there is a sequence (x_n) such that $\overline{d}(y)=r/2^t$, $\overline{d}(z)=s/2^t$ (i.e. when we are taking the limit with respect to Cesaro matrix) and*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}x_i=L.$$

Proof. Let y' and z' be such that $yr/2^t+y'(1-r)/2^t=zs/2^t+z'(1-s)/2^t=L$. Define n_k as follows. Let $n_0=0$ and $n_k=2^t10^{k^2}$, $k\in\mathbb{N}$. Let $B_k=[n_{k-1}+1,n_k]$, $k\in\mathbb{N}$. Note that $|B_k|=2^t(10^{k^2}-10^{(k-1)^2})$. Let $A_k\subset B_k$ be defined as follows

$$A_{2k}=\bigcup_{m=1}^{10^{(2k)^2}-10^{(2k-1)^2}}[n_{2k-1}+1+m2^t,n_{2k-1}+1+m2^t+r]$$

and

$$A_{2k+1}=\bigcup_{m=1}^{10^{(2k+1)^2}-10^{(2k)^2}}[n_{2k}+1+m2^t,n_{2k}+1+m2^t+s].$$

Now, we are ready to define (x_n). Let $x_n=y$ if $n\in A_{2k}$, $x_n=y'$ if $n\in B_{2k}\setminus A_{2k}$, $x_n=z$ if $n\in A_{2k+1}$ and $x_n=z'$ if $n\in B_{2k+1}\setminus A_{2k-1}$. Note that

$$\overline{d}(y)=\overline{d}\Big(\bigcup_{k=1}^{\infty}A_{2k}\Big)=\frac{r}{2^t},\quad\overline{d}(y')=\overline{d}\Big(\bigcup_{k=1}^{\infty}B_{2k}\setminus A_{2k}\Big)=1-\frac{r}{2^t},$$

$$\overline{d}(z) = \overline{d}\Big(\bigcup_{k=0}^{\infty} A_{2k+1}\Big) = \frac{s}{2^t} \quad \text{and} \quad \overline{d}(z') = \overline{d}\Big(\bigcup_{k=0}^{\infty} B_{2k+1} \setminus A_{2k+1}\Big) = 1 - \frac{s}{2^t}.$$

Note that for any $k \in \mathbb{N}$ and $m = 1, \dots, 10^{(2k)^2} - 10^{(2k-1)^2}$ we have

$$\sum_{i=n_{2k-1}+1+m2^t}^{n_{2k-1}+1+(m+1)2^t} x_i = ry + (1-r)y' = 2^t L.$$

Similarly, for any $k \in \mathbb{N}$ and $m = 1, \dots, 10^{(2k+1)^2} - 10^{(2k)^2}$ we have

$$\sum_{i=n_{2k}+1+m2^t}^{n_{2k}+1+(m+1)2^t} x_i = sz + (1-s)z' = 2^t L.$$

From this we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = L.$$

□

One can improve Proposition 9 assuming that $\overline{d}(y)$ and $\overline{d}(z)$ are arbitrary numbers from $(0,1)$ not necessarily with finite dyadic expansion.

3. Relation between A-statistical limit points and points having positive A-density

We first recall some basic facts about ideal convergence which will be used in this section as also in the last section. Let $\mathcal{I}$ be a proper nontrivial admissible ideal in $\mathbb{N}$ and let $(x_n) \in \ell^\infty$. We say that a sequence (x_n) of real numbers tends to y with respect to $\mathcal{I}$ provided $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$, in symbols $y = \mathcal{I} - \lim\limits_{n\to\infty} x_n$. It is well-known that if $\mathcal{I}$ is maximal and $(x_n) \in \ell^\infty$, then $\mathcal{I} - \lim\limits_{n\to\infty} x_n$ exists [14]. A point y is called $\mathcal{I}$-cluster point of (x_n) if $\{n : |x_n - y| \leq \varepsilon\} \notin \mathcal{I}$ for every $\varepsilon > 0$. We say that y is an $\mathcal{I}$-limit point of (x_n) if there is a set $B \subset \mathbb{N}$, $B \notin \mathcal{I}$ such that $\lim\limits_{n\in B} x_n = y$ [14]. Since $\mathcal{I}$ contains all singletons, clearly $\mathcal{I}$-limit points are $\mathcal{I}$-cluster points. $\mathcal{I}_d$-cluster points and $\mathcal{I}_d$-limit points are called statistical cluster points and statistical limit points, respectively (see [11]) while $\mathcal{I}_A$-cluster points and $\mathcal{I}_A$-limit points are called A-statistical cluster points and A-statistical limit points, respectively where $\mathcal{I}_A = \{B \subset \mathbb{N} : \delta_A(B) = 0\}$ forms an admissible ideal in $\mathbb{N}$. Characterizations of the sets of $\mathcal{I}$-cluster points and $\mathcal{I}$-limit points can found in [3, 15, 14]. Let $\mathcal{I}_{\text{fin}}$ be the ideal of finite subsets of $\mathbb{N}$. The classical Bolzano–Weierstrass Theorem says that every sequence $(x_n) \in \ell^\infty$ possesses a limit point, that is an $\mathcal{I}_{\text{fin}}$-limit point.

We start with the following example.

