ON ALGEBRABILITY OF NONABSOLUTELY CONVERGENT SERIES

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ABSTRACT. We prove that the set of all complex series which are nonabsolutely convergent is $\mathfrak{c}$-algebrable. We establish a similar result for the set of all divergent complex series with bounded partial sums.

1. INTRODUCTION

In [APS] the authors presented the following construction. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be a family of almost disjoint subsets of $\mathbb{N}$, say $A_\alpha = \{k_1^\alpha < k_2^\alpha < ...\}$, $\alpha < \mathfrak{c}$. For a fixed sequence of real or complex numbers $(a_n)_{n\in\mathbb{N}}$ and $\alpha < \mathfrak{c}$, we define a new sequence $(b_n^\alpha)_{n\in\mathbb{N}}$ by formulas: $b_{k_n^\alpha}^\alpha = a_n$ for $n \in \mathbb{N}$, and $b_k^\alpha = 0$ for $k \notin A_\alpha$. Then we say that $(a_n)_{n\in\mathbb{N}}$ is inscribed in every set A_α, $\alpha < \mathfrak{c}$. For a fixed nonabsolutely convergent series (divergent series with bounded partial sums) $\sum_{n=1}^\infty a_n$, the set $\{\sum_{n=1}^\infty b_n^\alpha : \alpha < \mathfrak{c}\}$ is linearly independent. Hence the set $E = \text{span}(\{\sum_{n=1}^\infty b_n^\alpha : \alpha < \mathfrak{c}\})$ forms a $\mathfrak{c}$-dimentional linear space. Let c_{00} denote the linear space of series $\sum_{n=1}^\infty x_n$ with finite sets of non-zero terms. The authors of [APS] claim that $\text{span}(E \cup c_{00})$ generates an algebra which contains only two types of series: nonabsolutely convergent series (divergent series with bounded partial sums) and series from c_{00}.

However, to obtain such an algebra, we cannot start from an arbitrary series $\sum_{n=1}^\infty a_n$. The necessary and sufficient condition for $\sum_{n=1}^\infty a_n$ to be a generator of appropriate algebra is the following:
($\star$) For any positive $k_1, ..., k_m \in \mathbb{N}$ and any numbers $\beta_1, ..., \beta_m$ which do not vanish simultaneously, the series $\sum_{n=1}^\infty (\beta_1 a_n^{k_1} + ... + \beta_m a_n^{k_m})$ is nonabsolutely convergent (divergent with bounded partial sums).

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Observe that $\sum_{n=1}^{\infty} a_n$ cannot be series of reals. Our aim is to complete the cited result of [APS] (in Theorem 1) and to obtain some strengthening (in Theorem 3).

Theorem 1. *Let $x \in \mathbb{C}$, $|x| = 1$ and x is not a root of unity (that is $x^n \neq 1$ for every $n \geq 2$). Consider the following cases:*

(i) *the series $\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$ is inscribed in every set A_α, $\alpha < \mathfrak{c}$,*
(ii) *the series $\sum_{n=1}^{\infty} x^n$ is inscribed in every set A_α, $\alpha < \mathfrak{c}$.*

In each of these two cases, let A denote the algebra generated by a given series and all series from c_{00}. Then each element of A is

(1) *either a nonabsolutely convergent series or a series from c_{00}, in case (i),*
(2) *either a divergent series with bounded partial sums or a series from c_{00}, in case (ii).*

The condition $(\star)$ for the series (i) and (ii) will be proved in the next section as a special case of more involved reasoning.

2. ALGEBRABILITY

An algebra is called κ–generated if it has a system of generators of cardinality κ. We say that a subset E of an algebra is κ–algebrable if there is a κ–generated subalgebra A such that $A \subset E \setminus \{0\}$ and A is not τ–generated by any cardinal $\tau < \kappa$. (See [GPS], [AS], [APGD]).

The following lemma is probably known but we present and prove it for the sake of completeness.

