CARDINALITY OF SETS OF ρ-UPPER AND ρ-LOWER CONTINUOUS FUNCTIONS

MAREK BIENIAS, SZYMON GŁĄB AND WŁADYSŁAW WILCZYŃSKI

Abstract. We prove that the cardinality of the set of all 1-upper continuous functions $f:(0,1)\to\mathbb{R}$ equals $2^{\mathfrak{c}}$. In particular, there is a non-Borel 1-upper continuous function. We also prove that there are $2^{\mathfrak{c}}$ ρ-lower continuous functions for $\rho\in(0,\frac{1}{2})$.

1. Introduction

S. Kowalczyk and K. Nowakowska in [4] introduced the notion of ρ-upper continuous functions, where $\rho\in(0,1)$. The notion of ρ-upper continuity is an example of the so called path continuity, which was widely described in [1]. They prove that each function of that class is Lebesgue measurable and has the Denjoy property. They also show that for any $\rho\in(0,\frac{1}{2})$, there are ρ-continuous functions which are not of Baire class one. Similar class, of the so called $[\lambda,\rho]$-continuous functions, was studied by K. Nowakowska in [7]. In [5] and [6], S. Kowalczyk and K. Nowakowska studied the so-called maximal additive and multiplicative classes for $[\lambda,\rho]$-continuous and ρ-upper continuous functions. A. Karasińska and E. Wagner-Bojakowska (cf. [2]) showed that there exists a function which is 1-upper continuous (i.e. ρ-upper continuous for each $\rho\in[0,1)$) and is not approximately continuous. Moreover, they showed that there is a function which is 1-upper continuous but is not of Baire class one.

In this paper we prove that there are $2^{\mathfrak{c}}$ functions which are 1-upper continuous and $2^{\mathfrak{c}}$ functions which are ρ-lower continuous, for $\rho\in(0,\frac{1}{2})$. In particular, there are non-Borel 1-upper continuous and ρ-lower continuous functions. We also show that the class of all ρ-upper continuous functions for $\rho\in(0,1)$ is not closed under point-wise addition, and therefore it does not form a linear subspace of $\mathbb{R}^{\mathbb{R}}$.

We use standard set-theoretic notation – for any undefined notion we refer the reader to A. Kechris's monograph [3]. Let $\mathbb{N}=\{1,2,3,...\}$ stands for the set of all natural numbers and let m stands for Lebesgue measure on the real line. Let E be a measurable subset of $\mathbb{R}$ and let $x\in\mathbb{R}$. The numbers

$$\underline{d}^{+}(E,x)=\liminf_{t\to 0^{+}}\frac{m(E\cap[x,x+t])}{t}$$

and

$$\overline{d}^{+}(E,x)=\limsup_{t\to 0^{+}}\frac{m(E\cap[x,x+t])}{t}$$

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are called respectively the right lower density of E at x and right upper density of E at x. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E,x) = \overline{d}^+(E,x) \quad \text{and} \quad \underline{d}^-(E,x) = \overline{d}^-(E,x),$$

then we call these numbers the right density and left density of E at x, respectively. The numbers

$$\underline{d}(E,x) = \liminf_{t,k\to 0^+} \frac{m(E\cap[x-t,x+k])}{t+k}$$

and

$$\overline{d}(E,x) = \limsup_{t,k\to 0^+} \frac{m(E\cap[x-t,x+k])}{t+k}$$

are called the upper and lower density of E at x, respectively. Note that

$$\underline{d}(E,x) = \min\{\underline{d}^-(E,x), \underline{d}^+(E,x)\}$$

and

$$\overline{d}(E,x) = \max\{\overline{d}^-(E,x), \overline{d}^+(E,x)\}.$$

If $\underline{d}(E,x) = \overline{d}(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$. If $d(E,x) = 1$, then we say that x is a density point of E.

Let us recall the notion of ρ-upper and ρ-lower continuity.

