TWO POINT SETS WITH ADDITIONAL PROPERTIES

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Abstract. A subset of the plane is called two point set whether it intersects any line in exactly two points. We give constructions of two point sets possessing some additional properties. Among these properties we consider: being a Hamel base, belonging to some σ-ideal, being (completely) nonmeasurable with respect to different σ-ideals, being κ-covering.

We also give examples of properties that are not satisfied by any two point set: being Luzin, Sierpiński and Bernstein set.

We also consider a natural generalizations of two point set, namely: partial two point sets and n point sets for $n = 3, 4, \ldots, \aleph_0, \aleph_1$. We obtained consistent results connecting partial two point sets and some combinatorial properties (e.g. being m.a.d. family).

1. introduction

At the beginning of the XX century Mazurkiewicz in [11] constructed a set on the plane which meets any line in exactly two points. Any such set is called *a two point set*.

Any two point set must be somehow complex, namely Larman in [9] show that it cannot be F_σ. It is a long standing open problem whether there is a Borel two point set (see [10]). The best known approximation to that problem is due to Miller who, assuming $V = L$, proved that there is a coanalytic two point set [12].

The aim of this paper is to construct two point sets which posses some additional properties. First, we focus on being Hamel base and being completely $\mathbb{I}$-nonmeasurable. (A is completely $\mathbb{I}$-nonmeasurable if the intersection $A \cap B$ does not belong to $\mathbb{I}$ for any Borel set $B \notin \mathbb{I}$; see e.g. [3], [14], [15], [19].)
We also construct a two point set which does not belong to the σ-algebra s (of Marczewski measurable sets). In contrast, we prove that there exists a two point set which belongs to the σ-ideal s_0 (of Marczewski null sets). In particular, we generalize result from [13].
Recently Schmerl proved in [16] that there is a two point set which can be covered by countably many circles. In particular, there is a two point set which is meager and null.

We positively answer the question whether every n point set (for $n = 2, 3, \ldots$) can be represented as a union of n bijections. We also show that any two point set does not contain an additive function. We construct a two point set which does not contain any measurable function.

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We observe that a two point set cannot be any of the following: a Luzin set, a Sierpiński set, a Bernstein set. However, under CH, we construct a partial two point set which is a strong Luzin set (or a strong Sierpiński set).

We also compare the notion of κ point set with the notion of κ-covering and κ-I-covering. (A is κ-covering if for every subset X of size κ there exists a translation h of $\mathbb{R}^2$ such that $h[X] \subseteq A$; A is κ-I-covering if for every subset X of size κ there exists an isomorphism h of $\mathbb{R}^2$ such that $h[X] \subseteq A$; see [7].)

We give some consistent examples of partial two point sets which are, in a sense, m.a.d. families, maximal families of eventually different functions.

2. Completely $\mathbb{I}$-nonmeasurable Hamel base

We say that $\mathbb{I}$ is a *σ-ideal* of subsets of $\mathbb{R}^2$ if $\mathbb{I}$ is closed under taking subsets and closet under taking countable unions.

Let $\mathbb{I}$ be a σ-ideal of subsets of $\mathbb{R}^2$ containing all singletons and having a Borel base (i.e. for every $I \in \mathbb{I}$ there is a Borel set $B \in \mathbb{I}$ such that $I \subseteq B$). We recall the notion of completely $\mathbb{I}$-nonmeasurability which was studied in e.g. [3], [7], [14], [15], [19]. This notion is also known as $\mathbb{I}$-Bernstein set.

Definition 2.1. We say that a set $A \subseteq \mathbb{R}^2$ is *completely* $\mathbb{I}$*-nonmeasurable* iff it intersects all $\mathbb{I}$-positive Borel sets (i.e sets which are in $Borel \setminus \mathbb{I}$) but does not contain any of them.

When $\mathbb{I} = [\mathbb{R}^2]^{\leq\omega}$ then the notion of completely $\mathbb{I}$-nonmeasurable set coincide with the notion of a Bernstein set.

We will assume that $\mathbb{I}$ is a σ-ideal of subsets of $\mathbb{R}^2$ with the property that for every $\mathbb{I}$-positive Borel set there is $\mathfrak{c}$ many pairwise disjoint lines which intersect it on the set of cardinality $\mathfrak{c}$.

Let us observe that σ-ideal of null sets $\mathscr{N}$ and σ-ideal of meager sets $\mathscr{M}$ on the real plane (by Fubini Theorem and by Kuratowski-Ulam Theorem) fulfill this condition.

We say that $H \subseteq \mathbb{R}^2$ is a Hamel base if H is a base of $\mathbb{R}^2$ treated as a linear space over $\mathbb{Q}$.

Theorem 2.2. *There exists a two point set $A \subseteq \mathbb{R}^2$ that is completely $\mathbb{I}$-nonmeasurable Hamel base.*

Proof. Let $\{l_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$, let $\{B_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all $\mathbb{I}$-positive Borel sets on a plane $\mathbb{R}^2$ and let $\{h_\xi : \xi < \mathfrak{c}\}$ be a Hamel base of $\mathbb{R}^2$. We will define, by induction on $\xi < \mathfrak{c}$, the sequence $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$:

1. $|A_\xi| < \omega$,
2. $\bigcup_{\zeta\leq\xi} A_\zeta$ does not have three collinear points,
3. $\bigcup_{\zeta\leq\xi} A_\zeta$ contains precisely two points of l_ξ,
4. $B_\xi \cap \bigcup_{\zeta\leq\xi} A_\zeta \neq \emptyset$,
5. $\bigcup_{\zeta\leq\xi} A_\zeta$ is linearly independent over $\mathbb{Q}$,
6. $h_\xi \in \mathrm{span}_{\mathbb{Q}}(\bigcup_{\zeta\leq\xi} A_\zeta)$.

To make an inductive construction assume that for some $\xi < \mathfrak{c}$ we have already defined the sequence $\{A_\zeta : \zeta < \xi\}$ which fulfills (1)-(6). Let $A_{<\xi} = \bigcup_{\zeta<\xi} A_\zeta$. Clearly $|A_{<\xi}| < \mathfrak{c}$. Let $\mathcal{L}$ be the family of all lines which meet $A_{<\xi}$ in exactly two points. Then $|\mathcal{L}| \le |A^2_{<\xi}| < \mathfrak{c}$. Moreover $|\operatorname{span}_{\mathbb{Q}}(A_{<\xi})| < \mathfrak{c}$. We will define A_ξ in three steps. In each step we will focus on one of desired properties of A_ξ.