Lemma 2. *Let X be a linear algebra. Suppose that A is a subalgebra of X which is a $\mathfrak{c}$–dimensional vector space. Then A is not τ–generated for any $\tau < \mathfrak{c}$.*

Proof. Let $G = \{g_\alpha : \alpha < \kappa\}$ be a set which generates an algebra A. Each element of A is of the form $\sum_{k=1}^{n} c_k g_{\alpha_{1,k}} \cdot g_{\alpha_{2,k}} \cdots g_{\alpha_{m_k,k}}$ where $(g_{\alpha_{1,1}}, ..., g_{\alpha_{m_n,n}})$ is a finite sequence of elements of G, and c_k are arbitrary scalars. Hence A as a vector space can be generated by elements of the form $g_{\alpha_1} \cdot g_{\alpha_2} \cdots g_{\alpha_m}$. Since there are at most $|G^{<\omega}| = |\kappa^{<\omega}| = \kappa$ such elements, we have $\mathfrak{c} = \dim A \leq \kappa$. Therefore $\kappa = \mathfrak{c}$. □

Now, we are ready to establish our main result which is a strengthening of Theorem 1

Theorem 3. *The set of nonabsolutely convergent complex series and the set of divergent complex series with bounded partial sums are $\mathfrak{c}$-algebrable.*

Proof. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be complex numbers with $|x_\alpha| = 1$ such that the set $\{\mathrm{Arg}(x_\alpha) : \alpha < \mathfrak{c}\} \cup \{\pi\}$ is a linearly independent subset of reals. Let A_1 be a linear algebra generated by the set $\{\sum_{n=1}^\infty x_\alpha^n : \alpha < \mathfrak{c}\}$. We show that A_1 gives the $\mathfrak{c}$–algebrability of the set of divergent series with bounded partial sums. To prove it we need to show that, for any $x_{\alpha_1}, ..., x_{\alpha_j}$, for every matrix $(k_{il} : i \le m, l \le j)$ of naturals with non-zero and distinct rows, and any $\beta_1, ..., \beta_m \in \mathbb{C}$ which do not vanish simultaneously, the series

$$\sum_{n=1}^{\infty} \left(\beta_1 \left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}} \right)^n + ... + \beta_m \left(x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}} \right)^n \right) \tag{1}$$

is divergent with bounded partial sums. Note that each number $x_{\alpha_1}^{k_{i1}} \cdots x_{\alpha_j}^{k_{ij}}$ is in the unite circle and it is not a root of unity. Hence each $\sum_{n=1}^\infty (x_{\alpha_1}^{k_{i1}} \cdots x_{\alpha_j}^{k_{ij}})^n$ is a geometric series with bounded partial sums.

Suppose that the series (1) is convergent. Then the sequence of its partial sums

$$S_n = \beta_1 \frac{x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}} \left(1 - \left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n\right)}{1 - x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}} + ... + \beta_m \frac{x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}} \left(1 - \left(x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}\right)^n\right)}{1 - x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}}$$

$$= \gamma_1 \left(1 - \left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n\right) + ... + \gamma_m \left(1 - \left(x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}\right)^n\right)$$

tends to some complex number if $n \to \infty$. Hence the sequence

$$\gamma_1 \left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n + ... + \gamma_m \left(x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}\right)^n$$

also tends to some S if $n \to \infty$. Using the multidimensional Kroncker Lemma (see [HW, Theorem 442]) we infer that for any y with $|y| = 1$ there is a subsequence $(n_l)_{l \in \mathbb{N}}$ of naturals with $x_{\alpha_1}^{n_l} \to y, x_{\alpha_2}^{n_l} \to y^p, ..., x_{\alpha_j}^{n_l} \to y^{p^{j-1}}$, where $p = \max\{k_{il} + 1 : i \le m, l \le j\}$. Then

$$\gamma_1 y^{k_{11} + k_{12}p + ... + k_{1j}p^{j-1}} + ... + \gamma_m y^{k_{m1} + k_{m2}p + ... + k_{mj}p^{j-1}} = S.$$

The numbers $k_{11} + k_{12}p + ... + k_{1j}p^{j-1}, ..., k_{m1} + k_{m2}p + ... + k_{mj}p^{j-1}$ are distinct, since they have distinct expansions with respect to powers of p.

Hence

$$P(y) = \gamma_1 y^{k_{11}+k_{12}p+...+k_{1j}p^{j-1}} + ... + \gamma_m y^{k_{m1}+k_{m2}p+...+k_{mj}p^{j-1}}$$

is a non-constant polynomial. On the other hand, P is constant on the unit circle $\{y \in \mathbb{C} : |y| = 1\}$. A contradiction.