Definition 1. *Let $\rho \in (0,1)$ and let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval. We say that f is*

(i) ρ-upper continuous at $x \in I$ provided there exists a measurable set $E \subseteq I$ with $x \in E$, such that $\overline{d}(E,x) > \rho$ and $f|_E$ is continuous at x;

(ii) ρ-lower continuous at $x \in I$ provided there exists a measurable set $E \subseteq I$ with $x \in E$, such that $\underline{d}(E,x) > \rho$ and $f|_E$ is continuous at x;

If f is ρ-upper (ρ-lower resp.) continuous at every point of I, we say that f is ρ-upper (ρ-lower resp.) continuous.

We will denote the class of all ρ-upper (ρ-lower resp.) continuous functions defined on a unit interval $(0,1)$ by $\mathcal{UC}_\rho$ ($\mathcal{LC}_\rho$ resp.). We say that f is 1-upper continuous if it is ρ-upper continuous for every $\rho \in [0,1)$.

For any nonempty set A we will denote the family of all finite sequences of elements of A by $A^{<\mathbb{N}}$. For any finite sequence $s = (s_1, ..., s_n) \in A^{<\mathbb{N}}$ and $a \in A$ by $s\hat{\ }a$ we denote a concatenation of s and a, i.e. $s\hat{\ }a = (s_1, ..., s_n, a)$. By $|s|$ we denote the length of s. If $\alpha \in A^{\mathbb{N}}$, then let $\alpha|n = (\alpha(1), ..., \alpha(n))$ and $\alpha|0 = \emptyset$. Moreover, by $2^{<\mathbb{N}}$ (resp. $2^{\mathbb{N}}$) we mean the set $\{0,1\}^{<\mathbb{N}}$ (resp. $\{0,1\}^{\mathbb{N}}$). For $n \in \mathbb{N}$ we denote $2^n = \{s \in 2^{<\mathbb{N}} : |s| = n\}$ and $2^0 = \{\emptyset\}$.

2. Cardinality of the set $\mathcal{UC}_\rho$

Note that the definition of upper 1-continuous functions and approximately continuous functions are similar but not identical. We have that f is 1-upper continuous at x if there is a measurable set E such that $\overline{d}(E,x) = 1$ and $f|_{E\cup\{x\}}$ is continuous, and we say that f is approximately continuous at x if there is a measurable set E such that $d(E,x) = 1$ and $f|_{E\cup\{x\}}$ is continuous. This slight difference in the definition has a huge consequence. Since an approximately continuous function is of Baire class one, there are $\mathfrak{c}$ approximately continuous functions. In this section we show that there are $2^{\mathfrak{c}}$ functions which are 1-upper continuous.

The main idea is the following. We may define sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ of positive numbers such that $x_{n+1} < y_{n+1} < u_{n+1} < w_{n+1} < x_n$ for each $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ tends to 0 and

$$\overline{d}(\bigcup_{n\in\mathbb{N}}[x_n, y_n], 0) = \overline{d}(\bigcup_{n\in\mathbb{N}}[u_n, w_n], 0) = 1.$$

Then, we define $f : \mathbb{R} \to \mathbb{R}$ in the following way:

- put $f(x) = 1$ if $x \in [x_n, y_n]$ or $-x \in [x_n, y_n]$ or $x > x_1$ or $x < -x_1$
- put $f(x) = 0$ if $x \in [u_n, w_n]$ or $-x \in [u_n, w_n]$
- on $\mathbb{R} \setminus \{0\}$ define f to be (locally) affine.

The question is how to define f at 0? One can put $f(0) = 1$ or $f(0) = 0$. In both cases f is 1-upper continuous at 0, and consequently f is 1-upper continuous on its domain. Our plan is to make a similar construction of a function f for which the set A where we can freely put 0 or 1 is large, i.e. of cardinality $\mathfrak{c}$. Since there are $2^{\mathfrak{c}}$ functions from A to $\{0,1\}$ our construction will show that we may define f in $2^{\mathfrak{c}}$ ways to get a 1-upper continuous function.

