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ACHIEVEMENT SETS OF CONDITIONALLY CONVERGENT SERIES

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Abstract. Considering the sets of subsums of series (or achievement sets) we show that for conditionally convergent series the multidimensional case is much more complicated than that of the real line. Although we are far from the full topological classification of such sets, we present many surprising examples and capture the ideas standing behind them in general theorems.

1. Introduction. S. Kakeya [Ka] was probably the first one to consider topological properties of subsums of absolutely convergent series of real numbers. For an absolutely summable sequence (x_n), we call the set

$$\mathrm{A}(x_n) = \Big\{\sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}}\Big\}$$

the *set of subsums* or the *achievement set* of (x_n) (or of the series $\sum x_n$) [J]. Of course, for (x_n) with almost all terms equal to zero, the set $\mathrm{A}(x_n)$ is finite. Kakeya has shown:

THEOREM 1.1. *For an absolutely summable sequence (x_n) with infinitely many nonzero terms:*

- *$\mathrm{A}(x_n)$ is a compact perfect set.*
- *If $|x_n| > \sum_{k>n} |x_k|$ for almost all n then $\mathrm{A}(x_n)$ is homeomorphic to the Cantor set (after M. Morán we call such sequences* quickly convergent*).*
- *If $|x_n| \le \sum_{k>n} |x_k|$ for almost all n then $\mathrm{A}(x_n)$ is a finite union of closed intervals. Moreover the implication can be reversed for nonincreasing sequences $(|x_n|)$.*

Kakeya conjectured that Cantor-like sets and finite unions of closed intervals are the only possible achievement sets for sequences $(x_n) \in \ell_1 \setminus c_{00}$.

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Kakeya's results were rediscovered many times and his conjecture was repeated, even after the first counterexamples were given. The first counterexamples were published by Vaĭnshteĭn and Shapiro [VS], Ferens [F] and Guthrie and Nymann [GN]. Thanks to Guthrie, Nymann and Saenz [GN, NS] we know that the achievement set of an absolutely summable sequence can be a finite set, a finite union of intervals, homeomorphic to the Cantor set, or a so called Cantorval. A *Cantorval* is a set homeomorphic to the union of the Cantor set and sets which are removed from the unit segment at even steps of the Cantor set construction. That gives a partition of ℓ_1 into four disjoint sets. Topological and algebraic properties of these sets were recently considered in [BBGS1, BBGS2]. Some sufficient conditions for a given sequence to be a Cantorval were recently described in [BBFS, BFS, J]. The connections between achievement sets of some absolutely summable sequences and self-similar sets were observed in [J].

If $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series in a Banach space, then the function $\{0,1\}^{\mathbb{N}} \ni (\varepsilon_n) \mapsto \sum_{n=1}^{\infty} \varepsilon_n x_n$ maps continuously the Cantor space $\{0,1\}^{\mathbb{N}}$ onto $\mathrm{A}(x_n)$ (see for example [BG]). In particular, $\mathrm{A}(x_n)$ is compact. We will prove that this function is also continuous if the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent. This will follow immediately from Lemma 1.2 below. It is well-known [KK, Theorem 1.3.2, p. 10] that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space is unconditionally convergent if and only if each series of the form $\sum_{n \in A} x_n$, $A \subseteq \mathbb{N}$, is convergent.

LEMMA 1.2. *Assume that $\sum_{n=1}^{\infty} x_n$ is an unconditionally convergent series in a Banach space X. Then for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every set $A \subseteq \mathbb{N}$,*

$$\Big\| \sum_{k \in A \setminus \{1,\dots,N\}} x_k \Big\| \le \varepsilon.$$

Proof. Suppose to the contrary that there is $\varepsilon > 0$ such that for every N there is $A \subseteq \{N+1, N+2, \dots\}$ with

$$\Big\| \sum_{k \in A} x_k \Big\| > \varepsilon.$$

For $N = 1$ find A_1 with $\| \sum_{k \in A_1} x_k \| > \varepsilon$. There is a finite set $F_1 \subseteq A_1$ with $\| \sum_{k \in F_1} x_k \| > \varepsilon$. In the second step for $N = \max F_1$ find $A_2 \subseteq \{N+1, N+2, \dots\}$ with $\| \sum_{k \in A_2} x_k \| > \varepsilon$. As before we take a finite set $F_2 \subseteq A_2$ with $\| \sum_{k \in F_2} x_k \| > \varepsilon$. Proceeding inductively, we produce finite sets $F_1, F_2, \dots$ such that $\max F_i < \min F_{i+1}$ and $\| \sum_{k \in F_i} x_k \| > \varepsilon$. Set $A = \bigcup_{i \ge 1} F_i$. Then by the Cauchy condition the series $\sum_{n \in A} x_n$ diverges, which is a contradiction. ∎

One can also define the achievement sets for all sequences in Banach spaces. Then one should only consider those sequences $(\varepsilon_n) \in \{0,1\}^{\mathbb{N}}$ for which $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. We say that a series $\sum_{n=1}^{\infty} x_n$ is *potentially conditionally convergent* if it has a conditionally convergent rearrangement $\sum_{n=1}^{\infty} x_{\sigma(n)}$. On the real line, this means that both the series of positive and that of negative terms are divergent. Note that the difference between potentially conditionally convergent series and conditionally convergent series is slight: a potentially conditionally convergent series can be divergent. However, this notion allows us to formulate the following characterization.

THEOREM 1.3. *For sequences of reals with* $\lim_{n\to\infty} x_n = 0$*:*

- $\sum_{n=1}^{\infty} x_n$ *is potentially conditionally convergent if and only if* $\mathrm{A}(x_n) = \mathbb{R}$.
- *The subseries of negative terms is convergent and the subseries of positive terms is divergent (or vice versa) if and only if* $\mathrm{A}(x_n)$ *is a half-line.*

For simple proofs see for example [BFPW, J, N1]. If (x_n) does not converge to zero, then $\mathrm{A}(x_n)$ is always an F_σ-set [J]. So, for conditionally convergent series of reals the achievement set $\mathrm{A}(x_n)$ is $\mathbb{R}$, exactly as the *sum range* $\mathrm{SR}(x_n)$, the set of all rearrangements $\sum_{n=1}^{\infty} x_{\sigma(n)}$, by the classical Riemann Theorem on permutations of conditionally convergent series.

The Riemann Theorem can be generalized to finite-dimensional spaces. Before we state it precisely let us consider a couple of examples. Note that $\mathrm{SR}((-1)^n/n, (-1)^n/n) = \{(x,x) : x \in \mathbb{R}\}$, $\mathrm{SR}((-1)^n/n, 0) = \mathbb{R} \times \{0\}$ and $\mathrm{SR}((-1)^n/n, (-1)^n/\sqrt{n}) = \mathbb{R}^2$ (the latter is not obvious). This shows that a straightforward generalization of the Riemann Theorem is not true. However, we have the following theorem, the proof of which can be found in [KK].

THEOREM 1.4 (Steinitz). *Let* $\sum_{n=1}^{\infty} x_n$ *be a conditionally convergent series in* $\mathbb{R}^m$. *Then the set* $\mathrm{SR}(x_n) = \{\sum_{n=1}^{\infty} x_{\sigma(n)} : \sigma \in S_\infty\}$ *of all convergent rearrangements of* $\sum_{n=1}^{\infty} x_n$ *is an affine subspace of* $\mathbb{R}^m$. *More precisely, if* $\Gamma = \{f \in (\mathbb{R}^m)^* : \sum_{n=1}^{\infty} |f(x_n)| < \infty\}$, *and* $\Gamma^\perp = \{x \in \mathbb{R}^m : f(x) = 0$ *for all* $f \in \Gamma\}$ *is the annihilator of* Γ, *then*

$$\mathrm{SR}(x_n) = \sum_{n=1}^{\infty} x_n + \Gamma^\perp.$$

The aim of the present paper is to show that in multidimensional spaces $\mathrm{A}(x_n)$ can essentially differ from $\mathrm{SR}(x_n)$. We observe e.g. that for the achievement sets $\mathrm{A}(x_n)$ of conditionally convergent series in $\mathbb{R}^2$ the following situations are possible:

- $\mathrm{A}(x_n) \cap \mathrm{SR}(x_n)$ can be a singleton (Example 3.3), and it is always nonempty;
- $\mathrm{A}(x_n)$ can be the graph of a function (Example 3.10);
- $\mathrm{A}(x_n)$ can be a dense set in $\mathbb{R}^2$ with empty interior (Example 3.13);
- $\mathrm{A}(x_n)$ can be neither an F_σ nor a G_δ-set (Theorem 3.9);
- $\mathrm{A}(x_n)$ can be an open set $\neq \mathbb{R}^2$ (Theorem 4.2);

On the other hand, to obtain regular achievement sets let us make a simple observation. Let X be a Banach space. Suppose $A = \mathrm{A}(x_n)$ and $B = \mathrm{A}(y_n)$ are achievement sets in X. Then

$$A \times B = \mathrm{A}((x_1, 0), (0, y_1), (x_2, 0), (0, y_2), \dots).$$

If $T : X \to Y$ is a bounded linear operator from X to some other Banach space Y, then

$$T(A) = \mathrm{A}(Tx_1, Tx_2, \dots).$$

Now take any conditionally convergent series $\sum_{n=1}^{\infty} x_n$ and absolutely convergent series $\sum_{n=1}^{\infty} y_n$, both on the real line. Then $\mathrm{A}(x_n) = \mathbb{R}$ and $\mathrm{A}(y_n) = C$ is a compact set. By the above observation there are conditionally convergent series on the plane whose achievement sets equal $\mathbb{R} \times C$, $\mathbb{R}^2$ or any rotation of $\mathbb{R} \times C$.