Step I (two point set). Note that (2) implies $l_\xi \cap A_{<\xi}$ has at most two points.

If $|l_\xi \cap A_{<\xi}| = 2$, then set $A^{(1)}_\xi = \emptyset$.

Let us focus on $|l_\xi \cap A_{<\xi}| < 2$. Then $|l_\xi \cap l| \le 1$ for any $l \in \mathcal{L}$. Therefore $|l_\xi \setminus \bigcup \mathcal{L}| = \mathfrak{c}$. Choose

$$x^{(1)} \in l_\xi \setminus \operatorname{span}_{\mathbb{Q}}\left(A_{<\xi} \cup \bigcup\nolimits_{l\in\mathcal{L}} (l \cap l_\xi)\right),$$

$$y^{(1)} \in l_\xi \setminus \operatorname{span}_{\mathbb{Q}}\left(A_{<\xi} \cup \{x^{(1)}\} \cup \bigcup\nolimits_{l\in\mathcal{L}} (l \cap l_\xi)\right).$$

Set $A^{(1)}_\xi = \{x^{(1)}, y^{(1)}\}$ if $A_{<\xi} \cap l_\xi = \emptyset$ and set $A^{(1)}_\xi = \{x^{(1)}\}$ if $A_{<\xi} \cap l_\xi$ is a singleton.

Step II (complete $\mathbb{I}$-nonmeasurability). Let $\mathcal{L}'$ be the family of all lines which meet $A_{<\xi} \cup A^{(1)}_\xi$ in exactly two points. Then $|\mathcal{L}'| < \mathfrak{c}$ and $\mathcal{L} \subseteq \mathcal{L}'$. Since B_ξ is $\mathbb{I}$-positive Borel set, therefore we can find a line l such that $l \cap (A_{<\xi} \cup A^{(1)}_\xi) = \emptyset$ and $|l \cap B_\xi| = \mathfrak{c}$.

Choose

$$x^{(2)} \in (l \cap B_\xi) \setminus \operatorname{span}_{\mathbb{Q}}\left(A_{<\xi} \cup A^{(1)}_\xi \cup \bigcup\nolimits_{l\in\mathcal{L}'} (l \cap l_\xi)\right).$$

Set $A^{(2)}_\xi = \{x^{(2)}\}$.

Step III (Hamel base). Let us focus on the condition (6). If $h_\xi \in \operatorname{span}_{\mathbb{Q}}(A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi)$, then set $A^{(3)}_\xi = \emptyset$. Assume now that $h_\xi \notin \operatorname{span}_{\mathbb{Q}}(A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi)$. Let $\mathcal{L}''$ be the family of all lines which meet $A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi$ in exactly two points. Then $|\mathcal{L}''| < \mathfrak{c}$ and $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}''$. Choose the line l parallel to h_ξ, with $l \cap (A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi) = \emptyset$. Choose

$$x^{(3)} \in l \setminus \operatorname{span}_{\mathbb{Q}}\left(A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi \cup \{h_\xi\} \cup \bigcup\nolimits_{l\in\mathcal{L}''} (l \cap l_\xi)\right).$$

Set $y^{(3)} = x^{(3)} + h_\xi$. Then,

$$y^{(3)} \in l \setminus \operatorname{span}_{\mathbb{Q}}\left(A_{<\xi} \cup A^{(1)}_\xi \cup A^{(2)}_\xi \cup \bigcup\nolimits_{l\in\mathcal{L}''} (l \cap l_\xi)\right).$$

Set $A^{(3)}_\xi = \{x^{(3)}, y^{(3)}\}$.

Finally set $A_\xi = A^{(1)}_\xi \cup A^{(2)}_\xi \cup A^{(3)}_\xi$.

Clearly conditions (1)-(6) are satisfied. So, the inductive construction is finished.

The set $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$ will have desired property. Evidently, conditions (2) and (3) imply that the set A is a two point set. Since every $\mathbb{I}$-positive Borel set must have an uncountable section so the set A does not contain any set from $\{B_\xi : \xi < \mathfrak{c}\}$ and (4) makes sure it intersects all of them, so the set A is completely $\mathbb{I}$-nonmeasurable. Moreover, conditions (5) and (6) imply that A is a Hamel base of $\mathbb{R}^2$.

□

Considering $\mathbb{I} = \mathscr{N}$, we get the following corollary.

Corollary 2.3. *There exists a two point set $A \subseteq \mathbb{R}^2$, that is a Hamel base such that $\lambda_*(A) = \lambda_*(\mathbb{R}^2 \setminus A) = 0$, where λ_* denotes the inner Lebesgue measure on the plane.*

3. Marczewski null and Marczewski nonmeasurable set

In this section we will consider a σ-ideal s_0 and a σ-algebra s of subsets of $\mathbb{R}^2$ that were introduced by Marczewski (see e.g. [17], [6]).

Definition 3.1. We say that a set $A \subseteq \mathbb{R}$

(1) belongs to s_0 iff for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$.
(2) is s-measurable iff for every perfect set P there exists perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$ or $Q \subseteq A$.
(3) is s-nonmeasurable iff A is not s-measurable.

Definition 3.2. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Bernstein set* iff for every perfect set $P \subseteq \mathbb{R}^2$

$$A \cap P \neq \emptyset \wedge A^c \cap P \neq \emptyset.$$

Let us recall that every Bernstein set is s-nonmeasurable.

Let us start with the result connected with the σ-ideal s_0 of Marczewski null sets.

Theorem 3.3. *There exists a two point set $A \subseteq \mathbb{R}^2$ that belongs to s_0.*

Proof. Let $\{l_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$. Let $\{Q_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all perfect sets in $\mathbb{R}^2$ such that every perfect set occurs $\mathfrak{c}$ many times.

We will define, by induction on $\xi < \mathfrak{c}$ sequences $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and $\{P_\xi : \xi < \mathfrak{c}\}$ of perfect or empty sets such that

($\star$) for every perfect set Q there is $\xi_0 < \mathfrak{c}$ such that $P_{\xi_0} \neq \emptyset$ and $P_{\xi_0} \subseteq Q$;

and for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$,
(2) $\bigcup_{\zeta \le \xi} A_\zeta$ does not contain three collinear points,
(3) $\bigcup_{\zeta \le \xi} A_\zeta$ contains precisely two points of l_ξ,
(4) $P_\xi \subseteq Q_\xi$,
(5) $\bigcup_{\zeta \le \xi} P_\zeta \cap \bigcup_{\zeta \le \xi} A_\zeta = \emptyset$,
(6) $|l_\eta \setminus \bigcup_{\zeta \le \xi} P_\zeta| = \mathfrak{c}$ for every $\eta \ge \xi$.