Theorem 2. *The set $\mathcal{UC}_1$ has cardinality $2^{\mathfrak{c}}$. In particular there is a non-Borel 1-upper continuous function.*

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers from the interval $(0, \frac{1}{3})$ that it is convergent to 0. One can construct a sequence $\{I_s : s \in 2^{<\mathbb{N}}\}$ of closed subintervals of $[0,1]$ such that

1. $I_\emptyset = [0,1]$;
2. $\forall s \in 2^{<\mathbb{N}}\ \min I_{s\hat{}0} = \min I_s, \max I_{s\hat{}1} = \max I_s$;
3. $\forall s \in 2^{<\mathbb{N}}\ \forall i \in \{0,1\}\ |I_{s\hat{}i}| = q_{|s|+1}|I_s|$.

Let

$$\mathcal{C} = \bigcup_{\alpha\in 2^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} I_{\alpha|n}.$$

One can easily check, that $\mathcal{C}$ is a Cantor-like set (i.e. perfect, zero-dimensional and of measure zero).

The above construction is similar to the classical geometric construction of the ternary Cantor. Starting from the unit interval $I_\emptyset = [0,1]$, in the first step we remove the open middle subinterval leaving two subintervals: I_0, I_1 of lengths q_1. In the second step the open middle subintervals of the remaining intervals I_0, I_1 are removed, leaving four segments: $I_{00}, I_{01}, I_{10}, I_{11}$ of lengths $q_2|I_0| = q_2 \cdot q_1$ and so on.

Let $A, B \subseteq [0,1]$ be the unions of the closures of intervals removed in the odd and even steps respectively, i.e.

$$A = \bigcup_{k\in\mathbb{N}} \bigcup_{s\in 2^{2k-2}} \overline{I_s \setminus (I_{s\hat{}0} \cup I_{s\hat{}1})}$$

$$B = \bigcup_{k\in\mathbb{N}} \bigcup_{s\in 2^{2k-1}} \overline{I_s \setminus (I_{s\hat{}0} \cup I_{s\hat{}1})}$$

Observe, that $A \cup B = [0,1] \setminus \mathcal{C}'$, where $\mathcal{C}' \subseteq \mathcal{C}$ is created by removing the boundary of the set $[0,1] \setminus \mathcal{C}$ from $\mathcal{C}$. In particular $\mathcal{C}'$ has cardinality $\mathfrak{c}$. Let $g : A \cup B \to \{0,1\}$ be a characteristic function of A. To illustrate the idea, the function g is created by putting 1 over the closure of the intervals removed in the odd step and putting 0 elsewhere. Let $\mathcal{C}'' = \mathcal{C}' \setminus \{0,1\}$, let $F \subseteq \mathcal{C}''$ and define a function $g_F : (0,1) \to \{0,1\}$ by the formula

$$g_F(x) = \begin{cases} 1, \text{ when } x \in F \\ g(x), \text{ when } x \in A \cup B \\ 0, \text{ otherwise.} \end{cases}$$

We will show that g_F is 1-upper continuous. Let $x \in (0,1)$ and let $E = g_F^{-1}(\{g_F(x)\})$. We have the following possibilities:

(1) $x \in A$, then $\overline{d}^+(E,x) = 1$ or $\overline{d}^-(E,x) = 1$. Hence, $\overline{d}(E,x) = 1$ and g_F is 1-upper continuous at x.

(2) $x \in B$, then $\overline{d}^+(E,x) = 1$ or $\overline{d}^-(E,x) = 1$. Hence, $\overline{d}(E,x) = 1$ and g_F is 1-upper continuous at x.