As far as we know, the achievement sets of series in multidimensional spaces were considered in a few papers only. For example, Morán [M1, M2] studied quickly convergent series. The series considered by Bartoszewicz and Głąb [BG] are also absolutely convergent. So our paper is probably the first one on achievement sets of conditionally convergent series in $\mathbb{R}^n$ for $n > 1$. On the other hand, properties of sum range sets are well-studied [KK]. A strong suggestion to consider achievement sets of series in multidimensional spaces was given by Nitecki in his nice lecture [N1] (a shorter version of this survey is [N2]).

2. Cardinality of achievement sets. In this section we study the cardinality of achievement sets in Banach spaces. Then we prove that the achievement set of a conditionally convergent series is *perfectly dense in itself*, that is, for any point x which can be achieved and any $\varepsilon > 0$, the intersection of the achievement set and the ball $B(x, \varepsilon)$ contains a perfect set.

PROPOSITION 2.1. *Let X be a Banach space and let (x_n) be a sequence of elements of X.*

(i) *If there are finitely many nonzero x_n's then $\mathrm{A}(x_n)$ is finite.*
(ii) *If there are infinitely many nonzero x_n's and there is $\delta > 0$ such that $\|x_k\| \geq \delta$ for any nonzero x_k then $\mathrm{A}(x_n)$ is infinite and countable. Moreover if X is finite-dimensional then $\mathrm{A}(x_n)$ is unbounded.*
(iii) *$\mathrm{A}(x_n)$ contains a perfect set otherwise.*

Proof. (i) Let $x_{k_1}, \dots, x_{k_m}$ be all the nonzero terms of (x_n). Then $\operatorname{card} \mathrm{A}(x_n) \leq 2^m$.

(ii) Since zero terms do not affect the achievement set, we may assume that (x_n) consists of nonzero elements. First, we consider the case of $X = \mathbb{R}^k$ with the supremum norm. Then $x_n = (x_n(1), \dots, x_n(k))$. Without loss of generality we may assume that the set $F_j = \{n \in \mathbb{N} : x_n(j) \geq \delta\}$ is infinite for some $j \leq k$. Hence $\sum_{n \in F_j} x_n(j) = \infty$, so the achievement set is unbounded and therefore infinite.

In general, if $\{x_n : n \in \mathbb{N}\}$ spans a finite-dimensional space, then we may assume that X is finite-dimensional and therefore isomorphic to $\mathbb{R}^k$ with the supremum norm. If $\{x_n : n \in \mathbb{N}\}$ spans an infinite-dimensional space, then $\{x_n : n \in \mathbb{N}\}$ is an infinite set contained in $\mathrm{A}(x_n)$.

To see that $\mathrm{A}(x_n)$ is countable, note that no subsequence of x_n converges to zero. Therefore every element of $\mathrm{A}(x_n)$ is a sum of finitely many x_n's.

(iii) The negation of the first two conditions means that there exists an infinite subsequence (x_{n_l}) of nonzero terms which tends to 0. We may assume $\|x_{n_{l+1}}\| < \|x_{n_l}\|/3$ for every $l \in \mathbb{N}$. Then $\{0,1\}^{\mathbb{N}} \ni (\varepsilon_l) \overset{f}{\mapsto} \sum_{l=1}^{\infty} \varepsilon_l x_{n_l}$ is injective: To see this, assume that $\overline{\varepsilon_j} \neq \varepsilon_j$ for some $j \in \mathbb{N}$. We have

$$\Big\| \sum_{l=j+1}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=j+1}^{\infty} \overline{\varepsilon}_l x_{n_l} \Big\| = \Big\| \sum_{l=j+1}^{\infty} (\varepsilon_l - \overline{\varepsilon}_l) x_{n_l} \Big\| \leq \sum_{l=j+1}^{\infty} \|x_{n_l}\|$$
$$\leq \tfrac{3}{2} \|x_{n_{j+1}}\| < \|x_{n_j}\|.$$

Hence

$$\Big\|\sum_{l=j}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=j}^{\infty} \overline{\varepsilon}_l x_{n_l}\Big\| = \Big\|\varepsilon_j x_{n_j} - \overline{\varepsilon_j} x_{n_j} + \sum_{l=j+1}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=j+1}^{\infty} \overline{\varepsilon}_l x_{n_l}\Big\|$$
$$\geq \|\varepsilon_j x_{n_j} - \overline{\varepsilon_j} x_{n_j}\| - \Big\| \sum_{l=j+1}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=j+1}^{\infty} \overline{\varepsilon}_l x_{n_l} \Big\|$$
$$= \|x_{n_j}\| - \Big\| \sum_{l=j+1}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=j+1}^{\infty} \overline{\varepsilon}_l x_{n_l} \Big\| > 0.$$

This implies that there exists $r < j$ such that $\overline{\varepsilon_r} \neq \varepsilon_r$. After finitely many steps we get $\overline{\varepsilon_1} \neq \varepsilon_1$. But

$$\Big\|\sum_{l=1}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=1}^{\infty} \overline{\varepsilon}_l x_{n_l}\Big\| \geq \|x_{n_1}\| - \Big\|\sum_{l=2}^{\infty} \varepsilon_l x_{n_l} - \sum_{l=2}^{\infty} \overline{\varepsilon}_l x_{n_l}\Big\| > 0.$$

Since $\sum_{l=1}^{\infty} x_{n_l}$ is absolutely convergent, the mapping f is continuous. Hence $\mathrm{A}(x_{n_l}) \subseteq \mathrm{A}(x_n)$ is a continuous injective image of the Cantor space $\{0,1\}^{\mathbb{N}}$. ∎

PROPOSITION 2.2. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in a Banach space X. Then $\mathrm{A}(x_n)$ is perfectly dense in itself.*

Proof. Note that the set $\{\sum_{n \in F} x_n : F \text{ is finite}\}$ is dense in $\mathrm{A}(x_n)$. Let $x \in \mathrm{A}(x_n)$ and $\varepsilon > 0$. There is a finite set F such that $\| \sum_{n \in F} x_n - x\| < \varepsilon/2$. Using the same method as in the proof of Proposition 2.1(iii), we find a subsequence x_{n_l} of x_n such that $\{n_l : l \in \mathbb{N}\} \cap F = \emptyset$, $\sum_{l=1}^{\infty} \|x_{n_l}\| < \varepsilon/2$ and $\mathrm{A}(x_{n_l})$ is a perfect set. Then $\sum_{n \in F} x_n + \mathrm{A}(x_{n_l}) \subseteq B(x, \varepsilon) \cap \mathrm{A}(x_n)$. ∎

3. Achievement sets of conditionally convergent series. We start this section by considering the instructive example of the conditionally convergent series $\sum_{n=1}^{\infty}((-1)^{n+1}/n, 1/2^n)$ in the plane. The achievement set of this series has several properties which show that the theory of achievement sets of conditionally convergent series in multidimensional spaces is much more complicated and interesting than that of one-dimensional series. The analysis of this example will lead us to some general observations:

- First we note in Proposition 3.1 that every conditionally convergent series in $\mathbb{R}^m$ either has sum range $\mathbb{R}^m$ (*first type series*), or it is, up to linear isometry, of the form $\sum_{n=1}^{\infty}(x_n, y_n)$ where $x_n \in \mathbb{R}^k$, $y_n \in \mathbb{R}^{m-k}$, $\sum_{n=1}^{\infty} x_n$ is conditionally convergent with sum range $\mathbb{R}^k$, and $\sum_{n=1}^{\infty} y_n$ is absolutely convergent (*second type series*).
- We will observe that the closure of $\mathrm{A}((-1)^{n+1}/n, 1/2^n)$ contains the sum range $\mathrm{SR}((-1)^{n+1}/n, 1/2^n)$. This is a general fact in every Banach space (Lemma 3.2).
- We will observe that the closure of $\mathrm{A}((-1)^{n+1}/n, 1/2^n)$ equals $\mathbb{R}\times\mathrm{A}(1/2^n)$. A similar general fact is true in Euclidean spaces (Theorem 3.5).
- The achievement set $\mathrm{A}((-1)^{n+1}/n, 1/2^n)$ is not closed. This phenomenon is generalized in Theorem 3.7.
- The series $\sum_{n=1}^{\infty}(2/3^n, (-1)^{n+1}/n)$, considered in Example 3.10, a slight modification of that from Example 3.3, has an achievement set which is neither F_σ nor G_δ. A wide class of series with that property is given in Theorem 3.9. In the proof we observe that the achievement set of a conditionally convergent series is always an analytic (or Σ_1^1) set.

The series from Example 3.3 is of the first type. By Lemma 3.2 the achievement set of a series of the second type is dense in the whole space. One can easily give an example of a series whose achievement set is actually the whole space. It is much harder to give an example of a series of the second type whose achievement set is smaller. We will present it in Example 3.13; its achievement set will be a null subset of the plane.

The following observation, which is probably mathematical folklore, allows us to consider only two types of conditionally convergent series in Euclidean spaces.

PROPOSITION 3.1. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in $\mathbb{R}^m$ such that $\dim(\Gamma^\perp) = k < m$. Then there exists an isomorphism $(T_1, T_2) \in L(\mathbb{R}^m, \mathbb{R}^k \times \mathbb{R}^{m-k})$ such that $\sum_{n=1}^{\infty} T_1(x_n)$ is conditionally convergent with $\mathrm{SR}(\sum_{n=1}^{\infty} T_1(x_n)) = \mathbb{R}^k$ and $\sum_{n=1}^{\infty} T_2(x_n)$ is absolutely convergent in $\mathbb{R}^{m-k}$.*

Proof. Let $k = \dim \Gamma^\perp$ and let Y be the orthogonal to $\Gamma^\perp$ so that $\mathbb{R}^m = \Gamma^\perp \oplus Y$. Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^m$ and let $e'_1, \dots, e'_m$ be an orthogonal basis of $\Gamma^\perp \oplus Y$ such that $\Gamma^\perp = \mathrm{span}\{e'_1, \dots, e'_k\}$ and $Y =$

$\operatorname{span}\{e'_{k+1},\dots,e'_m\}$. Let $T:\mathbb{R}^m\to\Gamma^\perp\oplus Y$ be a linear isomorphism such that $T(e_i)=e'_i$ for every $i=1,\dots,m$. For $x=\sum_{i=1}^m x(i)e_i$ let $T_1(x)=\sum_{i=1}^k x(i)e'_i$ and $T_2(x)=\sum_{i=k+1}^m x(i)e'_i$. Then $T=(T_1,T_2)$. Let $f_i(x)=x(i)$. Then $f_i\in\Gamma$ for $i>k$, which means that $\sum_{n=1}^\infty\|T_2(x_n)\|<\infty$.