Assume that for some $\xi < \mathfrak{c}$ sequences $\{A_\zeta : \zeta < \xi\}$ and $\{P_\zeta : \zeta < \xi\}$ are already constructed. Set $A_{<\xi} = \bigcup_{\zeta < \xi} A_\zeta$.

Assume first that for every line l in a plane, $|Q_\xi \cap l| < \mathfrak{c}$. Then $|Q_\xi \cap l| \le \omega$. Since $|A_{<\xi}| < \mathfrak{c}$ we can choose a perfect set $P_\xi \subseteq Q_\xi$ such that $P_\xi \cap A_{<\xi} = \emptyset$ and $|P_\xi \cap l| \le \omega$ for every line l. Since intersection of P_ξ with any line is at most countable then $|l_\eta \setminus \bigcup_{\zeta \le \xi} P_\zeta| = \mathfrak{c}$, for every $\eta \ge \xi$ and $\bigcup_{\zeta \le \xi} P_\zeta \cap \bigcup_{\zeta < \xi} A_\zeta = \emptyset$.

Assume now that there exist a line l such that $|l \cap Q_\xi| = \mathfrak{c}$. If $l = l_\alpha$ for some $\alpha \ge \xi$, then put $P_\xi = \emptyset$. If $l = l_\alpha$ for some $\alpha < \xi$, then $|l \cap A_{<\xi}| = 2$ and since $l \cap Q_\xi$ is closed with $|l \cap Q_\xi| = \mathfrak{c}$ one can choose a perfect set $P_\xi \subseteq Q_\xi \cap l$ disjoint with $A_{<\xi}$. Then $|l_\eta \setminus \bigcup_{\zeta \le \xi} P_\zeta| = \mathfrak{c}$ for every $\eta \ge \xi$ and $\bigcup_{\zeta \le \xi} P_\zeta \cap \bigcup_{\zeta < \xi} A_\zeta = \emptyset$.

As in Theorem 2.3 we can choose a set A_ξ satisfying (1) - (3) outside the set $\bigcup_{\zeta \le \xi} P_\zeta$ what finishes inductive construction.

Finally, there exist sequences $\{A_\xi : \xi < \mathfrak{c}\}$ and $\{P_\xi : \xi < \mathfrak{c}\}$, satisfying (1) - (6) and by the construction they fulfill the condition ($\star$).

Then, the set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ will have the desired property. □

Let us note here that the unit circle intersects any line in at most two points but it cannot be extended to a two point set. In [5] and [4] it was investigated how small should be a subset of the unit circle to be extendable to a two point set. It turns out that sets of inner positive measure on the unit circle cannot be extended to two point sets. We show that there is a subset of the unit circle of full outer measure which can be extended to a two point set.

Theorem 3.4. *There exists a two point set $A \subseteq \mathbb{R}^2$ that is s-nonmeasurable. Moreover, A contains a subset of the unit circle of full outer measure.*

Proof. Let us observe that if B is a Bernstein set in some uncountable closed set C then B is s-nonmeasurable. Moreover, if a set D is such that $D \cap C = B$ then D is also s-nonmeasurable.

We construct a two point set A such that its intersection with the unit circle is a Bernstein subset of the unit circle. Let $\{l_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in $\mathbb{R}^2$. Let $\{P_\xi : \ \xi < \mathfrak{c}\}$ be an enumeration of all perfect subsets of unit circle.

We will define inductively a sequence $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and a sequence $\{y_\xi : \xi < \mathfrak{c}\}$ of points from the unit circle such that for every $\xi < \mathfrak{c}$:

(1) $|A_\xi| < \omega$,
(2) $\bigcup_{\zeta \leq \xi} A_\zeta$ does not contain three collinear points,
(3) $\bigcup_{\zeta \leq \xi} A_\zeta$ contains precisely two points of l_ξ,
(4) $P_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset$,
(5) $y_\xi \in P_\xi$,
(6) $A_\xi \cap \{y_\zeta : \ \zeta \leq \xi\} = \emptyset$.

The existence of the sequence $\{A_\xi : \xi < \mathfrak{c}\}$ follows in the similar way as in Theorem 2.3. Here, the key observation is that for each perfect set P_ξ of unit circle there exist $\mathfrak{c}$ many straight lines passing through P_ξ and the origin.

Setting $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ we obtain a two point s-nonmeasurable set. Clearly, A is of full outer measure on the unit circle. □

Using the method from the previous section we can strengthen the results in the following way.

Theorem 3.5. *Let $\mathbb{I}$ a σ-ideal of subsets of $\mathbb{R}^2$ with the property that for every $\mathbb{I}$-positive Borel set there is $\mathfrak{c}$ many pairwise disjoint lines which intersect it on the set of cardinality $\mathfrak{c}$.*

(1) *There exists a two point set $A \subseteq \mathbb{R}^2$, that is completely $\mathbb{I}$-nonmeasurable, s_0 Hamel base.*
(2) *There exists a two point set $B \subseteq \mathbb{R}^2$, that is completely $\mathbb{I}$-nonmeasurable, s-nonmeasurable Hamel base.*

To prove it one should combine the ideas of Theorems 2.3, 3.3 and 3.4.

The first part of the above theorem generalize the result from [13].

4. A union of graphs of functions

In this section we will focus on the question whether a two point set can be decomposed into a union of two functions having some additional properties.

Let us start with a simple observation.

Proposition 4.1. *Every two point set is an union of two functions.*

Proof. Let A be a two point set. In particular it intersects every vertical line in exactly two points. For $x \in \mathbb{R}$ let us denote by $A^x = A \cap (\{x\} \times \mathbb{R})$. Clearly A^x has two elements, so $A^x = \{(x, y_1), (x, y_2)\}$. Define the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ as follows $f_1(x) = y_1, f_2(x) = y_2$. Then we get that $A = f_1 \cup f_2$. This finishes the proof. □

Let us introduce a notion which generalize in a natural way the notion of two point set.

Definition 4.2. Let κ be a cardinal number, $\kappa \geq 2$. We say that a subset of the plane is a κ *point set* iff it meets any line in exactly κ points.