(3) $x \in F$, then there are sequences: $(n_k)_{k\in\mathbb{N}}$ of odd numbers, $(J_{n_k})_{k\in\mathbb{N}}$ of intervals, $(s_{n_k})_{k\in\mathbb{N}}$ of finite $0-1$ sequences, such that for every $k \in \mathbb{N}$:
   a) $|s_{n_k}| = n_k - 1$;
   b) $\{x\} = \bigcap_{k\in\mathbb{N}} I_{s_{n_k}}$;
   c) J_{n_k} is a connected component of A;
   d) $\frac{|J_{n_k}|}{|I_{s_{n_k}}|} = 1 - 2q_{n_k}$;
   e) both sequences $(\min J_{n_k})_{k\in\mathbb{N}}, (\max J_{n_k})_{k\in\mathbb{N}}$ converge to x from the right.

   The above sequences can be chosen in the following way: there is a sequence $(J_{n_k})_{k\in\mathbb{N}}$ of intervals removed in the odd step (i.e. connected components of A) that is convergent (in the sense of (e)) to x from the right and such that the interval J_{n_k} was removed exactly from

$I_{s_{n_k}}$. By c) and the definition of g, we have that $J_{n_k} \subseteq E$. Hence, by b), d) and e) we obtain that $\overline{d}^+(E,x) = 1$ and g_F is 1-upper continuous at x.

(4) $x \in \mathcal{C}'' \setminus F$, then by a similar reasoning as in the case when $x \in F$, the function g_F is 1-upper continuous at x.

Since there are exactly $2^{\mathfrak{c}}$ subsets of $\mathcal{C}''$, the set $\mathcal{UC}_1$ has cardinality $2^{\mathfrak{c}}$. □

3. Cardinality of the set $\mathcal{LC}_\rho$

Let $\rho \in (0, \frac{1}{2})$. This section is devoted to the proof that the cardinality of the set $\mathcal{LC}_\rho$ is $2^{\mathfrak{c}}$. The idea of a construction of $2^{\mathfrak{c}}$ ρ-lower continuous functions is similar to the one from the previous section and uses a geometric construction of the ternary Cantor set.

Lemma 3. *Let $b_n = \frac{1}{2n-1}$ and $a_n = \frac{1}{2n}$, for $n \in \mathbb{N}$. Let $A = \bigcup_{n\in\mathbb{N}} [a_n, b_n)$, then $d^+(A,0) = \frac{1}{2}$.*

The above fact is probably folklore, but for the reader's convenience we state the proof.

Proof. For $h \in (0,1)$ let $\varphi(h) = \frac{m(A\cap(0,h))}{h}$. Let $n \in \mathbb{N}$ and observe that

$$\varphi(\frac{1}{2n-1}) > \frac{1}{2}$$

and

$$\varphi(\frac{1}{2n}) < \frac{1}{2}.$$

By the simple properties of the function φ, we have that $d^+(A,0) = \lim_{n\to\infty} \varphi(\frac{1}{n}) = \frac{1}{2}$. □

Lemma 4. *For every open interval $(a,b) \subseteq (0,1)$ and every $\varepsilon > 0$ there exists a set $E \subseteq (a,b)$ such that for all $h \in (0, b-a)$*

$$\left| \frac{m(E \cap (a, a+h))}{h} - \frac{1}{2} \right| < \varepsilon.$$

Proof. Let $(a,b) \subseteq (0,1), \varepsilon > 0$ and let $A \subseteq (0,1)$ be as in Lemma 3. There exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{m(A \cap (0, \frac{1}{n_0}))}{\frac{1}{n_0}} - \frac{1}{2} \right| < \varepsilon.$$

In particular for every $n \geq n_0$

$$\left| \frac{m(A \cap (0, \frac{1}{n}))}{\frac{1}{n}} - \frac{1}{2} \right| < \varepsilon.$$

Let us put $A_{n_0} = A \cap (0, \frac{1}{n_0}), B_{n_0} = (0, \frac{1}{n_0}) \setminus A_{n_0}$. The idea is to fit the set A_{n_0} into the first half of (a,b) and to fit the set $-B_{n_0}$ into the second one, i.e. let $\tilde{E} = (\alpha \cdot A_{n_0} + a) \cup (b - \alpha \cdot B_{n_0})$, where $\alpha = \frac{(b-a)n_0}{2}$.