Let $\Lambda=\{f\in(\Gamma^\perp)^*:\sum_{n=1}^\infty|f(T_1(x_n))|<\infty\}$. Let $\pi_{\le k}:\mathbb{R}^m\to\mathbb{R}^k$ be the projection onto the first k coordinates. For $f\in(\Gamma^\perp)^*$ define $\tilde f\in(\mathbb{R}^m)^*$ by $\tilde f(x)=f(\pi_{\le k}(x))$. Then $f\in\Lambda\Leftrightarrow\tilde f\in\Gamma\Leftrightarrow\tilde f=0\Leftrightarrow f=0$. Thus $\Lambda=\{0\}$, and by the Steinitz Theorem, $\mathrm{SR}(T_1(x_n))=\Lambda^\perp=\mathbb{R}^k$. ∎

Note that a linear isomorphism T does not change the geometrical or topological properties of subsets of $\mathbb{R}^m$. Therefore we will assume that a conditionally convergent series in $\mathbb{R}^m$ either can be rearranged to get any point of $\mathbb{R}^m$, or it is of the form $\sum_{n=1}^\infty(x_n,y_n)$ where $x_n\in\mathbb{R}^k$, $y_n\in\mathbb{R}^{m-k}$, $\mathrm{SR}(x_n)=\mathbb{R}^k$ and $\sum_{n=1}^\infty y_n$ is absolutely convergent.

The following lemma shows a relation between the sum range and the achievement set of a series.

Lemma 3.2. *Let $\sum_{n=1}^\infty x_n$ be a conditionally convergent series in a Banach space X. Then $\mathrm{SR}(x_n)\subseteq\overline{\mathrm{A}(x_n)}$.*

Proof. Let $\varepsilon>0$ and $x\in\mathrm{SR}(x_n)$. Then $x=\sum_{n=1}^\infty x_{\sigma(n)}$ for some $\sigma\in S_\infty$. One can find a natural number k such that $\|x-\sum_{n=1}^k x_{\sigma(n)}\|<\varepsilon$. Denote $A=\{m:\sigma(n)=m,\,n\le k\}=\sigma(\{1,\dots,k\})$ and define $\varepsilon_n=1$ for $n\in A$ and $\varepsilon_n=0$ otherwise. Then $\sum_{n=1}^k x_{\sigma(n)}=\sum_{n=1}^\infty\varepsilon_n x_n$, so $\|x-\sum_{n=1}^\infty\varepsilon_n x_n\|<\varepsilon$. Hence $x\in\overline{\mathrm{A}(x_n)}$. ∎

Example 3.3. Let $x_n=(x_n^{(1)},x_n^{(2)})=((-1)^{n+1}/n,1/2^n)\in\mathbb{R}^2$. Then clearly $\sum_{n=1}^\infty x_n=(\log 2,1)$. By the Riemann Theorem for every $x\in\mathbb{R}$ one can find $\sigma\in S_\infty$ such that $x=\sum_{n=1}^\infty x_{\sigma(n)}^{(1)}$. Since permutating indices does not affect the sum of an absolutely convergent series, we have $\sum_{n=1}^\infty x_{\sigma(n)}^{(2)}=1$. Hence $\mathrm{SR}(x_n)=\mathbb{R}\times\{1\}$. Let $D=\{\sum_{n=1}^k\varepsilon_n/2^n:(\varepsilon_n)\in\{0,1\}^k,\,k\in\mathbb{N}\}$ be the set of all dyadic numbers in $[0,1)$. Then for every $d\in D$ there are $k\in\mathbb{N}$ and $(\varepsilon_n)\in\{0,1\}^k$ with $d=\sum_{n=1}^k\varepsilon_n/2^n$. Set $F_d:=\{n\le k:\varepsilon_n=1\}$. After removing finitely many terms from a conditionally convergent series, we still have a conditionally convergent series. Therefore $\mathrm{SR}((x_n)_{n\in\mathbb{N}\setminus F_d})=\mathbb{R}\times\{1-d\}$.

From Lemma 3.2 we get $\mathrm{SR}((x_n)_{n\in\mathbb{N}\setminus F_d})\subseteq\overline{\mathrm{A}((x_n)_{n\in\mathbb{N}\setminus F_d})}\subseteq\overline{\mathrm{A}(x_n)}$. Since $d\in D\Leftrightarrow 1-d\in D$, we have $\bigcup_{d\in D}(\mathbb{R}\times\{d\})\subseteq\overline{\mathrm{A}(x_n)}$, and consequently $\overline{\bigcup_{d\in D}(\mathbb{R}\times\{d\})}\subseteq\overline{\mathrm{A}(x_n)}$. But D is dense in $[0,1]$, so

$$\overline{\bigcup_{d\in D}(\mathbb{R}\times\{d\})}=\mathbb{R}\times\overline{\bigcup_{d\in D}\{d\}}=\mathbb{R}\times\overline{D}=\mathbb{R}\times[0,1].$$

Thus $\mathbb{R}\times[0,1]\subseteq\overline{\mathrm{A}(x_n)}$. The reverse inclusion is obvious; therefore $\overline{\mathrm{A}(x_n)}=\mathbb{R}\times[0,1]$. Suppose $(z,1)\in\mathrm{A}(x_n)$. The only way to get $\sum_{n=1}^{\infty}\varepsilon_n x_n^{(2)}=1$ for some $(\varepsilon_n)\subseteq\{0,1\}^{\mathbb{N}}$ is to take $\varepsilon_n=1$ for each $n\in\mathbb{N}$. Hence $z=\log 2$. This proves that the only point with second coordinate 1 which belongs to $\mathrm{A}(x_n)$ is $(\log 2,1)$. Thus $\mathrm{A}(x_n)$ is not closed. Finally, recall that $\mathrm{SR}(x_n)=\mathbb{R}\times\{1\}$. Therefore $\mathrm{A}(x_n)\cap\mathrm{SR}(x_n)=\{(\log 2,1)\}$.

Lemma 3.2 implies that the achievement sets of conditionally convergent series in finite-dimensional spaces are unbounded. The situation changes in infinite-dimensional Banach spaces.

EXAMPLE 3.4. Let (e_n) be the standard basis of c_0. Let (x_n) be the sequence

$$e_1,-e_1,\tfrac{1}{2}e_2,-\tfrac{1}{2}e_2,\tfrac{1}{2}e_2,-\tfrac{1}{2}e_2,\tfrac{1}{3}e_3,-\tfrac{1}{3}e_3,\tfrac{1}{3}e_3,-\tfrac{1}{3}e_3,\tfrac{1}{3}e_3,-\tfrac{1}{3}e_3,\dots.$$

Then $\sum_{n=1}^{\infty}x_n$ is convergent to zero, while its rearrangement

$$e_1-e_1+\tfrac{1}{2}e_2+\tfrac{1}{2}e_2-\tfrac{1}{2}e_2-\tfrac{1}{2}e_2+\tfrac{1}{3}e_3+\tfrac{1}{3}e_3+\tfrac{1}{3}e_3-\tfrac{1}{3}e_3-\tfrac{1}{3}e_3-\tfrac{1}{3}e_3+\cdots$$

is divergent, since the sequence of its partial sums contains each e_i. Since the projection of the series on each coordinate contains only finitely many nonzero terms and a finite sum does not change under rearrangements, we have $\mathrm{SR}(\sum_{n=1}^{\infty}x_n)=\{0\}$. Let $X_n=\{k/n : k\in[-n,n]\cap\mathbb{Z}\}$. Note that $\mathrm{A}(\sum_{n=1}^{\infty}x_n)=(\prod_{n=1}^{\infty}X_n)\cap c_0$. Therefore $\mathrm{A}(\sum_{n=1}^{\infty}x_n)$ is closed and bounded.

It can happen that $\mathrm{SR}(x_n)=\mathrm{A}(x_n)$. For example:

- $\mathrm{A}((-1)^n/n,0)=\mathbb{R}\times\{0\}=\mathrm{SR}((-1)^n/n,0)$.
- $\mathrm{A}((-1)^n/n,(-1)^n/n)=\{(x,x)\in\mathbb{R}\}=\mathrm{SR}((-1)^n/n,(-1)^n/n)$.
- If $\sum_{n=1}^{\infty}x_n$ and $\sum_{n=1}^{\infty}y_n$ are conditionally convergent on the real line, then
$$\begin{aligned}&\mathrm{A}((x_1,0),(0,y_1),(x_2,0),(0,y_2),\dots)\\&\qquad=\mathrm{SR}((x_1,0),(0,y_1),(x_2,0),(0,y_2),\dots)=\mathbb{R}^2.\end{aligned}$$

Note that if $\sum_{n=1}^{\infty}x_n$ is a conditionally convergent series in a Banach space X, then $\mathrm{A}(x_n)$ and $\mathrm{SR}(x_n)$ intersect, namely $\sum_{n=1}^{\infty}x_n\in\mathrm{A}(x_n)\cap\mathrm{SR}(x_n)$. Note that in Example 3.3, $\mathrm{A}(x_n)\cap\mathrm{SR}(x_n)$ is actually a singleton.

Using the idea from Example 3.3 we will prove the following.