Proposition 10. *There is (x_n) such that $d(y) = 0$ for all $y \in \mathbb{R}$ and (x_n) is Cesáro summable.*

Proof. Define (x_n) in the following way

$$(\underbrace{0,1}_{B_1}, \underbrace{0,1,\frac{1}{2}}_{B_2}, \underbrace{0,1,\frac{1}{4},\frac{3}{4},\frac{1}{2}}_{B_3}, \underbrace{0,1,\frac{1}{8},\frac{7}{8},\frac{2}{8},\frac{6}{8},\frac{3}{8},\frac{5}{8},\frac{4}{8}}_{B_4},\dots)$$

which consists of blocks $B_1, B_2, \dots$. For k and $m = 0, 1, \dots, 2^k - 1$ let $A = \{n : \frac{m}{2^k} \leq x_n \leq \frac{m+1}{2^k}\}$. Then $|A \cap B_n| = 2^{n-k} + 1$ for each $n \geq k$. Hence $d(A) = \frac{1}{2^k}$. Therefore $d(y) = 0$ for $y \in [0,1]$. □

Note that the set of limit points of (x_n) defined in the proof of Proposition 10 equals $[0,1]$, and therefore is uncountable. This is a consequence of the assumption that $d(y) = 0$ for every y, or the fact that (x_n) does not have any statistical limit point. The next two results are proved in the more general settings of ideals. We will prove that if (x_n) does not have any $\mathcal{I}$-limit point, for some ideal $\mathcal{I}$ with a special property, then its set of limit points is uncountable.

Lemma 11. *Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$. Assume that $X := \{n : x_n \in [a,b]\} \notin \mathcal{I}$. Suppose that*

$$\{n : a \leq x_n \leq t - \varepsilon\} \in \mathcal{I} \text{ or } \{n : t + \varepsilon \leq x_n \leq b\} \in \mathcal{I}$$

for any $t \in (a,b)$ and any $\varepsilon > 0$ such that $\varepsilon < \min\{t-a, b-t\}$. Then there is $y \in [a,b]$ such that $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

Proof. Suppose that for any $y \in [a,b]$ there is $\varepsilon_y > 0$ with $\{n : |x_n - y| \geq \varepsilon_y\} \notin \mathcal{I}$. Since $[a,b]$ is compact, there are $a \leq y_1 < y_2 < \dots < y_k \leq b$ such that $\{(y_i - \varepsilon_i, y_i + \varepsilon_i) : i = 1, \dots, k\}$ is an open cover of $[a,b]$, where $\varepsilon_i = \varepsilon_{y_i}$. We may assume that none element of this cover contains over element of this subcover. Let $A_i = \{n : |x_n - y_i| < \varepsilon_i\}$. Note that $A_1 \cup \dots \cup A_k = X$ and therefore there is i with $A_i \notin \mathcal{I}$. Since $X \setminus A_i \notin \mathcal{I}$, there is $j \neq i$ such that $A_j \setminus A_i \notin \mathcal{I}$.

Assume that $i < j$. Since $y_i < y_j$ and $(y_j - \varepsilon_j, y_j + \varepsilon_j)$ is not contained in $(y_i - \varepsilon_i, y_i + \varepsilon_i)$, then $y_i + \varepsilon_i < y_j + \varepsilon_j$. Let $t = y_i + \varepsilon_i$. There is $\varepsilon_t > 0$ with $B := \{n : |x_n - t| \geq \varepsilon_t\} \notin \mathcal{I}$. Consider two cases.

Case 1. If there is $\varepsilon > 0$ such that $B' := \{n : |x_n - t| < \varepsilon\} \in \mathcal{I}$, then $A_i \setminus B' \notin \mathcal{I}$ and $A_j \setminus (A_i \cup B') \notin \mathcal{I}$. Hence

$$\{n : a \leq x_n \leq t - \varepsilon_t\} \notin \mathcal{I} \text{ and } \{n : t + \varepsilon_t \leq x_n \leq b\} \notin \mathcal{I}.$$

Case 2. If $\{n : |x_n - t| < \varepsilon\} \notin \mathcal{I}$ for any $\varepsilon > 0$, then $\{n : |x_n - t| < \varepsilon_t/2\} \notin \mathcal{I}$. Since $B \notin \mathcal{I}$, we have either $\{n : a \leq x_n \leq t - \varepsilon_t\} \notin \mathcal{I}$ or $\{n : t + \varepsilon_t \leq x_n \leq b\} \notin \mathcal{I}$. Assume that $\{n : a \leq x_n \leq t - \varepsilon_t\} \notin \mathcal{I}$. Then

$$\left\{n : a \leq x_n \leq (t - \frac{3}{4}\varepsilon_t) - \frac{1}{4}\varepsilon_t\right\} \notin \mathcal{I} \text{ and } \left\{n : (t - \frac{3}{4}\varepsilon_t) + \frac{1}{4}\varepsilon_t \leq x_n \leq b\right\} \notin \mathcal{I}.$$

□

For any nonempty set A, we will denote by $A^{<\mathbb{N}}$ the family of all finite sequences of elements of A. For any finite sequence $s = (s_1, \dots, s_n) \in A^{<\mathbb{N}}$ and $a \in A$ by $s\hat{\ }a$ we denote a concatenation of s and a, i.e. $s\hat{\ }a = (s_1, \dots, s_n, a)$. By $|s|$ we denote the length of s. If $\alpha \in A^{\mathbb{N}}$, then let $\alpha|n = (\alpha(1), \dots, \alpha(n))$ and $\alpha|0 = \emptyset$, where $\emptyset$ stands for empty sequence.