Proposition 4.3. *Let $n \geq 2$ be a natural number. For any n point set A there is no additive function $f \subseteq A$.*

Proof. Let A be an n point set and suppose that there is an additive function $f \subseteq A$. Notice that $f(2) = f(1 + 1) = f(1) + f(1) = 2f(1)$ and, more generally for $k \geq 1$, $f(k) = kf(1)$. So points $(1, f(1)),\ (2, 2f(1)), \ldots, (n+1, (n+1)f(1))$ are members of A which lies on the same line. This leads to a contradiction. □

Now, let us focus on the class of bijections.

We will use the following theorem (see e.g. [1]).

Theorem 4.4 (Hall)**.** *Assume that X, Y are infinite sets. Let $R \subseteq X \times Y$ be a relation that fulfills the following property*

$$(\forall k \in \mathbb{N})(\forall X' \subseteq X)(|X'| = k \longrightarrow |R[X']| \geq k),$$

where $R[X'] = \{y :\ (\exists x \in X')((x, y) \in R)\}$. Then there exists an injection $h : X \to Y$ such that $h \subseteq R$.

We will also use the following theorem (see e.g. [6]).

Theorem 4.5 (Cantor, Bernstein)**.** *Let X, Y be any sets. Assume that $f : X \to Y$ and $g : Y \to X$ are injections. Then there exists $A \subseteq X$ and $B \subseteq Y$ such that $f \restriction A : A \to Y \setminus B$ and $g \restriction B : B \to X \setminus A$ are bijections.*

Theorem 4.6. *Fix a natural number n. Let $A \subseteq \mathbb{R}^2$ be such that its intersection with every horizontal and vertical line has exactly n elements. Then there exist n bijections $F_0, ..., F_{n-1} : \mathbb{R} \to \mathbb{R}$ such that $A = F_0 \cup ... \cup F_{n-1}$.*

Proof. Let us notice that $A \subseteq \mathbb{R} \times \mathbb{R}$ fulfills the assumptions of Theorem 4.4. So there exists an injection $f : \mathbb{R} \to \mathbb{R}$ such that $f \subseteq A$.

A set $A^{-1} = \{(x, y) :\ (y, x) \in A\}$ also fulfills the assumptions of Theorem 4.4. So there exists an injection $g : \mathbb{R} \to \mathbb{R}$ such that $g \subseteq A^{-1}$.

By Theorem 4.5 we can construct a bijection $F_0 : \mathbb{R} \to \mathbb{R}$ of the form $F_0 = (f \restriction A) \cup (g^{-1} \restriction (\mathbb{R} \setminus A))$. So, $F_0 \subseteq A$.

Let us notice that $A \setminus F_0$ is such that its intersection with every horizontal and vertical line has exactly $n - 1$ elements. So, the proof can be finished by a simple induction. □

We get the immediate corollary.

Corollary 4.7. *Let $n \geq 2$ be a natural number. Any n point set can be decomposed into n bijections.*

One can ask if any two point set can be decomposed into two measurable (with Baire property) functions. We will prove that this is not the case. Moreover, there is a two point set which does not admit a measurable (with Baire property) uniformization.

We will use the following, probably well-known, lemma. We give a short proof of it for reader's convenience.

Lemma 4.8. *There exists an unbounded F_σ set $C \subseteq \mathbb{R}_+$ of measure zero such that its intersection with any interval in $\mathbb{R}_+$ is of cardinality $\mathfrak{c}$. (In particular, C is meager.)*

Proof. Let $\mathbb{C}$ denote the standard ternary Cantor set. Let $\mathbb{Q}_+$ denote the set of positive rationals. Set

$$C = \mathbb{C} + \mathbb{Q}_+ = \{x + y : \ x \in \mathbb{C} \wedge y \in \mathbb{Q}_+\}.$$

This finishes the proof. □

Theorem 4.9. *For any Bernstein set $B \subseteq \mathbb{R}$ there exists a two point set $A \subseteq \mathbb{R}^2$ which is null and meager such that for any function $f \subseteq A$, $f^{-1}((0,1))$ is B.*

Proof. Let $B \subseteq \mathbb{R}$ be a Bernstein set and let $\{l_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$. Let $C^* = \{r \cdot e^{it} : t \in [0, 2\pi], r \in C\}$ where C is the set from Lemma 4.8. Notice that C^* is F_σ-set. By Fubini's Theorem and Ulam's Theorem the set C^* is meager and of measure zero in the plane $\mathbb{R}^2$. Notice that $|l_\xi \cap C^*| = \mathfrak{c}$ for any $\xi < \mathfrak{c}$. We will define, by induction on $\xi < \mathfrak{c}$, the sequence $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of C^* such that for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$;
(2) $\bigcup_{\zeta \le \xi} A_\zeta$ does not have three collinear points;
(3) $\bigcup_{\zeta \le \xi} A_\zeta$ contains precisely two points of l_ξ;
(4) If l_ξ is a vertical line with $x-$coordinate $x_\xi \in B$ then $\bigcup_{\zeta \le \xi} A_\zeta \cap l_\xi \subseteq \{x_\xi\} \times (0,1)$;
(5) If l_ξ is a horizontal line with $y-$coordinate $y_\xi \in (0,1)$ then $\bigcup_{\zeta \le \xi} A_\zeta \cap l_\xi \subseteq B \times \{y_\xi\}$;
(6) If neither (4) nor (5) then $\left(\bigcup_{\zeta \le \xi} A_\zeta \cap l_\xi\right) \cap (B \times (0,1)) = \emptyset$.

Assume that for some $\xi < \mathfrak{c}$ the sequence $\{A_\zeta : \zeta < \xi\}$ is already defined. Set $A_{<\xi} = \bigcup_{\zeta < \xi} A_\zeta$. Let $\mathcal{L}$ be the family of all lines which meet $A_{<\xi}$ in exactly two points. Then $|\mathcal{L}| \le |A^2_{<\xi}| < \mathfrak{c}$. Note that $L_\xi \cap A_{<\xi}$ has at most two elements. Consider three cases.