THEOREM 3.5. *Let $(x_n)\subset\mathbb{R}^k$ be such that $\sum_{n=1}^{\infty}x_n$ is conditionally convergent with $\mathrm{SR}(x_n)=\mathbb{R}^k$, and let $(y_n)\subset\mathbb{R}^m$ be such that $\sum_{n=1}^{\infty}y_n$ is absolutely convergent. Then $\overline{\mathrm{A}(x_n,y_n)}=\mathbb{R}^k\times\mathrm{A}(y_n)$.*

Proof. "$\subseteq$". It is easy to see that $\mathrm{A}(x_n,y_n)\subseteq\mathrm{A}(x_n)\times\mathrm{A}(y_n)$. Therefore

$$\overline{\mathrm{A}(x_n,y_n)}\subseteq\overline{\mathrm{A}(x_n)\times\mathrm{A}(y_n)}=\overline{\mathrm{A}(x_n)}\times\overline{\mathrm{A}(y_n)}.$$

Since $\mathrm{SR}(x_n)=\mathbb{R}^k$, by Lemma 3.2 we get $\overline{\mathrm{A}(x_n)}=\mathbb{R}^k$. By the absolute convergence of $\sum_{n=1}^{\infty} y_n$ the set $\mathrm{A}(y_n)$ is compact, so $\overline{\mathrm{A}(y_n)}=\mathrm{A}(y_n)$.

"$\supseteq$". Let $(x,y)\in\mathbb{R}^k\times \mathrm{A}(y_n)$ and $\varepsilon>0$. Since $y=\sum_{n=1}^{\infty}\varepsilon_n y_n$ for some $(\varepsilon_n)\in\{0,1\}^{\mathbb{N}}$, there exists $k_\varepsilon\in\mathbb{N}$ such that $\|y-\sum_{n=1}^{N}\varepsilon_n y_n\|<\varepsilon$ for every $N\geq k_\varepsilon$. Since $\sum_{n=1}^{\infty} y_n$ is absolutely convergent, we may assume that $\sum_{n=k_\varepsilon+1}^{\infty}\|y_n\|<\varepsilon$. Let $K=\{n\leq k_\varepsilon : \varepsilon_n=1\}=\{k_1<\cdots<k_l\}$. Define $\sigma(n)=k_n$ for $n\in\{1,\ldots,l\}$.

Note that if $\mathrm{SR}(x_n)=\mathbb{R}^k$, then $\mathrm{SR}((x_n)_{n\geq k_\varepsilon+1})=\mathbb{R}^k$ as well. In particular $x-\sum_{n=1}^{l}x_{k_n}\in \mathrm{SR}((x_n)_{n\geq k_\varepsilon+1})$. Therefore there exist $M\in\mathbb{N}$ and a one-to-one mapping $\tau:\{k_\varepsilon+1,\ldots,M\}\to\{k_\varepsilon+1,k_\varepsilon+2,\ldots\}$ such that $\|x-\sum_{n=1}^{l}x_{k_n}-\sum_{n=k_\varepsilon+1}^{M}x_{\tau(n)}\|<\varepsilon$. Enumerate the range $\tau(\{k_\varepsilon+1,\ldots,M\})$ as $\{k_{l+1}<\cdots<k_{l+l'}\}$ and define

$$\delta_n=\begin{cases}\varepsilon_n & n\leq k_\varepsilon,\\ 1 & n=k_{l+i} \text{ for } i\in\{1,\ldots,l'\},\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\Big\|\sum_{n=1}^{\infty}\varepsilon_n y_n-\sum_{n=1}^{\infty}\delta_n y_n\Big\|=\Big\|\sum_{n=k_\varepsilon+1}^{\infty}\varepsilon_n y_n-\sum_{n=k_\varepsilon+1}^{\infty}\delta_n y_n\Big\|$$
$$=\Big\|\sum_{n=k_\varepsilon+1}^{\infty}(\varepsilon_n-\delta_n)y_n\Big\|\leq\sum_{n=k_\varepsilon+1}^{\infty}\|y_n\|<\varepsilon.$$

Hence $\|y-\sum_{n=1}^{\infty}\delta_n y_n\|<\varepsilon$. Moreover

$$\Big\|x-\sum_{n=1}^{\infty}\delta_n x_n\Big\|=\Big\|x-\sum_{n\leq k_\varepsilon}\delta_n x_n-\sum_{n=1}^{l'}\delta_{k_{l+n}}x_{k_{l+n}}\Big\|$$
$$=\Big\|x-\sum_{n\leq k_\varepsilon}\varepsilon_n x_n-\sum_{n=l+1}^{l+l'}x_{k_n}\Big\|=\Big\|x-\sum_{n\in K}x_n-\sum_{n=l+1}^{l+l'}x_{k_n}\Big\|$$
$$=\Big\|x-\sum_{n=1}^{l}x_{k_n}-\sum_{n=k_\varepsilon+1}^{M}x_{\tau(n)}\Big\|<\varepsilon.$$

Since all norms in a finite-dimensional space are equivalent, there is $C>0$ such that

$$\Big\|(x,y)-\sum_{n=1}^{\infty}\delta_n(x_n,y_n)\Big\|\leq C\max\Big\{\Big\|x-\sum_{n=1}^{\infty}\delta_n x_n\Big\|,\Big\|y-\sum_{n=1}^{\infty}\delta_n y_n\Big\|\Big\}<C\varepsilon.$$

Thus $(x,y)\in\overline{\mathrm{A}(x_n,y_n)}$, which finishes the proof. ■

Now, we present a sufficient condition for a conditionally convergent series to have an achievement set which is not closed. Here we again use the idea from Example 3.3.

LEMMA 3.6. *Let $\sum_{n=1}^{\infty} y_n$ be an absolutely convergent series in $\mathbb{R}^m$ with $y_n \neq 0$ for each $n \in \mathbb{N}$. Then for every extreme point a of $\mathrm{A}(y_n)$, there is a unique sequence $(\varepsilon_n) \in \{0,1\}^{\mathbb{N}}$ such that $a = \sum_{n=1}^{\infty} \varepsilon_n y_n$.*

Proof. We will show that a is achieved for a unique $(\varepsilon_n) \in \{0,1\}^{\mathbb{N}}$. Suppose on the contrary that $a = \sum_{n=1}^{\infty} \varepsilon_n y_n = \sum_{n=1}^{\infty} \delta_n y_n$ for two distinct sequences (ε_n) and (δ_n), and set $M = \{n \in \mathbb{N} : \varepsilon_n \neq \delta_n\}$. Divide M into $M_\varepsilon = \{n \in M : \varepsilon_n = 1,\, \delta_n = 0\}$ and $M_\delta = \{n \in M : \varepsilon_n = 0,\, \delta_n = 1\}$. Then $a = \sum_{n \in M_\varepsilon} y_n + \sum_{n \in M^c} \varepsilon_n y_n = \sum_{n \in M_\delta} y_n + \sum_{n \in M^c} \varepsilon_n y_n$, so

$$a = \frac{1}{2} \sum_{n \in M^c} \varepsilon_n y_n + \frac{1}{2}\Big(\sum_{n \in M} y_n + \sum_{n \in M^c} \varepsilon_n y_n \Big) = \frac{1}{2} b + \frac{1}{2} c.$$

As $b = \sum_{n \in M^c} \varepsilon_n y_n$ and $c = \sum_{n \in M} y_n + \sum_{n \in M^c} \varepsilon_n y_n$, we have $b, c \in \mathrm{A}(y_n)$. Since a is an extreme point of $\mathrm{A}(y_n)$, we get $b = c$, so $\sum_{n \in M} y_n = 0$. Since each y_n is nonzero, M has at least two elements. Assume that $n_0 \in M$; then $y_{n_0} + \sum_{l \in M \setminus \{n_0\}} y_l = 0$. Define $b' = b + y_{n_0}$ and $c' = c - y_{n_0}$. Then $b', c' \in \mathrm{A}(y_n)$. We have $a = \frac{1}{2} b' + \frac{1}{2} c'$, and using again the assumption that a is an extreme point, we get $b' = c'$. But then $y_{n_0} = \sum_{l \in M \setminus \{n_0\}} y_l$ and $\sum_{n \in M} y_n = 0$, and so $y_{n_0} = 0$, a contradiction. ∎

THEOREM 3.7. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in $\mathbb{R}^k$ with $\mathrm{SR}(x_n) = \mathbb{R}^k$, and $\sum_{n=1}^{\infty} y_n$ an absolutely convergent series with $y_n \neq 0$ for each $n \in \mathbb{N}$. Then $\mathrm{A}(x_n, y_n)$ is not closed.*

Proof. Let a be an extreme point of $\mathrm{A}(y_n)$; such a point exists because $\mathrm{A}(y_n)$ is compact. By Lemma 3.6, a has a unique representation $a = \sum_{n=1}^{\infty} \varepsilon_n y_n$. Hence, the a-section of $\mathrm{A}(x_n, y_n)$ is a singleton if $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges, and is empty otherwise. On the other hand, $\{(x, a) : x \in \mathbb{R}^k\} \subseteq \mathbb{R}^k \times \mathrm{A}(y_n) = \overline{\mathrm{A}(x_n, y_n)}$. Hence, $\mathrm{A}(x_n, y_n)$ is not closed. ∎

Now, we will prove that the achievement set of a conditionally convergent series need not be F_σ or G_δ. This will apply to the series from Example 3.10.

Let us start by recalling some notions from descriptive set theory. A topological space is *Polish* if it is completely metrizable and separable. An $F_{\sigma\delta}$ subset A of a Polish space X is called Π^0_3-*complete* if for any zero-dimensional Polish space Y and for any $F_{\sigma\delta}$ subset B of Y there is a continuous function $f : Y \to X$ such that $f^{-1}(A) = B$. It is known that Π^0_3-complete sets are $F_{\sigma\delta}$ but not $G_{\delta\sigma}$. To prove that an $F_{\sigma\delta}$ subset C of a Polish space Z is Π^0_3-complete, it is enough to take a known example A of a Π^0_3-complete subset of a Polish space X and find a continuous function $g : X \to Z$ such that $g^{-1}(C) = A$. For more information, see [Ke].