We will also need the following facts: An ideal $\mathcal{I}$ of $\mathbb{N}$ is called a P-ideal if for any sequences of sets (D_n) from $\mathcal{I}$ there is another sequence of sets (C_n) in $\mathcal{I}$ such that $D_n \bigtriangleup C_n$ is finite for every n and $\bigcup_n C_n \in \mathcal{I}$. Equivalently if for each sequence (A_n) of sets from $\mathcal{I}$ there exists $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$. If $\mathcal{I}$ is a P-ideal then a sequence (x_n) is $\mathcal{I}$-convergent to x if and only if there is a $M \in \mathcal{F}(\mathcal{I})$ (where $\mathcal{F}(\mathcal{I}) = \{B \subset \mathbb{N} : B^c \in \mathcal{I}\}$ is the dual filter) such that $(x_n)_{n \in M}$ is usually convergent to x (see [14]).

A function $\varphi : 2^{\mathbb{N}} \to [0, \infty]$ is called a submeasure if $\varphi(E) \leq \varphi(E \cup F) \leq \varphi(E) + \varphi(F)$ for any $E, F \in 2^{\mathbb{N}}$. A submeasure φ is called lower semicontinuous if $\lim_{m \to \infty} \varphi(E \cap [1,m]) = \varphi(E)$. By $\mathrm{Exh}(\varphi)$ denote the set of all $E \subset \mathbb{N}$ with $\lim_{m \to \infty} \varphi(E \setminus [1,m]) = 0$. The celebrated Solecki's characterization [21] states that an ideal $\mathcal{I}$ is an analytic P-ideal if and only if it is of the form $\mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ on $\mathbb{N}$.

The following result seems to be a mathematical folklore but we present its short proof for sake of completeness.

Proposition 12. *Let A be a non-negative regular matrix. Then $\mathcal{I}_A$ is a P-ideal.*

Proof. Let $\varphi_A(E) = \sup_{n \in \mathbb{N}} \sum_{k \in E} a_{n,k}$. We need to show that φ is a lower semicontinuous submeasure on $\mathbb{N}$ and $\mathrm{Exh}(\varphi) = \mathcal{I}_A$. Clearly φ_A is monotonous and subadditive. We will show that it is lower semicontinuous. Fix $E \subset \mathbb{N}$. Let $s := \varphi_A(E) = \sup_{n \in \mathbb{N}} \sum_{k \in E} a_{n,k}$ and let $\varepsilon > 0$. We can find $n \in \mathbb{N}$ such that $\sum_{k \in E} a_{n,k} > s - \varepsilon/2$. Then there is $m \in \mathbb{N}$ with

$$s - \varepsilon = s - \varepsilon/2 - \varepsilon/2 < \sum_{k \in E, k \leq m} a_{n,k} \leq \sup_{n \in \mathbb{N}} \sum_{k \in E, k \leq m} a_{n,k} = \varphi_A(E \cap [1,m]).$$

Since the sequence $(\varphi_A(E \cap [1,m]))$ is non-decreasing, then $\lim_{m \to \infty} \varphi_A(E \cap [1,m]) = \varphi_A(E)$.

Assume that $\delta_A(E) = 0$. Let $\varepsilon > 0$. There is n_0 such that $\sum_{k \in E} a_{n,k} < \varepsilon$ for $n \geq n_0$. Since $\sum_{k \in E} a_{n,k}$ is convergent for any $n < n_0$, there is m such that $\sum_{k \in E, k > m} a_{n,k} < \varepsilon$ for $n < n_0$.

Consequently

$$\varphi_A(E \setminus [1,m]) = \sup_{n\in\mathbb{N}} \sum_{k\in E, k>m} a_{n,k} < \varepsilon.$$

Hence $\mathcal{I}_A \subset \mathrm{Exh}(\varphi_A)$.

Assume now that $\lim_{m\to\infty} \varphi_A(E \setminus [1,m]) = 0$. Let $\varepsilon > 0$. There is $m \in \mathbb{N}$ such that $\sum_{k\in E,k>m} a_{n,k} < \varepsilon/2$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_{n,k} = 0$ for every $n \in \mathbb{N}$, there is n_0 such that $a_{n,k} < \varepsilon/(2m)$ for $n \geq n_0$ and $k \leq m$. Therefore $\sum_{k\in E} a_{n,k} < \varepsilon$ for $n \geq n_0$. That means that $\delta_A(E) = 0$. Thus $\mathcal{I}_A \supset \mathrm{Exh}(\varphi_A)$. □

Proposition 13. *Let $\mathcal{I}$ be a P-ideal. Assume that $(x_n) \in \ell^\infty$ does not have any $\mathcal{I}$-limit points. Then the set of limit points of (x_n), i.e. the set*

$$\{y \in \mathbb{R} : x_{n_k} \to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

Proof. Let $I_\emptyset = [a,b]$ be such that $(x_n) \subset [a,b]$. Then there are $t \in (a,b)$ and $\varepsilon > 0$ such that

$$\{n : a \leq x_n \leq t-\varepsilon\} \notin \mathcal{I} \text{ and } \{n : t+\varepsilon \leq x_n \leq b\} \notin \mathcal{I}.$$

If there are not such t and ε, then by Lemma 11 there is $y \in [a,b]$ such that $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. This means that (x_n) is $\mathcal{I}$ convergent to y. Since $\mathcal{I}$ is a P-ideal, so y is an $\mathcal{I}$-limit point of (x_n) which yields a contradiction. Let $I_{(0)} = [a, t-\varepsilon]$ and $I_{(1)} = [t+\varepsilon, b]$. Proceeding inductively we define a family $\{I_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ of nontrivial compact intervals such that

(i) $I_{s\hat{\ }i} \subset I_s$ for $i = 0,1$ and $s \in \mathbb{N}^{<\mathbb{N}}$;
(ii) $\{n : x_n \in I_s\} \notin \mathcal{I}$ for $s \in \mathbb{N}^{<\mathbb{N}}$.