Case 1 (l_ξ is a vertical line with $x-$coordinate $x_\xi \in B$). If $|l_\xi \cap A_{<\xi}| = 2$ then put $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$, then $|l_\xi \cap l| \le 1$ for any $l \in \mathcal{L}$. Choose two numbers $y^1_\xi, y^2_\xi \in (0,1)$ such that $(x_\xi, y^1_\xi), (x_\xi, y^2_\xi) \in (C^* \cap l_\xi) \setminus \left(\bigcup_{l \in \mathcal{L}} l \cap l_\xi\right)$. It is possible since $|C^* \cap l_\xi| = \mathfrak{c}$ and $\left|\bigcup_{l \in \mathcal{L}} l \cap l_\xi\right| < \mathfrak{c}$. Set $A_\xi = \{(x_\xi, y^1_\xi), (x_\xi, y^2_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$;

Case 2 (l_ξ is a horizontal line with $y-$coordinate $y_\xi \in (0,1)$.) Since $l_\xi \cap C^*$ is uncountable F_σ, it contains a perfect set and $|\pi_1[l_\xi \cap C^*] \cap B| = \mathfrak{c}$. If $|l_\xi \cap A_{<\xi}| = 2$ then put $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$, then $|l_\xi \cap l| \le 1$ for any $l \in \mathcal{L}$ and choose an arbitrary two points $x^1_\xi, x^2_\xi \in B$ such that $(x^1_\xi, y_\xi), (x^2_\xi, y_\xi) \in (C^* \cap l_\xi) \setminus \left(\bigcup_{l \in \mathcal{L}} l \cap l_\xi\right)$. Set $A_\xi = \{(x^1_\xi, y_\xi), (x^2_\xi, y_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$;

Case 3 (otherwise). If $|l_\xi \cap A_{<\xi}| = 2$ then set $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$ then $|l_\xi \cap l| \leq 1$ for any $l \in \mathcal{L}$ and choose an arbitrary $(x^1_\xi, y^1_\xi), (x^2_\xi, y^2_\xi) \in (C^* \cap l_\xi) \setminus \left(\bigcup_{l \in \mathcal{L}} l \cap l_\xi\right)$ with $x^1_\xi, x^2_\xi \notin B$ and $y^1_\xi, y^2_\xi \notin (0,1)$. It is possible since $|\pi_1[l_\xi \cap C^*] \cap (\mathbb{R} \setminus B)| = \mathfrak{c}$. Set $A_\xi = \{(x^1_\xi, y^1_\xi), (x^2_\xi, y^2_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x^1_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$;

At the end set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$. Since $A \subseteq C^*$, it is meager and null. By (4)-(6) if $f \subseteq A$ then $f^{-1}((0,1)) = B$.

□

5. Luzin and Sierpiński set

We start this section with the definitions of special subsets of the real plane $\mathbb{R}^2$.

Definition 5.1. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Luzin set* iff intersection of the set A with every meager set is countable.

Moreover, a set $A \subseteq \mathbb{R}^2$ is a *strongly Luzin set* iff A is a Luzin set and the intersection of A with every Borel nonmeager set has cardinality $\mathfrak{c}$.

Definition 5.2. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Sierpiński set* iff intersection of the set A with every null set is countable.

Moreover, a set $A \subseteq \mathbb{R}^2$ is a *strongly Sierpinński set* iff A is a Sierpiński set and the intersection of A with every Borel of positive Lebesgue measure set has cardinality $\mathfrak{c}$.

The following remark holds.

Remark 5.3. Assume $A \subseteq \mathbb{R}^2$ is two point set. Then

(1) A is not Bernstein,
(2) A is not Luzin,
(3) A is not Sierpiński.

Proof. 1) Each line l is a perfect set such that $|A \cap l| = 2$, so A cannot be a Bernstein set.
2) Let M be a perfect meager subset of $\mathbb{R}$. Then $M \times \mathbb{R}$ is meager and

$$|(M \times \mathbb{R}) \cap A| = 2|M| = \mathfrak{c}.$$

So, A cannot be a Luzin set.
3) Let N be a perfect null subset of $\mathbb{R}$. Then $N \times \mathbb{R}$ is null and

$$|(N \times \mathbb{R}) \cap A| = 2|N| = \mathfrak{c}.$$

So, A cannot be a Sierpiński set. □

Let us give the following definition.

Definition 5.4. A set $A \subseteq \mathbb{R}^2$ is a *partial two point set* iff A intersects every line in at most two points.

Theorem 5.5. *(CH)*

(1) *There exists a partial two point set A that is a strong Luzin set.*
(2) *There exists a partial two point set B that is a strong Sierpiński set.*

Proof. Let us focus on the Luzin set. The case of the Sierpiński set is similar.

Fix a base $\{B_\alpha : \alpha < \omega_1\}$ of the ideal of meager sets and let $\{D_\alpha : \alpha < \omega_1\}$ be the enumeration of Borel nonmeager sets such that each set appears ω_1 many times. We will construct a sequence $\{x_\alpha : \alpha < \omega_1\}$ satisfying the following properties:

(1) $A_\alpha = \{x_\xi : \xi \leq \alpha\}$ does not contain three collinear points,
(2) $x_\alpha \in D_\alpha \setminus \bigcup_{\xi<\alpha} B_\xi$.

We will show that at any α step we can pick x_α such that (1) and (2) are fulfilled. Since A_ξ is countable so is $\bigcup_{\xi<\alpha} A_\xi$. Therefore the set

$$\mathcal{L}_{<\alpha} = \{l : l \text{ is a line and } |l \cup \bigcup_{\xi<\alpha} A_\xi| = 2\}$$

is countable. Hence, both $\mathcal{L}_{<\alpha}$ and $\bigcup_{\xi<\alpha} B_\xi$ are meager. Consequently, one can pick a point x_α from D_α that meets neither $\mathcal{L}_{<\alpha}$ nor $\bigcup_{\xi<\alpha} B_\xi$. So, the inductive construction in done.

Finally, set $A = \{x_\alpha : \alpha < \omega_1\}$. It is a required partial two point set that is strong Luzin. □

Let us remark that sets A and B constructed in Theorem 5.5 are s_0. Moreover A is strongly null and B is strongly meager. For the definitions of strongly meager and strongly null we refer the reader to [2].

Theorem 5.5 can be strengthen. If we assume that $\text{add}(\mathscr{M}) = \text{cof}(\mathscr{M}) = \kappa$ then we can construct partial two point set A such that $|A| = \kappa$ and for every Borel set B, $|B \cap A| < \kappa$ if and only if $B \in \mathscr{M}$.

The analogous observation is true in the case of null sets $\mathscr{N}$.

6. κ-covering

At the beginning of this section we will recall the notion of κ-covering and κ-I-covering (see [7]).