Proposition 3.8. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in $\mathbb{R}$ such that $\mathrm{A}(x_n) = \mathbb{R}$. Then $E := \{(\varepsilon_n) \in 2^{\mathbb{N}} : \sum_{n=1}^{\infty} \varepsilon_n x_n$ converges$\}$ is a Π_3^0-complete subset of $\{0,1\}^{\mathbb{N}}$.*

Proof. Note that

$$(\varepsilon_n) \in E \;\Leftrightarrow\; \forall k \in \mathbb{N}\; \exists l \in \mathbb{N}\; \forall M > m \geq l \left(\Big|\sum_{n=m}^{M} \varepsilon_n x_n\Big| \leq \frac{1}{k}\right).$$

Therefore E is an $F_{\sigma\delta}$ subset of $2^{\mathbb{N}}$.

To prove that E is Π_3^0-complete, we will use the fact that the set

$$C_3 := \left\{v \in \mathbb{N}^{\mathbb{N}} : \lim_{n\to\infty} v(n) = \infty\right\}$$

is Π_3^0-complete (for details see [Ke, Section 23A]). It is enough to construct a continuous function $\psi : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that $v \in C_3 \;\Leftrightarrow\; \psi(v) \in E$. The spaces $\{0,1\}^{\mathbb{N}}$ of all 0-1 sequences and $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers are considered with the metric $d(x,y) = 2^{-n}$ where $n = \min\{k : x(k) \neq y(k)\}$.

One can inductively define sets $F_n = F_n(v)$ and $H_n = H_n(v)$ such that $F_0 = H_0 = \emptyset$ and for every $n \geq 1$:

(i) $F_n < H_n < F_{n+1}$, that is, $\max F_n < \min H_n$ and $\max H_n < \min F_n$;
(ii) $|\sum_{k\in F_1\cup H_1\cup\cdots\cup F_n} x_k| < 2^{-n}$;
(iii) $|\sum_{k\in F_1\cup H_1\cup\cdots\cup F_n\cup H_n} x_k - 2^{-v(n)}| < 2^{-n}$;
(iv) $x_k < 0$ for $k \in \bigcup_{n\geq 1} F_n$, and $x_k > 0$ for $k \in \bigcup_{n\geq 1} H_n$.

Note that the above construction can be done uniformly in the sense that if $v(i) = v'(i)$ for $i \leq n$, then $F_i(v) = F_i(v')$ and $H_i(v) = H_i(v')$ for $i \leq n$.

Now, we define $\psi : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ as follows. Let $\psi(v)$ be the characteristic function of $\bigcup_{n=1}^{\infty}(F_n(v) \cup H_n(v))$. Since the construction is uniform, for $v, v' \in \mathbb{N}^{\mathbb{N}}$ such that $v(i) = v'(i)$ for $i \leq n$ we have $d(\psi(v), \psi(v')) \leq 2^{-n}$. Therefore ψ is continuous. We will prove that $v \in C_3 \;\Leftrightarrow\; \sum_{n=1}^{\infty} \psi(v)(n) x_n$ is convergent.

If $v \notin C_3$, then there are $m \in \mathbb{N}$ and an infinite set $L \subseteq \mathbb{N}$ such that $v(l) = m$ for all $l \in L$. Thus by construction, the series $\sum_{n=1}^{\infty} \psi(v)(n) x_n$ diverges, since the sequence $(\sum_{n=1}^{N} \psi(v)(n) x_n)$ of partial sums has, by (ii) and (iii), two accumulation points, 0 and 2^{-m}.

If $v \in C_3$, then $2^{-v(n)} \to 0$. Let $\varepsilon > 0$. There is $l \in \mathbb{N}$ such that $2^{-v(n)}, 2^{-n} < \varepsilon/2$ for $n \geq l$. Let $n \geq l$. By (ii) and (iii) we obtain

$$\Big|\sum_{k\in F_1\cup H_1\cup\cdots\cup F_n} x_k\Big| = \Big|\sum_{k=1}^{\max F_n} \psi(v)(k) x_k\Big| < 2^{-n} < \varepsilon$$

and

$$\Big|\sum_{k\in F_1\cup H_1\cup\cdots\cup F_n\cup H_n} x_k - 2^{-v(n)}\Big| = \Big|\sum_{k=1}^{\max H_n}\psi(v)(k)x_k - 2^{-v(n)}\Big| < 2^{-n} + 2^{-v(n)} < \varepsilon.$$

By (iv) we also have $|\sum_{k=1}^{M}\psi(v)(k)x_k| < 2^{-n}$ for every $M \geq \max H_l$. Thus $\sum_{n=1}^{\infty}\psi(v)(n)x_n$ converges to zero. ∎

A result similar to Proposition 3.8 was proved by Cohen [C]: if $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series in $\mathbb{R}^n$, then the set of all permutations σ such that $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges is a Π^0_3-complete subset of S_∞. The example of a Π^0_3-complete set in Proposition 3.8 will be used to prove the following.

THEOREM 3.9. *Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series in $\mathbb{R}$, and let $\sum_{n=1}^{\infty} y_n$ be an absolutely convergent series in $\mathbb{R}$ such that the function $\{0,1\}^{\mathbb{N}} \ni (\varepsilon_n) \mapsto \sum_{n=1}^{\infty}\varepsilon_n y_n$ is one-to-one. Then $\mathrm{A}(x_n, y_n)$ is a Borel subset of $\mathbb{R}^2$ which is neither G_δ nor F_σ.*

Proof. Note that

$$(x,y)\in \mathrm{A}(x_n,y_n) \;\Leftrightarrow$$
$$\exists(\varepsilon_n)\in 2^{\mathbb{N}}\ \forall m\ \exists l\ \forall k\geq l\ \left(\Big|x-\sum_{n=1}^{k}\varepsilon_n x_n\Big| < \frac{1}{m} \text{ and } \Big|y-\sum_{n=1}^{k}\varepsilon_n y_n\Big| < \frac{1}{m}\right),$$

which shows that $\mathrm{A}(x_n, y_n)$ is Σ^1_1 as a projection of a Borel set, and

$$(x,y)\in \mathrm{A}(x_n,y_n) \;\Leftrightarrow$$
$$\exists!(\varepsilon_n)\in 2^{\mathbb{N}}\ \forall m\ \exists l\ \forall k\geq l\ \left(\Big|x-\sum_{n=1}^{k}\varepsilon_n x_n\Big| < \frac{1}{m} \text{ and } \Big|y-\sum_{n=1}^{k}\varepsilon_n y_n\Big| < \frac{1}{m}\right),$$

which shows that $\mathrm{A}(x_n, y_n)$ is Π^1_1 [Ke, 18.11]. The first equivalence follows from the definition of achievement set, and the second by the injectivity of $(\varepsilon_n) \mapsto \sum_{n=1}^{\infty}\varepsilon_n y_n$. By the Suslin Theorem, $\mathrm{A}(x_n, y_n)$ is Borel.

By Theorem 3.5, $\mathrm{A}(x_n, y_n)$ is dense in $\mathbb{R}\times\mathrm{A}(y_n)$. However, every horizontal section of $\mathrm{A}(x_n, y_n)$ consists of at most one point. Suppose that $\mathrm{A}(x_n, y_n)$ is a G_δ. Then it would be a comeager subset of $\mathbb{R}\times\mathrm{A}(y_n)$, and therefore almost all of its horizontal sections in the sense of category would be comeager in $\mathbb{R}$. That would be a contradiction since each of the horizontal sections has at most one point.

Suppose that $\mathrm{A}(x_n, y_n)$ is an F_σ subset of $\mathbb{R}^2$. Then it would be a countable union of compact sets, and consequently its projection $\mathrm{proj}_2(\mathrm{A}(x_n, y_n))$ would be a countable union of compact sets, that is, an F_σ set. But $\mathrm{proj}_2(\mathrm{A}(x_n, y_n))$ is homeomorphic to the set E from Lemma 3.8, which is not F_σ; a contradiction. ∎

The next example is a slight modification of Example 3.3.

Example 3.10. Let $(x_k, y_k) = (2/3^k, (-1)^{k+1}/k)$. Since the mapping $\{0,1\}^{\mathbb{N}} \ni (\varepsilon_k) \mapsto \sum_{k=1}^{\infty} \varepsilon_k x_k$ is one-to-one, $\mathrm{A}(x_k, y_k)$ is the graph of a function f with domain contained in the ternary Cantor set. Moreover f maps onto $\mathbb{R}$, and its domain is an $F_{\sigma\delta}$ which is not $G_{\delta\sigma}$, by Theorem 3.9.

We can also consider whether or not in Theorem 3.5 the condition that the sum range of a conditionally convergent series is the whole space can be replaced by the same condition for the achievement set. The following proposition shows that in some cases we can reverse Theorem 3.5.

Proposition 3.11. *Assume that $\sum_{n=1}^{\infty} x_n$ is conditionally convergent in $\mathbb{R}^k$. If $\mathrm{A}(x_n)$ is a dense subset of $\mathbb{R}^k$, then $\mathrm{SR}(x_n) = \mathbb{R}^k$.*

Proof. Suppose that $\mathrm{SR}(x_n) \neq \mathbb{R}^k$. By Proposition 3.1 we have $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} T(y_n, z_n)$, where $T : \mathbb{R}^m \times \mathbb{R}^{k-m} \to \mathbb{R}^k$ is an isomorphism and $\mathrm{SR}(y_n) = \mathbb{R}^m$ for some m with $1 \leq m < k$ and $\sum_{n=1}^{\infty} z_n$ is absolutely convergent in $\mathbb{R}^{k-m}$. Without loss of generality we may assume $x_n = (y_n, z_n)$. Hence $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (y_n, z_n)$. Thus by Theorem 3.5 and compactness of $\mathrm{A}(z_n)$ we have $\overline{\mathrm{A}(x_n)} = \mathbb{R}^m \times \mathrm{A}(z_n) \neq \mathbb{R}^k$, a contradiction. ∎

By Theorem 3.5 and Proposition 3.11 we immediately obtain the following.