Note that for any $\alpha \in \{0,1\}^{\mathbb{N}}$ and any $k \in \mathbb{N}$ there are infinitely many x_n's with $x_n \in I_{\alpha|k}$. Therefore there is $x_\alpha \in \bigcap_k I_{\alpha|k}$ which is a limit point of (x_n). Note that $x_\alpha \neq x_\beta$ for distinct $\alpha, \beta \in \{0,1\}^{\mathbb{N}}$. Therefore the set of limit points of (x_n) is uncountable. Note that the set of limit points of (x_n) is always closed. □

Using the same reasoning one can prove a slightly stronger assertion, which will be used in the sequel.

Corollary 14. *Let $[a,b]$ be a fixed interval and $\mathcal{I}$ be a P-ideal. Assume that $\{n : x_n \in [a,b]\} \notin \mathcal{I}$ and any point $y \in (a,b)$ is not an $\mathcal{I}$-limit point of (x_n). Then the set of limit points of (x_n) in $[a,b]$, i.e. the set*

$$\{y \in (a,b) : x_{n_k} \to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

The next Corollary is a counterpart of Bolzano–Weierstrass Theorem for $\mathcal{I}$-limit points.

Corollary 15. *Let $(x_n) \in \ell^\infty$. Assume that the set of limit points of (x_n) is countable. Then the sequence (x_n) has at least one $\mathcal{I}$-limit for every P-ideal $\mathcal{I}$.*

Filipów, Mrożek, Recław and Szuca have introduced in [6] the notion of Bolzano–Weierstrass property (in short BW property) for ideals defined on $\mathbb{N}$. An ideal $\mathcal{I}$ satisfies BW if for any bounded sequence (x_n) there is a set of indexes $E \subset \mathbb{N}$ such that $(x_n)_{n\in E}$ is $\mathcal{I}|E$-convergent where $\mathcal{I}|E = \{X \cap E : X \in \mathcal{I}\}$. It was mentioned in [6] that the density zero ideal does not satisfy BW. Very probably none ideal of the form $\mathcal{I}_A$ satisfy BW. For other Bolzano–Weierstrass properties of ideals we refer the reader to [6].

Note that if $\mathcal{I}$ is maximal and y is an $\mathcal{I}$-cluster point of (x_n), then y is an $\mathcal{I}$-limit of (x_n).

Fridy observed in [11] that statistical limit points of a sequence are its statistical cluster points, and there is a sequence (x_n) such that 0 is statistical cluster point of (x_n) but 0 is not a statistical limit point of (x_n). Below we present a characterization of A-statistical limit points in terms of points with positive A-density. But before that we prove the following lemma which helps us to characterize those A-statistical cluster points of a sequence which are its A-statistical limit points.

Lemma 16. *Let $r \in (0,1)$, $r_1 \geq r_2 \geq r_3 \geq \dots$, $\lim_{n\to\infty} r_n = r$ and let (E_n) be a decreasing sequence of subsets of $\mathbb{N}$.*

(i) *If $\underline{\delta_A}(E_n) = r_n$, $n \in \mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\underline{\delta_A}(E) = r$ and such that $E \subset^* E_n$, $n \in \mathbb{N}$, i.e. $E_n \setminus E$ is finite for every $n \in \mathbb{N}$. Moreover, if $\overline{\delta_A}(E_n) \to r$, then $\delta_A(E) = r$.*

(ii) *If $\overline{\delta_A}(E_n) = r_n$, $n \in \mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\overline{\delta_A}(E) = r$ and such that $E \subset^* E_n$, $n \in \mathbb{N}$.*

Proof. (i) Let (p_n) be an increasing sequence of natural numbers such that $\sum_{k\in E_n} a_{j,k} > r_n - \frac{1}{3n}$ for every $j \geq p_n$. For each $n \in \mathbb{N}$ now choose $m_n > p_n$ such that $\sum_{k\in E_n\cap[1,m_n]} a_{j,k} > r_n - \frac{1}{3n} - \frac{1}{3n} > r_n - \frac{1}{n}$ for all $j, p_n \leq j \leq p_{n+1}$. Thus we have two increasing sequences of natural numbers (p_n) and (m_n) such that $\forall\ j \in [p_n, p_{n+1}]$ we have

$$\sum_{k\in E_n\cap[1,m_n]} a_{j,k} > r_n - \frac{1}{n}.$$

Put $E = \bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}]$. Take $p_n \leq j < p_{n+1}$. Then

$$\sum_{k\in E} a_{j,k} \geq \sum_{k\in E_n\cap[1,m_n]} a_{j,k} > r_n - \frac{1}{n}.$$