Definition 6.1. Let κ be a cardinal number. A set $A \subseteq \mathbb{R}^2$ is called a *κ-covering* iff

$$(\forall X \in [\mathbb{R}^2]^\kappa)(\exists y \in \mathbb{R}^2)\ \ y + X \subseteq A.$$

where $y + X$ denotes $\{y + x : x \in X\}$.

Let $\text{Iso}(\mathbb{R}^2)$ be the group of all isometries of the real plane $\mathbb{R}^2$.

Definition 6.2. Let κ be a cardinal number. A set $A \subseteq \mathbb{R}^2$ is called a *κ-I-covering* iff

$$(\forall X \in [\mathbb{R}^2]^\kappa)(\exists g \in \text{Iso}(\mathbb{R}^2))\ \ g[X] \subseteq A.$$

Obviously, if A is κ-covering then A is κ-I-covering and if $\kappa < \lambda$ then A is κ-covering (κ-I-covering) implies that A is λ-covering (λ-I-covering).

Let us start with the following result.

Theorem 6.3. *There exists a $\aleph_0$ point set which is not 2-I-covering.*

Proof. Let us enumerate the set of all lines $\text{Lines} = \{l_\xi : \xi < \mathfrak{c}\}$ in $\mathbb{R}^2$. We construct the transfinite sequence $(A_\xi : \xi < \mathfrak{c})$ of countable subsets of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$:

(1) $l \cap A_\xi = \emptyset$ for every $l \in \mathcal{L}_{<\xi}$,
(2) if $l_\xi \notin \mathcal{L}_{<\xi}$ then $|l_\xi \cap A_\xi| = \aleph_0$,
(3) $d(a, b) \neq 1$ for every $a, b \in \bigcup_{\zeta<\xi} A_\zeta$.

where $\mathcal{L}_{<\xi} = \{l \in \text{Lines} : |l \cap \bigcup_{\zeta<\xi} A_\zeta| = \aleph_0\}$ and $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ denotes the standard metric on $\mathbb{R}^2$.

Let us notice that $\mathcal{L}_{<\xi} \subseteq \{l \in \text{Lines} : |l \cap \bigcup_{\zeta<\xi} A_\zeta| \geq 2\}$. So, $|\mathcal{L}_{<\xi}| < \mathfrak{c}$ and The inductive construction can be done.

Now, setting $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$, we obtain the requested set. Indeed, (1) and (2) implies that A is an $\aleph_0$ point set and (3) guaranties that A is not 2-I-covering. □

Theorem 6.4. *There exists a $\aleph_0$ point set which is $\aleph_0$-covering.*

Proof. Let us enumerate the set of all lines $\text{Lines} = \{l_\xi : \xi < \mathfrak{c}\}$ and the family of all countable subsets of the real plane $[\mathbb{R}^2]^\omega = \{X_\xi : \xi < \mathfrak{c}\}$. We construct the transfinite sequence $((A_\xi, y_\xi) \in [\mathbb{R}^2]^\omega \times \mathbb{R}^2 : \ \xi < \mathfrak{c})$ with the following properties:

(1) $l \cap A_\xi = \emptyset$ for every $l \in \mathcal{L}_{<\xi}$,
(2) if $l_\xi \notin \mathcal{L}_{<\xi}$ then $|l_\xi \cap A_\xi| = \aleph_0$,
(3) $y_\xi + X_\xi \subseteq A_\xi$.

where $\mathcal{L}_{<\xi} = \{l \in \text{Lines} : \ |l \cap \bigcup_{\zeta<\xi} A_\zeta| = \aleph_0\}$.

Let us notice that

$$\{y : \ y + X_\xi \cap \bigcup \mathcal{L}_{<\xi} \neq \emptyset\} = \{y : \ \exists x \in X_\xi \ \exists l \in \mathcal{L}_{<\xi} \ y + x \in l\} = \bigcup_{l\in\mathcal{L}_{<\xi}} \bigcup_{x\in X_\xi} l - x.$$

The latter set, as a union of $< \mathfrak{c}$ many lines, does not cover the whole $\mathbb{R}^2$. Set y_ξ in such a way that $y_\xi \notin \bigcup_{l\in\mathcal{L}_{<\xi}} \bigcup_{x\in X_\xi} \ l - x$. The rest of inductive construction is similar as in Theorem 6.6.

The resulting set $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$ is an $\aleph_0$ point set by (1) and (2). y_ξ's constructed in (3) witness that A is $\aleph_0$-covering. □

Theorem 6.5. *It is relatively consistent with ZFC that $\aleph_1 < \mathfrak{c}$ and there exists a $\aleph_1$ point set which is also $\aleph_1$-covering.*

Proof. Let us consider V a model of ZFC such that $V \models \mathfrak{c} = 2^{\aleph_1} = \aleph_2$. Such a model can be obtained by adding ω_2 Cohen reals to the constructible universe L. The rest of the proof goes in the similar way as the proof of Theorem 6.4. □

We can obtain the following result.

Theorem 6.6. *Fix an integer $n \geq 2$.*

- *There exists an n point set which is not 2-I-covering.*
- *There exists an n point set which is n-covering.*

Proof. The proof of this theorem is similar to the proofs of the Theorem 6.3 and Theorem 6.4. □

Let us recall that A is 2-covering iff $A - A = \mathbb{R}^2$. This gives the following result.

Corollary 6.7. *There exists a two point set A such that $A - A = \mathbb{R}^2$.*

7. Combinatorial properties

Let us recall that a family $\mathcal{A}$ of infinite subsets of ω is an *almost disjoint family* (ad) iff any two distinct members of $\mathcal{A}$ has finite intersection. $\mathcal{A}$ is a *maximal almost disjoint family* (mad) iff it is ad family which is maximal with respect to inclusion.

Analogously, we say that $\mathcal{B} \subseteq \omega^\omega$ is a *family of eventually different functions* iff every two distinct members $x, y \in \mathcal{B}$ are equal only on a finite subset of ω.

Let κ be a cardinal number. We say that the family $\{A_\xi \in P(\omega) : \xi < \kappa\}$ is a *tower* iff

- $(\forall \xi, \eta < \kappa) \ \xi < \eta \longrightarrow A_\eta \subseteq^* A_\xi$ and
- there is no $B \in [\omega]^\omega$ $(\forall \xi < \kappa)$ $B \subseteq^* A_\xi$.

Here, $A \subseteq^* B$ means that $|A \setminus B| < \omega$.