For the further applications we have to modify the set $\tilde{E}$. Let $\{..., x_{-1}, x_0, x_1, x_2, ...\}$ be an increasing sequence of all endpoints of the intervals that form the set $\tilde{E}$, where $x_0 = \frac{a+b}{2}$. For every $i \in \mathbb{Z}$ one can choose a small enough $h_i > 0$ such that for the set $F = \bigcup_{i\in\mathbb{Z}} (x_i - h_i, x_i + h_i)$ we have

- $d^+(F,a)=0$;
- $d^-(F,b)=0$.

It is easy to see that $E=\tilde{E}\setminus F$ has the desired properties. □

Remark 5. *In the sequel we will be using the following construction: for an interval $(a,b)\subseteq(0,1)$ and $\varepsilon>0$, let $E\subseteq(a,b)$ be as in Lemma 4 and define a function $f:(a,b)\to[0,1]$ by the formula*

$$f(x)=\begin{cases}1, \text{ when } x\in E\\ 0, \text{ when } x\in(a,b)\setminus(E\cup F)\\ \text{locally affine, when } x\in F\end{cases}$$

where F is as in the proof of Lemma 4 and the locally affine mappings over F ensure that f is continuous.

Theorem 6. *Let $\rho\in(0,\frac{1}{2})$. The set $\mathcal{LC}_\rho$ has cardinality $2^{\mathfrak{c}}$. In particular there is a non-Borel ρ-lower continuous function.*

Proof. Consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, where $\varepsilon_n=\frac{1}{10^n}$ for $n\in\mathbb{N}$. One can construct a sequence $\{I_s : s\in 2^{<\mathbb{N}}\}$ of closed subintervals of $[0,1]$ such that

1. $I_\emptyset=[0,1]$;
2. $\forall s\in 2^{<\mathbb{N}}\ \min I_{s\hat{}0}=\min I_s, \max I_{s\hat{}1}=\max I_s$;
3. $\forall s\in 2^{<\mathbb{N}}\ \forall i\in\{0,1\}\ |I_{s\hat{}i}|=\frac{1}{3}|I_s|$.

Let

$$\mathcal{C}=\bigcup_{\alpha\in 2^{\mathbb{N}}}\bigcap_{n\in\mathbb{N}} I_{\alpha|n},$$

then $\mathcal{C}$ is the ternary Cantor set.

The idea of the proof is to define a function $f:(0,1)\setminus\mathcal{C}\to[0,1]$ by putting functions as in Remark 5 for $\varepsilon_n>0$ inside all of the intervals removed in the n^{th} step. To formalize this concept: for every $n\in\mathbb{N}$ and $s\in 2^{n-1}$ let $J_s=I_s\setminus(I_{s\hat{}0}\cup I_{s\hat{}1})$ and $f_s:J_s\to[0,1]$ be as in Remark 5 for $\varepsilon_{|s|+1}=\varepsilon_n$. Let us put $f=\bigcup_{s\in 2^{<\mathbb{N}}} f_s$. We will show that for any $D\subseteq\mathcal{C}\setminus\{0,1\}$, the function $f_D:(0,1)\to[0,1]$ defined by

$$f_D(x)=\begin{cases}1, \text{ when } x\in D\\ f(x), \text{ when } x\in(0,1)\setminus\mathcal{C}\\ 0, \text{ otherwise,}\end{cases}$$

is ρ-lower continuous. Let $D\subseteq\mathcal{C}\setminus\{0,1\}$ and fix a point $x\in(0,1)$ and let $B=f_D^{-1}(\{f_D(x)\})$. We have the following possibilities:

(1) $x\in(0,1)\setminus\mathcal{C}$, then f_D is continuous at x. Hence, it is ρ-lower continuous.