Thus $\liminf_{n\to\infty} \sum_{k\in E} a_{n,k} \geq r$ which means that $\underline{\delta_A}(E) \geq r$. Since $E_1 \supset E_2 \supset E_3 \supset \dots$, so $\bigcup_{n=j}^{\infty} E_n \cap [1, m_{n+1}] \subset E_j$ and $E_j \setminus E \subset \bigcup_{n=1}^{j-1} E_n \cap [1, m_{n+1}]$. Therefore $E \subset^* E_j$ and consequently $\underline{\delta_A}(E) \leq \underline{\delta_A}(E_j)$ and $\overline{\delta_A}(E) \leq \overline{\delta_A}(E_j)$. Hence $\underline{\delta_A}(E) = r$ and if $\overline{\delta_A}(E_n) \to r$, then $\delta_A(E) = r$.

(ii) As before we can choose two increasing sequences of natural numbers (p_n) and (m_n) such that

$$\sum_{k\in E_n\cap[1,m_n]} a_{p_n,k} > r_n - \frac{1}{n}$$

for every n. Put $E = \bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}]$. Then

$$\sum_{k\in E} a_{p_{n+1},k} \geq \sum_{k\in E_n\cap[1,m_{n+1}]} a_{p_{n+1},k} > r_{n+1} - \frac{1}{n+1}.$$

Thus $\overline{\delta_A}(E) = r$ and $E \subset^* E_n\ \forall\ n \in \mathbb{N}$. □

We now have the following characterization of A-statistical limit points.

Theorem 17. *Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an A-statistical limit point of (x_n) if and only if $\overline{\delta_A}(y) > 0$. Moreover if $\delta_A(y) > 0$, then there is $E \subset \mathbb{N}$ with $\delta_A(E) = \delta_A(y)$ and $\lim_{n\in E} x_n = y$.*

Proof. Assume that $\overline{\delta_A}(y) = 0$ and suppose that y is an A-statistical limit point of (x_n). Then there is $E \subset \mathbb{N}$ such that $\overline{\delta_A}(E) > 0$ and $\lim_{n\in E} x_n = y$. Note that $E \subset^* \{j : |x_j - y| \leq \varepsilon\}$ for every $\varepsilon > 0$. Hence $\overline{\delta_A}(E) \leq \overline{\delta_A}(\{j : |x_j - y| \leq \varepsilon\})$ for every $\varepsilon > 0$. Therefore $\overline{\delta_A}(E) = 0$ which yields a contradiction.

Assume that $\overline{\delta_A}(y) > 0$. Let $E_n = \{j : |x_j - y| \leq 1/n\}$. Then (E_n) is a decreasing sequence with $\overline{\delta_A}(E_n) \to \overline{\delta_A}(y)$. By Lemma 16 there is $E \subset^* E_n$, $n \in \mathbb{N}$, with $\overline{\delta_A}(E) = \overline{\delta_A}(y)$. Since almost all elements of E are contained in E_n, then clearly $\lim_{j\to\infty, j\in E} x_j = y$. Hence y is an A-statistical limit point of (x_n).

The last part of the assertion follows in a similar way. □

As a corollary we obtain a characterization of those A-statistical cluster points which are not A-statistical limit points.

Corollary 18. *Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an A-statistical cluster point of (x_n) and it is not an A-statistical limit point if and only if*

- $\overline{\delta_A}(\{j : |x_j - y| \le 1/n\}) > 0$ *for every n;*
- $\delta_A(y) = \lim\limits_{n\to\infty} \overline{\delta_A}(\{j : |x_j - y| \le 1/n\}) = 0$.

4. When a sequence is a splice and the proof of the main result

In this section we show how to divide the set of indices of a sequence (x_n) to obtain a partition $\{E_i\}$ such that (x_n) becomes a splice.

Proposition 19. *Let $(x_n) \in \ell^\infty$. Assume that there exist distinct real numbers $y_1, \dots, y_m$ with $\delta_A(y_i) > 0$ $\forall i$ such that $\sum\limits_{i=1}^{m} \delta_A(y_i) = 1$. Then there exists a partition $E_1, \dots E_m, E_{m+1}$ such that $\delta_A(E_i) = \delta_A(y_i)$, for $i = 1, \dots, m$, $\delta_A(E_{m+1}) = 0$ and $\lim\limits_{n\in E_i} x_n = y_i$.*

Proof. By Theorem 17 there are $E_1', \dots, E_m'$ with $\lim\limits_{n\in E_i'} x_n = y_i$. Let $\varepsilon = \min\{|y_i - y_j|/3 : i, j = 1, \dots, m, i \ne j\}$. Put $E_i = \{n : |x_n - y_i| \le \varepsilon\} \cap E_i'$, $i = 1, \dots, m$. Clearly $\delta_A(E_i) = \delta_A(y_i)$, $E_{m+1} = \mathbb{N} \setminus \bigcup\limits_{i=1}^{m} E_i$ has A-density zero, $E_1, \dots E_m, E_{m+1}$ are pairwise disjoint, and $\lim\limits_{n\in E_i} x_n = y_i$. □

Consider the following example. Let $E_1, \dots, E_k, E_{k+1}$ be such that $\delta_A(E_1) + \dots + \delta_A(E_k) = 1$, $\delta_A(E_{k+1}) = 0$ and E_{k+1} is infinite. Define (x_n) in the following way. Put $x_n = i$ if $n \in E_i$, $i = 1, \dots, k$ and $(x_n)_{n\in E_{k+1}}$ be dense in the unit interval $[0, 1]$. For a such sequence (x_n) we can apply Theorem 6. It turns out that the Osikiewicz Theorem also can be applied in this situation. To do this we need a partition of E_{k+1} into infinitely many infinite subsets $F_1, F_2, \dots$ such that $\lim\limits_{n\in F_k} x_k$ exists and apply the following Proposition 20.