Corollary 3.12. *Let $(x_n) \subset \mathbb{R}^k$ be such that $\sum_{n=1}^{\infty} x_n$ is conditionally convergent and $\mathrm{A}(x_n)$ is dense in $\mathbb{R}^k$. Let $(y_n) \subset \mathbb{R}^m$ be such that $\sum_{n=1}^{\infty} y_n$ is absolutely convergent. Then $\overline{\mathrm{A}(x_n, y_n)} = \mathbb{R}^k \times \mathrm{A}(y_n)$.*

On the other hand, one can construct a series in the plane whose sum range is $\mathbb{R}^2$ but the achievement set is a dense set of measure zero.

Example 3.13. Let $\sum_{n=1}^{\infty} (x_n, y_n)$ be defined by $x_i = (-1)^i / 2^{10^{k^2}}$ and $y_i = (-1)^i / 2^k$ for $i \in (n_{k-1}, n_k]$, where $n_0 = 0$ and $n_{k+1} = n_k + 2^{10^{k^2}+1}$. Since $\sum_{n=1}^{\infty} (x_n, y_n)$ is alternating and $(x_n, y_n) \to (0,0)$, the Leibniz criterion shows that $\sum_{n=1}^{\infty} (x_n, y_n)$ is convergent, and its sum equals $(0,0)$.

First, we will show that $\mathrm{SR}(x_n, y_n) = \mathbb{R}^2$. Let $\Gamma = \{f \in (\mathbb{R}^2)^* : \sum_{n=1}^{\infty} |f(x_n, y_n)| < \infty\}$. We need to prove that Γ contains no nontrivial functionals. Every $f \in (\mathbb{R}^2)^*$ is of the form $f(x, y) = ax + by$ for some $a, b \in \mathbb{R}$. We have

$$\sum_{i=n_{k-1}+1}^{n_k} |y_i| = (n_k - n_{k-1}) \frac{1}{2^k} = 2^{10^{k^2}+1-k},$$

$$\sum_{i=n_{k-1}+1}^{n_k} |x_i| = (n_k - n_{k-1}) \frac{1}{2^{10^{k^2}}} = 2.$$

If $a = 0$ and $b \neq 0$, then

$$\sum_{i=1}^{\infty} |f(x_i, y_i)| = \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} |by_i| = |b| \sum_{k=1}^{\infty} 2^{10^{k^2}+1-k} = \infty.$$

If $a \neq 0$ and $b = 0$, then

$$\sum_{i=1}^{\infty} |f(x_i, y_i)| = \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} |ax_i| = |a| \sum_{k=1}^{\infty} 2 = \infty.$$

Now let $a \neq 0$ and $b \neq 0$. For large enough k which satisfies $|a|/2^{10^{k^2}-k} < |b|/2$, we have

$$\left| \frac{a}{2^{10^{k^2}}} + \frac{b}{2^k} \right| \geq \frac{|b|}{2^k} - \frac{|a|}{2^{10^{k^2}}} \geq \frac{|b|}{2^{k+1}}$$

and the series $\sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} |b|/2^{k+1}$ diverges. Consequently,

$$\sum_{i=1}^{\infty} |f(x_i, y_i)| = \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} |ax_i + by_i| = \infty.$$

From the Steinitz Theorem we have $\mathrm{SR}(x_n, y_n) = \mathbb{R}^2$.

Now we will show that $\mathrm{A}(x_n, y_n) \subseteq (L \cup \mathbb{Q}) \times \mathbb{R}$, where $L = \{x : \forall r \in \mathbb{N}\ \exists p, q \in \mathbb{Z}\ (0 < |x - p/q| < 1/q^r)\}$ is the set of all Liouville numbers on the real line. It is well-known that L has Lebesgue measure zero. Let $r \in \mathbb{N}$. Suppose that $(x, y) \in \mathrm{A}(x_n, y_n)$, that is, $(x, y) = \sum_{n=1}^{\infty} \varepsilon_n (x_n, y_n)$. Then there exists $l \in \mathbb{N}$ such that $|\sum_{n=N}^{M} \varepsilon_n y_n| \leq 1$ for every $M > N \geq l$. There exists $k_0 \in \mathbb{N}$ for which $n_{k_0-1} \geq l$, so $|\sum_{i=n_{k-1}+1}^{n_k} \varepsilon_i y_i| \leq 1$ for every $k \geq k_0$. Note that $\lim_{k\to\infty} \frac{10^{k^2}-k-1}{10^{(k-1)^2}} = \infty$ and the sequence $\left(\frac{10^{k^2}-k-1}{10^{(k-1)^2}}\right)$ is strictly increasing. Assume that k is the minimal natural number such that $k \geq k_0$ and $\frac{10^{k^2}-k-1}{10^{(k-1)^2}} \geq r$. Let $m \geq k$. Since $|\sum_{i=n_{m-1}+1}^{n_m} \varepsilon_i y_i| \leq 1$ and $|y_i| = 1/2^m$ for $i \in (n_{m-1}, n_m]$, we obtain

$$\Big| \sum_{i \in (n_{m-1}, n_m] \cap 2\mathbb{N}} \varepsilon_i - \sum_{i \in (n_{m-1}, n_m] \cap (2\mathbb{N}-1)} \varepsilon_i \Big| \leq 2^m.$$

This means that the excess of ones in the sequence $(\varepsilon_i)_{i \in (n_{m-1}, n_m]}$ with odd indices over those with even indices is less than 2^m and vice versa. Consequently, $|\sum_{i=n_{m-1}+1}^{n_m} \varepsilon_i x_i| \leq 2^m/2^{10^{m^2}}$. Moreover, $\sum_{i=1}^{n_{k-1}} \varepsilon_i x_i = p_0/2^{10^{(k-1)^2}}$ for some $p_0 \in \mathbb{Z}$. We have

$$\Big| \sum_{i=n_{k-1}+1}^{\infty} \varepsilon_i x_i \Big| \leq \sum_{m=k}^{\infty} \Big| \sum_{i=n_{m-1}+1}^{n_m} \varepsilon_i x_i \Big| \leq \sum_{m=k}^{\infty} \frac{2^m}{2^{10^{m^2}}} = \sum_{m=k}^{\infty} \frac{1}{2^{10^{m^2}-m}}.$$

Note that

$$\frac{\frac{1}{2^{10^{(m+1)^2}-m-1}}}{\frac{1}{2^{10^{m^2}-m}}} = \frac{2^{10^{m^2}-m}}{2^{10^{(m+1)^2}-m-1}}$$
$$= 2^{10^{m^2}-m-10^{(m+1)^2}+m+1} = 2^{1-(1-10^{2m+1})10^{m^2}} \leq 1/2$$

for every $m \in \mathbb{N}$. Hence

$$\sum_{m=k}^{\infty} \frac{1}{2^{10^{m^2}-m}} \leq \sum_{m=k}^{\infty} \frac{1}{2^{10^{k^2}-k}} \cdot \frac{1}{2^{m-k}} = \frac{2}{2^{10^{k^2}-k}} = 2^{1+k-10^{k^2}}.$$

Since $1 + k - 10^{k^2} \leq -10^{(k-1)^2} r$, we get

$$\Big|\sum_{i=n_{k-1}+1}^{\infty} \varepsilon_i x_i\Big| \leq 2^{-10^{(k-1)^2} \cdot r} = \frac{1}{(2^{10^{(k-1)^2}})^r}.$$

Hence

$$\Big|x - \sum_{i=1}^{n_{k-1}} \varepsilon_i x_i\Big| = \Big|\sum_{i=n_{k-1}+1}^{\infty} \varepsilon_i x_i\Big| \leq \frac{1}{(2^{10^{(k-1)^2}})^r}.$$

Thus $|x - p_0/q_0| \leq 1/q_0^r$ with $q_0 = 2^{10^{(k-1)^2}}$. That means that x is either a rational number or a Liouville number. Finally, $\mathrm{A}(x_n, y_n) \subseteq (L \cup \mathbb{Q}) \times \mathbb{R}$. Therefore $\mathrm{A}(x_n, y_n)$ is of measure zero.

4. Openness of achievement sets. In this section we show that for some series on the plane its achievement set can be an open set not equal to the whole plane or an open set with two additional points. These sets are unbounded, since bounded achievement sets are compact.

THEOREM 4.1. *Let $\sum_{k=1}^{\infty} x_k = X < \infty$ with $x_k > 0$ for every $k \in \mathbb{N}$, and suppose that*

($\star$) *for every $a \in (0, 2X)$ there exists an interval $I_a \subseteq \mathrm{A}(x_k)$ such that for all $t \in I_a$ there exists $z \in \mathrm{A}(x_k)$ with $t + z = a$.*

If (y_k) is conditionally convergent and $\sum_{k=1}^{\infty} y_k = Y$ then $\mathrm{A}(\overline{x}_k, \overline{y}_k) = (0, 2X) \times \mathbb{R} \cup \{(0,0), (2X, Y)\} =: B$, where $\overline{x}_{2k-1} = \overline{x}_{2k} = x_k$ and $\overline{y}_{2k-1} = y_k, \overline{y}_{2k} = 0$ for every $k \in \mathbb{N}$.