(2) $x \in \mathcal{C} \setminus \{0,1\}$ and x is not an endpoint of any interval J_s. We will show that $\underline{d}^+(B,x) > \rho$. Let $h > 0$ and consider the right neighborhood of x of the form $(x, x+h)$. We have two possibilities:

a) $x+h \in \mathcal{C}$, then for $\mathcal{J} = \{J_s : J_s \subseteq (x, x+h)\}$ and $\mathcal{S} = \{s \in 2^{<\mathbb{N}} : J_s \in \mathcal{J}\}$ we have that $h = m((x,x+h)) = \sum_{J \in \mathcal{J}} m(J)$. Moreover, for $N_h = \min\{|s|+1 : s \in \mathcal{S}\}$, N_h tends to ∞ whenever $h \to 0$. By Lemma 4 the following estimation holds

$$(\frac{1}{2} - \varepsilon_{N_h})h < m(B \cap (x, x+h)) < (\frac{1}{2} + \varepsilon_{N_h})h.$$

(b) or $x+h \notin \mathcal{C}$, then there is a finite $0-1$ sequence s_h such that $x+h \in J_{s_h}$. Let $\mathcal{J} = \{J_s : J_s \subseteq (x,x+h)\}$ and $\mathcal{S} = \{s \in 2^{<\mathbb{N}} : J_s \in \mathcal{J}\}$. We have that

$$h = m((x,x+h) \cap (\bigcup \mathcal{J} \cup J_{s_h})).$$

Moreover, let $N_h = \min\{|s|+1 : s \in \mathcal{S}\}$ and $N = \min\{N_h, |s_h|+1\}$. Observe that $N \to \infty$ whenever $h \to 0$. Let $d = \min J_{s_h}$. Since

$$m(B \cap (x,x+h)) = m(B \cap \bigcup \mathcal{J}) + m(B \cap (d, x+h)),$$

by Lemma 4 the following inequalities hold:

$$m(B \cap (x,x+h)) < (\frac{1}{2} + \varepsilon_{N_h})(d-x) + (\frac{1}{2} + \varepsilon_{|s_h|+1})(x+h-d) < (\frac{1}{2} + \varepsilon_N)h$$

and

$$m(B \cap (x,x+h)) > (\frac{1}{2} - \varepsilon_{N_h})(d-x) + (\frac{1}{2} - \varepsilon_{|s_h|+1})(x+h-d) > (\frac{1}{2} - \varepsilon_N)h.$$

By the above calculations, for a small enough $h > 0$, the number $\frac{m(B\cap(x,x+h))}{h}$ can be as close to $\frac{1}{2}$ as we want. In particular $\underline{d}^+(B,x) > \rho$. By a similar argument (taking the intervals from the left neighborhood of x) we may prove that $\underline{d}^-(B,x) > \rho$.

(3) $x \in \mathcal{C} \setminus \{0,1\}$ and x is an endpoint of some interval J_s. In this case $\underline{d}^+(B,x) > \rho$ and $\underline{d}^-(B,x) > \rho$ as well, where one of these inequalities follows from Lemma 4 and the other one can be proved by a similar argument as in the previous case.

Summarizing, $\underline{d}(B,x) > \rho$ and f_D is ρ-lower continuous at x. The fact that there are $2^{\mathfrak{c}}$ subsets of $\mathcal{C} \setminus \{0,1\}$ completes the proof.

□

We end the paper with two open questions:

(1) Does there exist a linear space $X \subseteq \mathbb{R}^{\mathbb{R}}$ of dimension $2^{\mathfrak{c}}$ such that any $f \in X \setminus \{0\}$ is 1-upper continuous? In other words: is the set $\mathcal{UC}_1$ $2^{\mathfrak{c}}$-lineable?

(2) What is the cardinality of the set of $[\lambda, \rho]$-continuous functions? Is it $2^{\mathfrak{c}}$ for some positive λ and ρ?

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Institute of Mathematics, Łódź University of Technology, Wólczańska 215, 93-005 Łódź, Poland

E-mail address: marek.bienias88@gmail.com, szymon.glab@p.lodz.pl

Department of Real Functions, Faculty of Mathematics, University of Łódź, Banacha 22, 90-238, Łódź, Poland

E-mail address: wwil@uni.lodz.pl