Proposition 20. *Let $(x_n) \in \ell^\infty$. Assume that there exist distinct real numbers $y_1, y_2, \dots$ such that $\delta_A(y_i) > 0$ $\forall i$ and $\sum\limits_{i=1}^{\infty} \delta_A(y_i) = 1$. Then there exists a partition $E_1, E_2, \dots$ such that $\delta_A(E_i) = \delta_A(y_i)$, $i = 1, 2, \dots$ and $\lim\limits_{n\in E_i} x_n = y_i$.*

Proof. By Theorem 17 there are $E_1', E_2', \dots$ with $\lim\limits_{n\in E_i'} x_n = y_i$. Note that $E_i' \cap E_j'$ is finite if $i \ne j$. Define $E_1, E_2, \dots$ in the following way. Let $E_1'' = E_1'$, $E_m'' = E_m' \setminus \bigcup\limits_{i=1}^{m-1} E_i'$, $k \ge 2$. Since $E_m' \cap \bigcup\limits_{i=1}^{m-1} E_i'$ is finite, then $\delta_A(E_m'') = \delta_A(E_m') = \delta_A(y_m)$, $m \in \mathbb{N}$. Let $E = \mathbb{N} \setminus \bigcup\limits_{m=1}^{\infty} E_m'$. If E is finite, then put $E_1 = E \cup E_1''$ and $E_m = E_m''$, $m \ge 2$. If the set E is infinite, then enumerate it as $\{n_1, n_2, \dots\}$ and put $E_m = E_m'' \cup \{n_m\}$. Clearly $\lim\limits_{n\in E_m} x_n = y_m$. □

Consider the following example. Let $F_1, F_2, \dots$ be a partition of $\mathbb{N}$ such that $\delta_A(F_i) = 1/2^i$, $i \in \mathbb{N}$. Define a new partition $E_1, E_2, \dots$ as follows. Let $F_1 = \{a_k : k \in \mathbb{N}\}$. Put $E_k = F_{k+1} \cup \{a_k\}$. Then $\bigcup\limits_{k=1}^{\infty} E_k = \mathbb{N}$ but $\sum\limits_{k=1}^{\infty} \delta_A(E_k) = 1/2$. The next lemma shows how to find a nice refinement of partitions of $\mathbb{N}$.

Lemma 21. *Assume that $\{E_n : n = 1, 2, \dots\}$ is a partition of $\mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \delta_A(E_n) < 1$. Then there is a partition $\{F_n : n = 0, 1, 2, \dots\}$ of $\mathbb{N}$ such that*

(i) $F_n \subset E_n$;
(ii) $\delta_A(F_n) = \delta_A(E_n)$;

(iii) $\delta_A(F_0) = 1 - \sum_{n=1}^{\infty} \delta_A(E_n)$.

Proof. Let (ε_n) be a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. We define inductively a strictly increasing sequence (m_n) of natural numbers such that

$$\sum_{k\in[m_{n-1},j]\setminus(E_1\cup E_2\cup\cdots\cup E_n)} a_{j,k} \geq 1 - (d(E_1) + d(E_2) + \cdots + d(E_n)) - \varepsilon_n$$

for every $j \geq m_n$. Let $F_0 = \bigcup_{n=1}^{\infty} \left([m_{n-1}, m_{n+1}] \setminus \bigcup_{i=1}^{n} E_i\right)$ where $m_0 = 0$. Let $m_n \leq j < m_{n+1}$. Then

$$\sum_{k\in F_0} a_{j,k} \geq \sum_{k\in[m_{n-1},j]\setminus(E_1\cup E_2\cup\cdots\cup E_n)} a_{j,k} \geq 1 - (d(E_1) + d(E_2) + \cdots + d(E_n)) - \varepsilon_n.$$

Therefore $\underline{\delta_A}(F_0) \geq 1 - \sum_{n=1}^{\infty} \delta_A(E_n)$. Let $F_n = E_n \setminus F_0$ for every $n \in \mathbb{N}$. Note that $F_0 \cap E_n$ is finite $\forall\, n \in \mathbb{N}$. Therefore $\delta_A(F_n) = \delta_A(E_n)$. Since $F_0 = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} F_n$, then $\overline{\delta_A}(F_0) \leq 1 - \sum_{n=1}^{\infty} \delta_A(E_n)$. Hence $\delta_A(F_0) = 1 - \sum_{n=1}^{\infty} \delta_A(E_n)$. □

The next theorem gives a sufficient condition for a sequence (x_n) to have $\sum_{y\in D} d(y) = 1$ for some countable set D.

Theorem 22. *Let $(x_n) \in \ell^\infty$. Suppose that the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for any $y \in \mathbb{R}$. Then*

$$\sum_{y\in D} \delta_A(y) = 1,$$

where $D = \{y \in \mathbb{R} : \delta_A(y) > 0\}$.