Theorem 7.1 (CH). *Let $h : \mathbb{R} \to \omega^\omega$ be a bijection. There exist a partial two point set $A \subseteq \mathbb{R}^2$ such that a family $h[\pi_1[A] \cup \pi_2[A]]$ forms a maximal family of the eventually different functions. (π_i denotes the projection on i-th coordinate.)*

Proof. Let $\omega^\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction we will construct a set $A = \{a_\xi : \xi < \omega_1\} \subseteq \mathbb{R}^2$ such that for every $\alpha < \omega_1$

(1) $A_\alpha = \{a_\xi : \xi < \alpha\}$ is a partial two point set,
(2) $F_\alpha = h[\pi_1[A_\alpha] \cup \pi_2[A_\alpha]]$ is a family of eventually different functions,
(3) $(\exists \xi \leq \alpha)(\exists i \in \{0,1\})\ |f_\alpha \cap h(\pi_i(a_\xi))| = \aleph_0$.

Assume now that we have already constructed a set A_α.

Case 1. (f_α is eventually different from every function of the form $h(\pi_i(a_\xi))$ for $\xi < \alpha$ and $i \in \{0,1\}$) Set $x_\alpha = h^{-1}(f_\alpha)$. We can find $y_\alpha \in \mathbb{R}$ such that

- (x_α, y_α) does not belong to any line from $\mathcal{L}(A_\alpha)$,
- $h(y_\alpha)$ is eventually different from every function from $F_\alpha \cup \{f_\alpha\}$,

where $\mathcal{L}(A_\alpha)$ denotes the family of all lines intersecting A_α in exactly two points. A point y_α can be found since A_α is countable.

Case 2. ($|f_\alpha \cap h(\pi_i(a_\xi))| = \aleph_0$ for some $\xi < \alpha$ and $i \in \{0,1\}$) Then we can find $x_\alpha,\ y_\alpha \in \mathbb{R}$ such that

- (x_α, y_α) does not belong to any line from $\mathcal{L}(A_\alpha)$,
- $F_\alpha \cup \{h(x_\alpha), h(y_\alpha)\}$ is a family of eventually different functions.

Again, construction is possible since A_α is countable.

Set $a_\alpha = (x_\alpha, y_\alpha)$. The inductive step is proved.

Let us notice that the resulting set $A = \bigcup_{\alpha<\omega_1} A_\alpha$ is a partial two point set by (1). $h[\pi_1[A] \cup \pi_2[A]]$ is a family of eventually different functions by (2). The maximality of this family follows from (3). □

Remark 7.2. The same result is true if we replace a maximal family of eventually different functions by a mad family. (In this case we consider a bijection $h : \mathbb{R} \to [\omega]^\omega$.)

In the proof of next theorem we adopt the method from Kunen's theorem about the existence of indestructible mad family (see [8]).

Theorem 7.3. *Let us fix a standard Borel bijection $h : \mathbb{R} \to [\omega]^\omega$. It is consistent with $ZFC + \neg CH$ that there exists a partial two point set A such that $h[\pi_1[A] \cup \pi_2[A]]$ forms a mad family of size ω_1.*

Proof. Let us consider a model V' obtained from $V \models CH$ by adding $\kappa > \omega_1$ Cohen reals (i.e. using forcing $\mathrm{Fn}(\kappa, 2)$). It suffices to construct a partial two point set A in V which remains maximal in the generic extension V'.

Let us notice that, since every new uncountable subset of ω has a name in $\mathrm{Fn}(I, 2)$ for some countable $I \subseteq \kappa$, it is enough to consider names in $\mathrm{Fn}(\omega, 2)$.

In V, let us enumerate all possible pairs $(p_\xi, \tau_\xi) : \omega \leq \xi < \omega_1$ (by CH), where $p_\xi \in \mathrm{Fn}(\omega, 2)$ and τ_ξ is a nice name for an infinite subset of ω. Take any countable sequence $(F^i_n : n \in \omega \wedge i \in \{0,1\})$ of pairwise disjoint countable subsets of ω.

Now we define a transfinite sequence $(F^i_\xi : \omega \leq \xi < \omega_1 \wedge i \in \{0,1\})$ satisfying the following conditions for every $\xi < \omega_1$:

(1) $(F^i_\zeta : \zeta < \xi \wedge i \in \{0,1\})$ is an almost disjoint family,

(2) if $(\forall \eta < \xi)(\forall i \in 2) p_\xi \Vdash |\tau_\xi \cap F^i_\eta| < \omega$
then $p_\xi \Vdash |\tau_\xi \cap F^0_\xi| = \omega$ or $p_\xi \Vdash |\tau_\xi \cap F^1_\xi| = \omega$,

(3) $\{a_\zeta = (h^{-1}[\{F^0_\zeta\}], h^{-1}[\{F^1_\zeta\}]) : \zeta < \xi\}$ forms a partial two point set.

To see that this recursion is possible let us assume that the construction at the step $\xi < \omega_1$ is done. Now let us enumerate $\{F^i_\eta : \eta < \xi \wedge i \in 2\} = \{B_n : n \in \omega\}$ by ω. If the assumptions in condition (2) is not fulfilled then choose any F^1_ξ almost disjoint with every F^i_η for $\eta < \xi$ and $i \in 2$ what is possible since $|\xi| = \omega$. Now, let us assume that the assumption of (2) is fulfilled. We show that

$$(\star\star) \qquad (\forall n \in \omega)(\forall q \le p_\xi)(\exists m > n)(\exists r < q) \ \ r \Vdash m \in \tau_\xi \setminus (B_0 \cup \dots B_n).$$

Let us fix any $n \in \omega$ and $q < p_\xi$. By assumption $p_\xi \Vdash |\tau_\xi \cap (B_0 \cup \dots B_n)| < \omega$. So

$$p_\xi \Vdash (\exists m > n) \ \ m \in \tau \setminus (B_0 \cup \ldots \cup B_n).$$

q is stronger than p_ξ, so it forces the same sentence. Now, we can find a stronger condition $r < q$ and positive integer $m > n$ such that

$$r \Vdash m \in \tau \setminus (B_0 \cup \dots B_n).$$

This finishes the proof of $(\star\star)$.