Proof. Observe that $\mathrm{A}(\overline{x}_k, \overline{y}_k) \subseteq B$. Indeed,

$$\sum_{k=1}^{\infty} (\overline{x}_k, \overline{y}_k) = \Big(2\sum_{k=1}^{\infty} x_k, \sum_{k=1}^{\infty} y_k\Big) = (2X, Y) \in \mathrm{A}(\overline{x}_k, \overline{y}_k)$$

and $(0,0) \in \mathrm{A}(\overline{x}_k, \overline{y}_k)$. Moreover $\sum_{k=1}^{\infty} \varepsilon_k \overline{x}_k \in (0, 2X)$ if in the sequence (ε_k) there is at least one 0 and at least one 1. Hence $\sum_{k=1}^{\infty} \varepsilon_k (\overline{x}_k, \overline{y}_k) \subseteq B$ for every $(\varepsilon_k) \in \{0,1\}^{\mathbb{N}}$.

To prove the reverse inclusion, it is sufficient to show that for every $(a,b)\in(0,2X)\times\mathbb{R}$ we have $(a,b)\in \mathrm{A}(\overline{x}_k,\overline{y}_k)$. Let $a\in(0,2X)$ and $b\in\mathbb{R}$. Let $I_a\subseteq \mathrm{A}(x_k)$ be an interval which satisfies $(\star)$. Then by the absolute convergence of $\sum_{k=1}^\infty x_k$ one can fix $(\varepsilon_1^a,\dots,\varepsilon_{k_a}^a)\in\{0,1\}^{k_a}$ for which $\sum_{k=1}^\infty \varepsilon_k^a x_k\in I_a$ for every $(\varepsilon_k^a)_{k=k_a+1}^\infty\in\{0,1\}^{\mathbb{N}}$. The conditional convergence of $\sum_{k=1}^\infty y_k$ yields $b=\sum_{k=1}^\infty \delta_k y_k$ for some $(\delta_k)\in\{0,1\}^{\mathbb{N}}$, where $\delta_k=\varepsilon_k^a$ for $k\le k_a$. Then for $t=\sum_{k=1}^\infty \delta_k x_k\in I_a$ we define $z=\sum_{k=1}^\infty \overline{\delta}_k x_k$, where $(\overline{\delta}_k)\in\{0,1\}^{\mathbb{N}}$ is such that $t+z=a$. Define $(\gamma_k)\in\{0,1\}^{\mathbb{N}}$ by alternating (δ_k) and $(\overline{\delta}_k)$, more precisely $\gamma_{2k-1}=\delta_k$ and $\gamma_{2k}=\overline{\delta}_k$ for every $k\in\mathbb{N}$. Then

$$\begin{aligned}\sum_{k=1}^\infty \gamma_k(\overline{x}_k,\overline{y}_k)&=\Big(\sum_{k=1}^\infty \gamma_k\overline{x}_k,\sum_{k=1}^\infty\gamma_k\overline{y}_k\Big)\\&=\Big(\sum_{k=1}^\infty \delta_k x_k+\sum_{k=1}^\infty\overline{\delta}_k x_k,\sum_{k=1}^\infty\delta_k y_k+\sum_{k=1}^\infty\overline{\delta}_k\cdot 0\Big)=(t+z,b+0)=(a,b).\end{aligned}$$

Finally, $\mathrm{A}(\overline{x}_k,\overline{y}_k)=(0,2X)\times\mathbb{R}\cup\{(0,0),(2X,Y)\}$. ∎

Now we will construct a series on the plane with an open achievement set.

THEOREM 4.2. *Let $\sum_{k=1}^\infty v_k=X<\infty$ with $v_k>0$ for every $k\in\mathbb{N}$ and such that $\mathrm{A}(v_k)-\mathrm{A}(v_k)=[-X,X]$. Let (w_k) be a decreasing null sequence of positive terms with $\sum_{k=1}^\infty w_k=\infty$. If $x_{4k-3}=x_{4k-2}=v_k$, $x_{4k-1}=x_{4k}=-v_k$ and $y_{4k-1}=y_{4k-3}=0$, $y_{4k-2}=w_{2k-1}$, $y_{4k}=-w_{2k}$ for every $k\in\mathbb{N}$ then $\mathrm{A}(x_k,y_k)=(-2X,2X)\times\mathbb{R}$.*

Proof. The inclusion $\mathrm{A}(x_k,y_k)\subset[-2X,2X]\times\mathbb{R}$ is obvious, since $\sum_{k=1}^\infty\varepsilon_k x_k\in[-2\sum_{k=1}^\infty v_k,2\sum_{k=1}^\infty v_k]$ for every $(\varepsilon_k)\in\{0,1\}^{\mathbb{N}}$. Moreover, the only way to obtain $2X$ in the first coordinate is to sum up all x_k's which are greater than 0, more precisely $2X=\sum_{k\in P}x_k$ where $P=\{k:x_k>0\}=\{4n-i:n\in\mathbb{N},\,i\in\{2,3\}\}$. We have $\sum_{k\in P}y_k=\sum_{k\in\mathbb{N}}w_{2k-1}=\infty$. Therefore $\mathrm{A}(x_k,y_k)\cap\{(2X,y):y\in\mathbb{R}\}=\emptyset$. In the same way we prove $\mathrm{A}(x_k,y_k)\cap\{(-2X,y):y\in\mathbb{R}\}=\emptyset$ by considering the indices $\mathbb{N}\setminus P$. Hence $\mathrm{A}(x_k,y_k)\subset(-2X,2X)\times\mathbb{R}$.

Now we prove the reverse inclusion. Fix $(a,r)\in(-2X,2X)\times\mathbb{R}$. Let (z_k) be the subsequence of (x_k) consisting of all even-numbered elements; clearly $(z_k)=(v_1,-v_1,v_2,-v_2,\dots)$. Then $\mathrm{A}(z_k)=\mathrm{A}(v_k)-\mathrm{A}(v_k)=[-X,X]$. Define $I_a=\{t:-X<t<X,\,-X<a-t<X\}$, which is a nonempty open interval. Then we can find a natural number N and $(\overline{\varepsilon_k})_{k=1}^N\in\{0,1\}^N$ such that $\sum_{k=1}^N\overline{\varepsilon_k}z_k+\sum_{k=N+1}^\infty\varepsilon_k z_k\in I_a$ for each $(\varepsilon_k)_{k=N+1}^\infty\in\{0,1\}^{\mathbb{N}}$. One can find $(\delta_k)_{k=N+1}^\infty\in\{0,1\}^{\mathbb{N}}$ such that $\sum_{k=1}^N\overline{\varepsilon_k}(-1)^{k+1}w_k+\sum_{k=N+1}^\infty\delta_k(-1)^{k+1}w_k=r$. Denote $t=\sum_{k=1}^N\overline{\varepsilon_k}z_k+\sum_{k=N+1}^\infty\delta_k z_k\in I_a$. Let $s_k=x_{2k-1}$ for $k\in\mathbb{N}$. Since $|a-t|<X$, one can find $(\gamma_k)_{k=1}^\infty\in\{0,1\}^{\mathbb{N}}$ such

that $\sum_{k=1}^{\infty}\gamma_k s_k = a - t$. Define $(\alpha_k)_{k=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ as follows: $\alpha_{2k} = \overline{\varepsilon_k}$ for $k \le N$, $\alpha_{2k} = \delta_k$ for $k > N$ and $\alpha_{2k-1} = \gamma_k$ for $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty}\alpha_k(x_k, y_k) = (a, r)$. Hence $\mathrm{A}(x_k, y_k) = (-2X, 2X) \times \mathbb{R}$. ∎

Remark 4.3. Let (v_k) be a sequence of positive real numbers which fulfills condition $(\star)$ from Theorem 4.1, which in turn implies $\mathrm{A}(v_k) + \mathrm{A}(v_k) = [0, 2X]$ (as follows immediately from Theorem 4.2). The latter equality implies that $\mathrm{A}(v_k) - \mathrm{A}(v_k) = [-X, X]$. To see this, assume that $\mathrm{A}(v_k) + \mathrm{A}(v_k) = [0, 2X]$ and fix $y \in [-X, X]$. Then $y + X \in [0, 2X]$, so $y + X = s + t$ for some $s, t \in \mathrm{A}(v_k)$. Hence $y = s + t - X = s - (X - t) \in \mathrm{A}(v_k) - \mathrm{A}(v_k)$. Note also that $\mathrm{A}(v_k) - \mathrm{A}(v_k) = [-X, X]$ implies $\mathrm{A}(v_k) + \mathrm{A}(v_k) = [0, 2X]$ provided $v_n \ge 0$.

On the other hand, if $v_k = 2/3^k$, then $\mathrm{A}(v_k)$ is the ternary Cantor set. It is well-known that $\mathrm{A}(v_k) + \mathrm{A}(v_k) = [0, 2]$, or equivalently $\mathrm{A}(v_k) - \mathrm{A}(v_k) = [-1, 1]$, but (v_k) does not satisfy $(\star)$ (the Cantor set contains no interval).

Example 4.4. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be

$$\left(\left(\tfrac{1}{2}, 1\right), \left(\tfrac{1}{2}, 0\right), \left(\tfrac{1}{4}, -\tfrac{1}{2}\right), \left(\tfrac{1}{4}, 0\right), \left(\tfrac{1}{8}, \tfrac{1}{3}\right), \left(\tfrac{1}{8}, 0\right), \left(\tfrac{1}{16}, -\tfrac{1}{4}\right), \left(\tfrac{1}{16}, 0\right), \ldots\right).$$

Then by Theorem 4.1 we have $\mathrm{A}(x_n, y_n) = (0, 2) \times \mathbb{R} \cup \{(0, 0), (2, \ln 2)\}$.

Example 4.5. If $(x_n, y_n)_{n\in\mathbb{N}}$ is

$$\left(\left(\tfrac{2}{3}, 0\right), \left(\tfrac{2}{3}, 1\right), \left(-\tfrac{2}{3}, 0\right), \left(-\tfrac{2}{3}, -\tfrac{1}{2}\right), \left(\tfrac{2}{9}, 0\right), \left(\tfrac{2}{9}, \tfrac{1}{3}\right), \left(-\tfrac{2}{9}, 0\right), \left(-\tfrac{2}{9}, -\tfrac{1}{4}\right), \ldots\right),$$

then by Theorem 4.2 we have $\mathrm{A}(x_n, y_n) = (-2, 2) \times \mathbb{R}$.