Proof. Suppose that

$$\sum_{y\in D} \delta_A(y) < 1.$$

Then by Corollary 14 the set D is non-empty. By Lemma 4 the set D is countable. Enumerate D as $\{y_1, y_2, \dots\}$. By Proposition 20 there is a partition $\{E_k : k = 1, 2, \dots\}$ of $\mathbb{N}$ such that $d(E_k) = d(y_k)$ and $\lim_{n\to\infty, n\in E_k} x_n = y_k$. By Lemma 21 there is a partition $\{F_k : k = 0, 1, 2, \dots\}$ such that $F_k \subset E_k$, $\delta_A(F_k) = \delta_A(E_k)$ and $\delta_A(F_0) = 1 - \sum_{k=1}^{\infty} \delta_A(F_k)$. Since $\delta_A(y) = 0$ for every $y \notin D$, then by Proposition 13 applied to the sequence $(x_n)_{n\in F_0}$ and to the ideal $\mathcal{I}_A|_{F_0} = \{E \cap F_0 : E \in \mathcal{I}_A\}$, we obtain that the sequence $(x_n)_{n\in F_0}$ has uncountably many limit points which contradicts the assumption. □

Finally note that combining Theorem 6 and Theorem 22 we obtain Theorem 3.

5. A different view of Osikiewicz Theorem

We first recall the following result of Henstock.

Theorem 23 (Henstock [12])**.** *Let $(x_n) \in \ell^\infty$. Assume that A is non-negative regular summability matrix. Assume that $G(t) = \delta_A(\{n : x_n \leq t\})$ exists for every $t \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \int_{-\infty}^{\infty} t\,dG(t).$$

We now show that The Osikiewicz Theorem (Theorem 2) follows easily from Theorem 23. The original Osikiewicz proof is relatively long but it is not based on Theorem 23.

Proof. At first we will show that $G(t) = \delta_A(\{n \in \mathbb{N} : x_n \leq t\})$ exists for every $t \in \mathbb{R}$. Fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $\sum_{i=1}^{k} \delta_A(E_i) > 1 - \varepsilon$. Let $t \in \mathbb{R} \setminus \{y_1, \ldots, y_k\}$. We may assume that $y_1 < y_2 < \cdots < y_k$. Then for any t there is $j = 1, \ldots, k-1$ such that $y_j < t < y_{j-1}$ or $t < y_1$ or $t > y_k$. Assume that $y_j < t < y_{j-1}$.

Put $\underline{\delta}_A(E) = \liminf_{n\to\infty} \sum_{n\in E} a_{n,k}$ and $\overline{\delta}_A(E) = \limsup_{n\to\infty} \sum_{n\in E} a_{n,k}$. Then $\underline{\delta}_A(\{n : x_n \leq t\}) \geq d(E_1) + \cdots + d(E_j)$ and $\overline{\delta}_A(\{n : x_n \leq t\}) \leq 1 - d(E_{j+1}) - \cdots - d(E_k)$. Thus $\overline{\delta}_A(\{n : x_n \leq t\}) - \underline{\delta}_A(\{n : x_n \leq t\}) \leq \varepsilon$. Therefore $\delta_A(\{n : x_n \leq t\})$ exists for every $t \in \mathbb{R}$.

To prove that $\int_{-\infty}^{\infty} t dG(t) = \sum_i y_i \delta_A(E_i)$ we need to show that $\lim_{t\to y_i^+} G(t) - \lim_{t\to y_i^-} G(t) = \delta_A(E_i)$. In fact it is enough to show that $\lim_{t\to y_i^+} G(t) - \lim_{t\to y_i^-} G(t) \geq \delta_A(E_i)$ since $\lim_{t\to\infty} G(t) = 1$, $\lim_{t\to-\infty} G(t) = 0$, G is non-decreasing and $\sum_i \delta_A(E_i) = 1$.

For any $\varepsilon > 0$ we have $E_i \setminus F_\varepsilon \subset \{n : y_i - \varepsilon < x_n < y_i + \varepsilon\}$ where F_ε is a finite subset of $\mathbb{N}$. Thus $\overline{\delta}_A(\{n : y_i - \varepsilon < x_n < y_i + \varepsilon\}) \geq \delta_A(E_i)$ for any $\varepsilon > 0$. Note that $\overline{\delta}_A(\{n : y_i - \varepsilon < x_n < y_i + \varepsilon\}) = G(y_i + \varepsilon) - G(y_i - \varepsilon) = \delta_A(\{n : y_i - \varepsilon < x_n < y_i + \varepsilon\})$. Thus $\lim_{t\to y_i^+} G(t) - \lim_{t\to y_i^-} G(t) \geq \delta_A(E_i)$. □

6. A generalization of the Henstock theorem

In this section our main goal is to generalize the Henstock theorem. Instead of the limit $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k$ in the matrix summability method, we will consider an ideal limit $\mathcal{I}-\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k$ which is called $A^{\mathcal{I}}$-summability method which has been very recently introduced by Savas, Das and Dutta [19] (one can see [18] for more related works).

After the study of the notion of $\mathcal{I}$-convergence by Kostyrko, Šalát, and Wilczyński [14] (which is a natural generalization of the usual convergence and the statistical convergence) a lot progress was done in recent years in applications of $\mathcal{I}$-convergence in analysis (see [1], [6], [7], [8], [16], [13], [2] and [4]).