Now let us enumerate the set $\omega \times \{q \in \mathrm{Fn}(\omega, 2) : \ \ q \le p_\xi\} = \{(n_j, q_j) : j < \omega\}$. Then for every $j < \omega$ there exist $m_j \in \omega$ and $r_j < q_j$ such that $n_j < m_j$ and

$$r_j \Vdash m_j \in \tau_\xi \setminus (B_0 \cup \dots B_{n_j}).$$

Let $F^1_\xi = \{m_j : \ j < \omega\}$. Then $F^i_\eta \cap F^1_\xi$ is finite, so $y_\xi = h^{-1}[\{F^1_\xi\}]$ is a real different from the other coordinates appeared in previous step construction.

Now we will construct the first coordinate of the new point. To do this, set $A_{<\xi} = \{(h^{-1}(F^0_\eta), h^{-1}(F^1_\eta)) : \eta < \xi\} \subset \mathbb{R}^2$. Denote by $\mathcal{L}_{<\xi}$ the set of all lines $l \subseteq \mathbb{R}^2$ on the real plane such that $|l \cap A_{<\xi}| = 2$. Let observe that the set

$$Y = \{z \in \mathbb{R} : \ \ (\exists l \in \mathcal{L}_{<\xi})(z, y_\xi) \in l\}.$$

is countable. Let us enumerate $Y = \{z_n : n < \omega\}$. Now, consider the following sequence $C_n = h(z_n), \ n \in \omega$.

To define the set F^0_ξ we will use the diagonal argument. Let us arrange elements of each set $C_n = \{c^n_i : i \in \omega\}$ in increasing sequence and let us define the increasing sequence $(d_n)_{n\in\omega}$ of nonnegative integers:

$$d_n = \max\{c^n_i : \ i \le n\}.$$

Now, let us choose an increasing sequence $(m_n)_{n\in\omega}$ such that for every $n \in \omega$ we have

- $d_n < m_n$ and
- $m_n \in \omega \setminus F^1_\xi \cup B_0 \cup \ldots \cup B_n$.

Set $F^0_\xi = \{m_n : \ n \in \omega\}$. It is easy to see that

(1) $F^0_\xi \ne C_n$ for every $n \in \omega$,

(2) $|F^0_\xi \cap B_n| < \omega$ for every $n \in \omega$,

(3) $|F^0_\xi \cap F^1_\xi| < \omega$.

The first property ensures that the set $A_{<\xi} \cup \{(h^{-1}(F^0_\xi), h^{-1}(F^1_\xi))\}$ doesn't contain three collinear points. The second and third properties implies that the set $\{F^i_\eta : \eta \le \xi \wedge i \in 2\}$ forms almost disjoint family.

Our construction of the sequence $(F^0_\xi : \xi < \omega)$ and $(F^1_\xi : \xi < \omega_1)$ finished. It remains to prove that

$$\Vdash_{\mathrm{Fn}(\omega,2)} \{F^0_\xi : \xi < \omega_1\} \cup \{F^1_\xi : \xi < \omega_1\} \text{ is mad family.}$$

If not then there exists condition $p \in \mathrm{Fn}(\omega, 2)$ and nice name $\tau \in V^{\mathrm{Fn}(\omega,2)}$ for element of $P(\omega)$ such that

$$p \Vdash (\forall \xi < \omega_1)(\forall (i \in 2))\ |\tau \cap F^i_\xi| < \omega.$$

There exists $\xi < \omega_1$ such that $(p, \tau) = (p_\xi, \tau_\xi)$. So, the assumptions in the condition (2) is fulfilled. We know that τ witness that there exists $q < p$ and $n \in \omega$ such that

$$q \Vdash \tau \cap F^i_\xi \subset n.$$

From the other hand, there exists $r < q$ and $m > n$ such that $r \Vdash m \in \tau \cap F^0_\xi$ or there exists $r' < q$ and $m' > n$ such that $r' \Vdash m' \in \tau \cap F^1_\xi$, a contradiction. □

Theorem 7.4. *Let us fix a standard Borel bijection $h : \mathbb{R} \to [\omega]^\omega$. It is consistent with $ZFC+\neg CH$ that there exists a partial two point set A such that $h[\pi_1[A] \cup \pi_2[A]]$ forms a tower of size ω_1.*

We will omit the proof because it is very similar to the proof of Theorem 7.3.

Theorem 7.5. *It is consistent with $ZFC + \neg CH$ that there exists a partial two point set $C \subseteq \mathbb{R}^2$ of size ω_2 such that C is a Luzin set and*

$$(\exists A \in \mathscr{N})(\forall D \in [C]^{\omega_1})\ A + D = \mathbb{R}^2.$$

Proof. Let us start with $V \vDash CH$. Consider the generic extension $V[c_\alpha : \alpha < \omega_2]$ obtained by adding ω_2 independent Cohen reals. We can assume that $c_\alpha \in \mathbb{R}^2$ for every $\alpha < \omega_2$. Set $C = \{c_\alpha : \alpha < \omega_2\}$.

C is a partial two point set. Indeed, take any line l which intersects two different points of C: c_α, c_β. Take any $\gamma \in \omega_2 \setminus \{\alpha, \beta\}$. c_γ is a Cohen real over $V[c_\alpha, c_\beta]$ and l is a meager set coded in $V[c_\alpha, c_\beta]$. So, $c_\gamma \notin l$.

C is a Luzin set. Take any Borel meager set M from $V[c_\alpha : \alpha < \omega_2]$. M is coded in $V[c_\alpha : \alpha \in I]$ for some countable I. So, $M \cap \{c_\alpha : \alpha \in \omega_2 \setminus I\} = \emptyset$.

Now, let us fix the Marczewski decomposition: $\mathbb{R}^2 = A \cup B$, where $A \in \mathscr{N}$, $B \in \mathscr{M}$ and $A \cap B = \emptyset$. Let us recall that A, B are coded in V. Take any $D \subseteq C$ of size ω_1. Take any $x \in \mathbb{R}^2$ (in $V[c_\alpha : \alpha < \omega_2]$). x is in $V[c_\alpha : \alpha \in J]$ for some countable J. So, $x - B$ is a meager set coded in $V[c_\alpha : \alpha \in J]$. Take $c \in D \setminus \{c_\alpha : \alpha \in J\}$. Then $c \notin x - B$. So, $x \in A + c$. This shows that $\mathbb{R}^2 \subseteq A + D$. □

In a similar way one can show the following result.

Theorem 7.6. *It is consistent with $ZFC + \neg CH$ that there exists a partial two point set $R \subseteq \mathbb{R}^2$ of size ω_2 such that R is a Sierpiński set and*

$$(\exists B \in \mathscr{M})(\forall D \in [R]^{\omega_1})\ B + D = \mathbb{R}^2.$$

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