Note that in particular we have that the set $\{\sum_{n=1}^{\infty} x_\alpha^n : \alpha < \mathfrak{c}\}$ is linearly independent. Hence by Lemma 2, algebra A_1 witnesses that the set of complex divergent series with bounded partial sums is $\mathfrak{c}$-algerable.

Now, we will prove $\mathfrak{c}$-algebrability of the set of all nonabsolutely convergent series of complex numbers. Let $\{r_\alpha : \alpha < \mathfrak{c}\}$ be a linearly independent subset of positive reals. Let A_2 be a linear algebra generated by the set

$$\left\{ \sum_{n=1}^{\infty} \frac{x_\alpha^n}{\ln^{r_\alpha}(n+1)} : \alpha < \mathfrak{c} \right\}.$$

We will show that A_2 gives the $\mathfrak{c}$-algebrability of the set of all nonabsolutely convergent series of complex numbers. To prove it we need to show that a series

(2)

$$\sum_{n=1}^{\infty} \left(\beta_1 \frac{\left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n}{\ln^{k_{11}r_{\alpha_1}+...+k_{1j}r_{\alpha_j}}(n+1)} + ... + \beta_m \frac{\left(x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}\right)^n}{\ln^{k_{m1}r_{\alpha_1}+...+k_{mj}r_{\alpha_j}}(n+1)} \right)$$

is nonabsolutely convergent. The boundedness of partial sums of a series

$$\sum_{n=1}^{\infty} \left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n$$

has been observed in the first part of the proof. Note that the sequence

$$\left(\frac{1}{\ln^{k_{11}r_{\alpha_1}+...+k_{1j}r_{\alpha_j}}(n+1)} \right)_{n\in\mathbb{N}}$$

tends to zero. Hence by Dirichlet's test we obtain the convergence of

$$\sum_{n=1}^{\infty} \frac{\left(x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}}\right)^n}{\ln^{k_{11}r_{\alpha_1}+...+k_{1j}r_{\alpha_j}}(n+1)}.$$

Now, we need to prove that the series (2) is not absolutely convergent. Since the set $\{r_\alpha : \alpha < \mathfrak{c}\}$ is linearly independent, the numbers $k_{11}r_{\alpha_1} + ... + k_{1j}r_{\alpha_j}$,..., $k_{m1}r_{\alpha_1} + ... + k_{mj}r_{\alpha_j}$ are distinct. To simplify the notation put

$x_i := x_{\alpha_1}^{k_{i1}} \cdots x_{\alpha_j}^{k_{ij}}$ and $k_i := k_{i1} r_{\alpha_1} + ... + k_{ij} r_{\alpha_j}$ for every $i = 1, ..., m$. We may assume that $k_1 < k_2 < ... < k_m$ and $\beta_1 \neq 0$. Then

$$\left|\beta_1 \frac{x_1^n}{\ln^{k_1}(n+1)} + \beta_2 \frac{x_2^n}{\ln^{k_2}(n+1)} + ... + \beta_m \frac{x_m^n}{\ln^{k_m}(n+1)}\right| \geq$$

$$\geq \frac{|\beta_1|}{\ln^{k_1}(n+1)} - \frac{|\beta_2|}{\ln^{k_2}(n+1)} - ... - \frac{|\beta_m|}{\ln^{k_m}(n+1)}.$$

Since k_1 is smaller than each $k_2, ..., k_m$, there is N such that

$$\frac{|\beta_2|}{\ln^{k_2}(n+1)} + ... + \frac{|\beta_m|}{\ln^{k_m}(n+1)} < \frac{|\beta_1|}{2\ln^{k_1}(n+1)}$$

for all $n \geq N$. Hence

$$\left|\beta_1 \frac{x_1^n}{\ln^{k_1}(n+1)} + \beta_2 \frac{x_2^n}{\ln^{k_2}(n+1)} + ... + \beta_m \frac{x_m^n}{\ln^{k_m}(n+1)}\right| \geq \frac{|\beta_1|}{2\ln^{k_1}(n+1)}$$

for all $n \geq N$. This shows that the series (2) is not absolutely convergent.

Note that in particular we obtain that the set $\{\sum_{n=1}^{\infty} \frac{x_\alpha^n}{\ln^{r_\alpha}(n+1)} : \alpha < \mathfrak{c}\}$ is linearly independent. Finally using Lemma 2 we obtain the assertion. □

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