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. For a non-negative regular matrix A and a set $E \subset \mathbb{N}$, we define the $\mathcal{I}$-extension of δ_A by

$$\mathcal{I} - \delta_A(E) = \mathcal{I} - \lim_{n\to\infty} \sum_{n\in E} a_{n,k}.$$

Note that if $\mathcal{I}$ is maximal, then $\mathcal{I} - \delta_A(E)$ is well-defined for any $E \subset \mathbb{N}$.

The following propostion is an ideal counterpart of a known result.

Proposition 24. *Let $\mathcal{I}$ be an ideal and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that $F, F_1, F_2, \ldots$ are non-decreasing functions such that $F_n(t)$ tends to $F(t)$ with respect to $\mathcal{I}$ for every point t of continuity of F. Then*

$$\int_{-\infty}^{\infty} f(t) dF(t) = \mathcal{I} - \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) dF_n(t).$$

Proof. Let $\varepsilon > 0$. There is $M > 0$ with $|\int_{-\infty}^{\infty} f(t)dF(t) - \int_{-M}^{M} f(t)dF(t)| < \varepsilon/4$. Let $\rho > 0$ be such that for any partition $P = \{-M = x_0 < x_1 < \cdots < x_m = M\}$ of $[-M, M]$ with mesh less than ρ we have

$$\Big| \int_{-\infty}^{\infty} f(t)dF(t) - \sum_{i=1}^{m} f(x_i)(F(x_i) - F(x_{i-1})) \Big| < \varepsilon/4.$$

Fix a partition $P = \{-M = x_0 < x_1 < \cdots < x_m = M\}$ of $[-M, M]$ with mesh less than ρ. We may assume that every x_i is a continuity point of F. Since $F(x_i) = \mathcal{I} - \lim_{n\to\infty} F_n(x_i)$, then $A_i = \{n :$

$|F(x_i)-F_n(x_i)| \geq \varepsilon/(4mK)\} \in \mathcal{I}$ where $K = \sup\{|f(x)| : x \in [-M, M]\}$. Let $E = E_0 \cup E_1 \cup \cdots \cup E_m$. Then for any $n \in \mathbb{N} \setminus E$ we have $|F(x_i) - F_n(x_i)| < \varepsilon/(4mK)$. Therefore for such n,

$$\Big|\sum_{i=1}^{m} f(x_i)(F(x_i) - F(x_{i-1})) - \sum_{i=1}^{m} f(x_i)(F_n(x_i) - F_n(x_{i-1}))\Big| =$$

$$\Big|\sum_{i=1}^{m} f(x_i)(F(x_i) - F_n(x_i)) - \sum_{i=1}^{m} f(x_i)(F(x_{i-1}) - F_n(x_{i-1}))\Big| \leq \varepsilon/2.$$

Finally

$$\Big|\int\limits_{-\infty}^{\infty} f(t)dF(t) - \sum_{i=1}^{m} f(x_i)(F_n(x_i) - F_n(x_{i-1}))\Big| \leq \varepsilon$$

for any $n \in \mathbb{N} \setminus E$. The result follows. $\square$

Let (x_k) be a sequence of real numbers. Let $G_n : \mathbb{R} \to [0, 1]$ be given by

$$G_n((x_k), t) = \sum_{k=1}^{\infty} a_{n,k} \mathbb{1}_{[x_k,\infty)}(t).$$

Lemma 25. *Let $t \in \mathbb{R}$ and let $\mathcal{I}$ be an ideal. Assume that $s = \mathcal{I}-\lim_{n\to\infty} G_n(t)$. Then $s = \mathcal{I}-\delta_A(\{n \in \mathbb{N} : x_n \leq t\})$.*

Proof. By E denote the set of those k's such that $x_k \leq t$. Then

$$G_n((x_k), t) = \sum_{k=1}^{\infty} a_{n,k} \mathbb{1}_{[x_k,\infty)}(t) = \sum_{k\in E} a_{n,k}.$$

Since $s = \mathcal{I}-\lim_{n\to\infty} G_n((x_k), t)$, then $\mathcal{I}-\lim_{n\to\infty} \sum_{k\in E} a_{n,k} = \mathcal{I}-\delta_A(E)$. From the uniqueness of $\mathcal{I}$-limit, we obtain $s = \mathcal{I} - \delta_A(\{n \in \mathbb{N} : x_n \leq t\})$. $\square$

Let $G((x_k), t) = \mathcal{I} - \lim_{n\to\infty} G_n((x_k), t)$ for $t \in \mathbb{R}$. Modifying, if necessary, G at some points of discontinuity of G, we may assume that G is a distribution function. For a non-negative regular summability matrix A we consider the $A^{\mathcal{I}}$-summability method as follows

$$\mathcal{I} - \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}x_k.$$

Now, using Proposition 24 and essentially the same method as in the previous section, we can prove a Henstock-type formula for the $A^{\mathcal{I}}$-summability method.

Theorem 26. *Let $\mathcal{I}$ be an ideal, let A be a non-negative regular summability matrix and $(x_n) \in \ell^\infty$. Then*

$$\mathcal{I} - \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}x_k = \int_{-\infty}^{\infty} t dG((x_k), t),$$

provided the $\mathcal{I}$-limit exists.

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