5. Achievement sets of potentially conditionally convergent series. In this section we consider the set $\bigcup_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)})$ for a series $\sum_{k=1}^{\infty} x_k$. If the series is absolutely convergent, then $\bigcup_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)}) = \mathrm{A}(x_k)$. Here we study the situation when the series is conditionally convergent. Clearly, only the multidimensional case is interesting.

Recall that $\sum_{k=1}^{\infty} x_k$ is potentially conditionally convergent if there exists a permutation σ for which $\sum_{k=1}^{\infty} x_{\sigma(k)}$ is conditionally convergent. We will also consider the set $\mathrm{A}_{\mathrm{abs}}(x_k) = \{\sum_{k=1}^{\infty}\varepsilon_k x_k : \sum_{k=1}^{\infty}\varepsilon_k\|x_k\| < \infty, \varepsilon_k \in \{0,1\}$ for each $k \in \mathbb{N}\}$.

Theorem 5.1. *Let $\sum_{k=1}^{\infty} y_k$ be an absolutely convergent series of real numbers such that the function $(\varepsilon_k) \mapsto \sum_{k=1}^{\infty}\varepsilon_k y_k$, where $(\varepsilon_k) \in \{0,1\}^{\mathbb{N}}$, is injective. Assume that $\sum_{k=1}^{\infty} x_k$ is conditionally convergent on the real line. Let $Y = \{\sum_{k=1}^{\infty}\varepsilon_k y_k : \sum_{k=1}^{\infty}\varepsilon_k x_k$ is potentially conditionally convergent$\}$. Then:*

(i) $\bigcap_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)}, y_{\sigma(k)}) = \mathrm{A}_{\mathrm{abs}}(x_k, y_k)$.
(ii) $\bigcup_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)}, y_{\sigma(k)}) = (\mathbb{R} \times Y) \cup \mathrm{A}_{\mathrm{abs}}(x_k, y_k)$.

Proof. (i) Let $(x,y)\in \mathrm{A}_{\mathrm{abs}}(x_k,y_k)$ and $\sigma\in S_\infty$. Then

$$(x,y)=\sum_{k=1}^{\infty}\varepsilon_k(x_k,y_k)=\sum_{k=1}^{\infty}\varepsilon_{\sigma(k)}(x_{\sigma(k)},y_{\sigma(k)}),$$

because $\sum_{k=1}^{\infty}\varepsilon_k\|(x_k,y_k)\| = \sum_{k=1}^{\infty}\|\varepsilon_k(x_k,y_k)\|<\infty$ and we know that every absolutely convergent series is unconditionally convergent. Hence (x,y) is in $\bigcap_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)},y_{\sigma(k)})$.

Conversely, let $(x,y)\in\bigcap_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)},y_{\sigma(k)})$. Fix $\sigma\in S_\infty$. There are $(\varepsilon_j),(\varepsilon'_k)\in\{0,1\}^{\mathbb N}$ such that

$$(x,y)=\sum_{j=1}^{\infty}\varepsilon_j(x_j,y_j)=\sum_{k=1}^{\infty}\varepsilon'_k(x_{\sigma(k)},y_{\sigma(k)})=\sum_{k=1}^{\infty}\varepsilon''_{\sigma(k)}(x_{\sigma(k)},y_{\sigma(k)})$$

where $\varepsilon''_k=\varepsilon'_{\sigma^{-1}(k)}$. Since $(\varepsilon_k)\in\{0,1\}^{\mathbb N}$ is injective, the sequence (y_k) is injective as well, and if $y_j=y_{\sigma(k)}$, then $j=\sigma(k)$ and $\varepsilon_j=\varepsilon''_{\sigma(k)}$. Thus $\varepsilon_{\sigma(k)}=\varepsilon''_{\sigma(k)}$. Therefore $x=\sum_{k=1}^{\infty}\varepsilon_{\sigma(k)}x_{\sigma(k)}$ for any $\sigma\in S_\infty$, and consequently $\sum_{k=1}^{\infty}\varepsilon_k x_k$ is unconditionally convergent. Without loss of generality we may assume that $\|(x,y)\|=|x|+|y|$. We have $\sum_{k=1}^{\infty}\varepsilon_k\|(x_k,y_k)\|=\sum_{k=1}^{\infty}\varepsilon_k(|x_k|+|y_k|)<\infty$. Hence $(x,y)\in\mathrm{A}_{\mathrm{abs}}(x_k,y_k)$.

(ii) “$\subseteq$”. Let $(x,y)=\sum_{k=1}^{\infty}\varepsilon_k(x_{\sigma(k)},y_{\sigma(k)})$ for some $(\varepsilon_k)\in\{0,1\}^{\mathbb N}$ and $\sigma\in S_\infty$. We have two possibilities:

(1) If $\sum_{k=1}^{\infty}\varepsilon_k x_{\sigma(k)}$ is absolutely convergent, then

$$\sum_{k=1}^{\infty}\varepsilon_{\sigma^{-1}(k)}x_k=\sum_{k=1}^{\infty}\varepsilon_k x_{\sigma(k)}=x$$

and also $\sum_{k=1}^{\infty}\varepsilon_{\sigma^{-1}(k)}y_k=\sum_{k=1}^{\infty}\varepsilon_k y_{\sigma(k)}=y$. Hence $\sum_{k=1}^{\infty}\varepsilon_{\sigma^{-1}(k)}(x_k,y_k)$ is absolutely convergent, so $(x,y)\in\mathrm{A}_{\mathrm{abs}}(x_k,y_k)$.

(2) If $\sum_{k=1}^{\infty}\varepsilon_k x_{\sigma(k)}$ is conditionally convergent, then it is also potentially conditionally convergent, so $(x,y)\in\mathbb R\times Y$.

“$\supseteq$”. From (i) we know that

$$\mathrm{A}_{\mathrm{abs}}(x_k,y_k)=\bigcap_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)},y_{\sigma(k)})\subset\bigcup_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)},y_{\sigma(k)}),$$

so it is enough to show that $\mathbb R\times Y\subset\bigcup_{\sigma\in S_\infty}\mathrm{A}(x_{\sigma(k)},y_{\sigma(k)})$. Fix $a\in\mathbb R$ and $y\in Y$. Then there exist $(\varepsilon_k)\in\{0,1\}^{\mathbb N}$ and $\sigma\in S_\infty$ such that $y=\sum_{k=1}^{\infty}\varepsilon_k y_{\sigma(k)}$ and $\sum_{k=1}^{\infty}\varepsilon_k x_{\sigma(k)}$ converge conditionally. One can rearrange the terms of $\sum_{k=1}^{\infty}\varepsilon_k x_{\sigma(k)}$ so that $a=\sum_{k=1}^{\infty}\varepsilon_{\tau(k)}x_{\tau(\sigma(k))}$. Since $\sum_{k=1}^{\infty}y_k$ is absolutely convergent, we have

$$\sum_{k=1}^{\infty}\varepsilon_{\tau(k)}y_{\tau(\sigma(k))}=\sum_{k=1}^{\infty}\varepsilon_k y_{\sigma(k)}=y.$$

Hence $(a,y)\in\mathrm{A}(x_{\tau(\sigma(k))},y_{\tau(\sigma(k))})$. ∎

Take any (x_n, y_n) which fulfills the assumptions of Theorem 5.1. By Theorem 3.5, $\mathrm{A}(x_n, y_n)$ is dense in $\mathbb{R} \times \mathrm{A}(y_n)$. So is $S = \bigcup_{\sigma \in S_\infty} \mathrm{A}(x_{\sigma(k)}, y_{\sigma(k)})$. By Theorem 5.1 the horizontal section S_y equals $\mathbb{R}$ if $y \in Y$; it is a singleton if $x = \sum_{k=1}^\infty \varepsilon_k x_k$ is absolutely convergent; and it is empty if $\sum_{k=1}^\infty \varepsilon_k x_k$ is not potentially conditionally convergent.

Let us finish the paper with a list of open questions:

(1) Does there exist a conditionally convergent series $\sum_{n=1}^\infty x_n$ on the plane such that $\mathrm{A}(x_n)$ is the graph of a function with domain being a bounded interval?
(2) Let $\mathrm{SR}(x_n) = \mathbb{R}^k$. Is it true that either $\mathrm{A}(x_n) = \mathbb{R}^k$ or $\mathrm{A}(x_n)$ is of measure zero?
(3) Lemma 3.2 implies that the achievement sets of conditionally convergent series in finite-dimensional spaces are unbounded. On the other hand, there is an example of a conditionally convergent series in c_0 with a closed and bounded achievement set (Example 3.4). Obviously, such a series can be found in every Banach space containing an isomorphic copy of c_0. Note that the series from Example 3.4 has an unbounded achievement set in ℓ_1 (and it is well-known that ℓ_1 does not contain a copy of c_0). Is there a conditionally convergent series in ℓ_1 with a bounded and closed achievement set?
(4) Is there a conditionally convergent series in some Banach space whose achievement set is compact?
(5) In the proof of Theorem 3.9, we show that the achievement set is analytic. Moreover there are analytic sets which are not Borel. Is there a (conditionally convergent) series whose achievement set is non-Borel?

Recently, Professor Ajit Iqbal Singh has informed the authors about a series of papers devoted to ranges of vector measures (see [LS] and references therein). The range $\{\mu(A) : A \text{ measurable}\}$ of a purely atomic finite measure μ is the achievement set $\mathrm{A}(x_n)$ where x_n is the measure of the nth μ-atom (by finiteness of μ there are at most countably many μ-atoms). Hence, there is a strict connection between these studies and our paper (and other articles on multidimensional achievement sets [BG, M1, M2]).

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